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SOLUTION OF A RATIONAL RECURSIVE SEQUENCES OF ORDER THREE

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Abstract: We obtain in this paper the solutions of the difference equations

$$x_{n+1} = \frac{ax_n x_{n-2}}{x_{n-1}(-b + cx_n x_{n-2})}, \qquad n = 0, 1, ...,$$

where a, b, c are positive real numbers and the initial conditions are arbitrary positive real numbers.

Keywords: difference equations, recursive sequences, stability, periodic solution.

1. Introduction

In this paper we obtain the solutions of the following recursive sequences

$$x_{n+1} = \frac{ax_n x_{n-2}}{x_{n-1}(-b + cx_n x_{n-2})}, \qquad n = 0, 1, ..., \tag{1}$$

where a, b, c are positive real numbers and the initial conditions are arbitrary positive real numbers.

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, see [1–40] and references therein.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

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Aloqeili [4] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [5]-[7] investigated the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}, \qquad x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}, \qquad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Elabbasy et al. [8]-[9] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequences

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}, \qquad x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a.$$

Ibrahim [26] get the solutions of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}.$$

Karatas et al. [27] get the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Simsek et al. [32] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [29].

Definition 1 (equilibrium point). *A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if*

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geqslant 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is the fixed point of the map

$$x \to f(x, x, ..., x).$$

Definition 2 (stability).

(i) *The equilibrium point* $\overline{x}$ *of Eq.(2) is locally stable if for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ *with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \qquad \textit{for all} \quad n \geqslant -k.$$

(ii) *The equilibrium point* $\overline{x}$ *of Eq.(2) is locally asymptotically stable if* $\overline{x}$ *is locally stable solution of Eq.(2) and there exists* $\gamma > 0$, *such that for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ *with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) *The equilibrium point* $\overline{x}$ *of Eq.(2) is global attractor if for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, *we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) *The equilibrium point* $\overline{x}$ *of Eq.(2) is globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of Eq.(2).*

(v) *The equilibrium point* $\overline{x}$ *of Eq.(2) is unstable if* $\overline{x}$ *is not locally stable.*

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} \, y_{n-i}.$$

Theorem A ([29]). *Assume that* $p_i \in R$, $i = 1, 2, ..., k$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \qquad n = 0, 1, ...$$

Definition 3 (periodicity). *A sequence* $\{x_n\}_{n=-k}^{\infty}$ *is said to be periodic with period* p *if* $x_{n+p} = x_n$ *for all* $n \geqslant -k$.

2. Local stability of the equilibrium points

Here we study the local stability character of the solutions of Eq.(1).

The equilibrium points of Eq.(1) are given by the relation

$$\overline{x} = \frac{a\overline{x}^2}{\overline{x}(-b + c\overline{x}^2)},$$

then Eq.(1) has a positive equilibrium point

$$\overline{x} = \sqrt{\frac{a+b}{c}}.$$

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a function defined by

$$f(u, v, w) = \frac{auw}{v(-b + cuw)}.$$

Thus

$$\frac{\partial f(u,v,w)}{\partial u} = \frac{-abw}{v(-b+cuw)^2}, \qquad \frac{\partial f(u,v,w)}{\partial v} = \frac{-auw}{v^2(-b+cuw)},$$

$$\frac{\partial f(u,v,w)}{\partial w} = \frac{-abu}{v(-b+cuw)^2}.$$

Then

$$\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial u} = -\frac{b}{a}, \qquad \frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial v} = -1, \qquad \frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial w} = \frac{-b}{a}.$$

The linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} + \frac{b}{a}y_n + y_{n-1} + \frac{b}{a}y_{n-2} = 0. \tag{3}$$

Theorem 1. *The equilibrium point $\overline{x}$ of Eq.(1) is not locally stable.*

Proof. If the equilibrium point $\overline{x}$ stable, then it follows by Theorem A that, Eq.(3) is asymptotically stable if

$$\left|\frac{b}{a}\right| + 1 + \left|\frac{b}{a}\right| < 1,$$

which is contradiction. The proof is complete. ■

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq.(1).

Example 1. Consider $a = 8$, $b = 6$, $c = 9$, $x_{-2} = 2$, $x_{-1} = 6$, $x_0 = 11$. See Fig. 1.

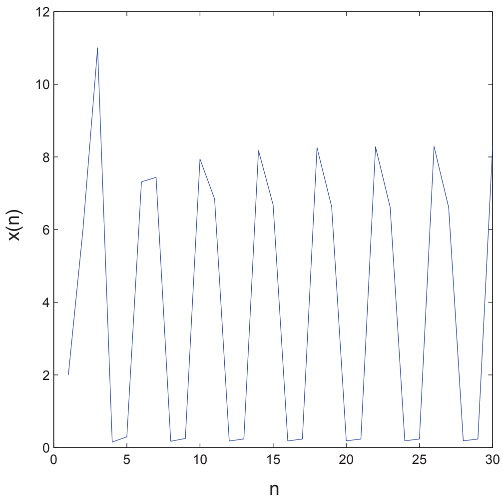

Figure 1: Plot of $x_{n+1} = (x_n x_{n-2})/(a x_{n-1}(-b + c x_n x_{n-2}))$

Example 2. See Fig. 2, since $a = 14,\ b = 11,\ c = 2$, $x_{-2} = 5,\ x_{-1} = 13,\ x_0 = 7$.

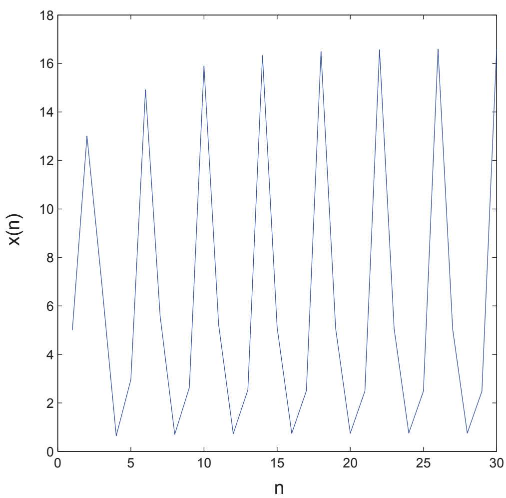

Figure 2: Plot of $x_{n+1} = (x_n x_{n-2})/(a x_{n-1}(-b + c x_n x_{n-2}))$

3. Solution of the difference equation $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(-1+x_n x_{n-2})}$

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(-1+x_n x_{n-2})}, \qquad n = 0, 1, ..., \tag{4}$$

where the initial conditions are arbitrary nonzero positive real numbers and $x_{-2}x_0 \neq 1$.

Theorem 2. *Every solution $\{x_n\}_{n=-2}^{\infty}$ of Eq.(4) is periodic with period 4; more precisely for $n = 0, 1, ...$*

$$x_{4n-2} = r, \qquad x_{4n-1} = k, \qquad x_{4n} = h, \qquad x_{4n+1} = \frac{hr}{k(-1+hr)},$$

where $x_{-2} = r,\ x_{-1} = k,\ x_0 = h.$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{4n-6} = r, \qquad x_{4n-5} = k, \qquad x_{4n-4} = h, \qquad x_{4n-3} = \frac{hr}{k(-1+hr)}.$$

Now, it follows from Eq.(4) that

$$\begin{aligned}
x_{4n-2} &= \frac{x_{4n-3}x_{4n-5}}{x_{4n-4}(-1+x_{4n-3}x_{4n-5})} = \frac{hrk}{k(-1+hr)h\left(-1+\frac{hrk}{k(-1+hr)}\right)} \\
&= \frac{r}{(1-hr+hr)} = r, \\
x_{4n-1} &= \frac{x_{4n-2}x_{4n-4}}{x_{4n-3}(-1+x_{4n-2}x_{4n-4})} = \frac{rh}{\left(\frac{hr}{k(-1+hr)}\right)(-1+hr)} = k, \\
x_{4n} &= \frac{x_{4n-1}x_{4n-3}}{x_{4n-2}(-1+x_{4n-1}x_{4n-3})} = \frac{k\left(\frac{hr}{k(-1+hr)}\right)}{r\left(-1+\frac{khr}{k(-1+hr)}\right)}\frac{(-1+hr)}{(-1+hr)} \\
&= \frac{h}{1-hr+hr} = h, \\
x_{4n+1} &= \frac{x_{4n}x_{4n-2}}{x_{4n-1}(-1+x_{4n}x_{4n-2})} = \frac{hr}{k(-1+hr)}.
\end{aligned}$$

Thus, the proof is complete. ∎

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq.(4).

Example 3. Consider $x_{-2} = 7,\ x_{-1} = 5,\ x_0 = 9$. See Fig. 3.

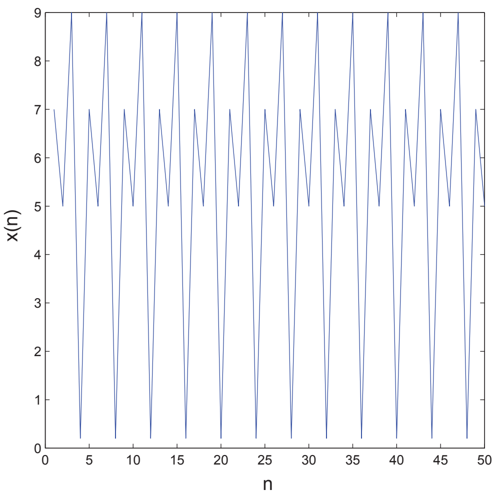

Figure 3: Plot of $x_{n+1} = (x_n x_{n-2})/(x_{n-1}(-1 + x_n x_{n-2}))$

Example 4. See Fig. 4, since $x_{-2} = -3,\ x_{-1} = 8,\ x_0 = 7$.

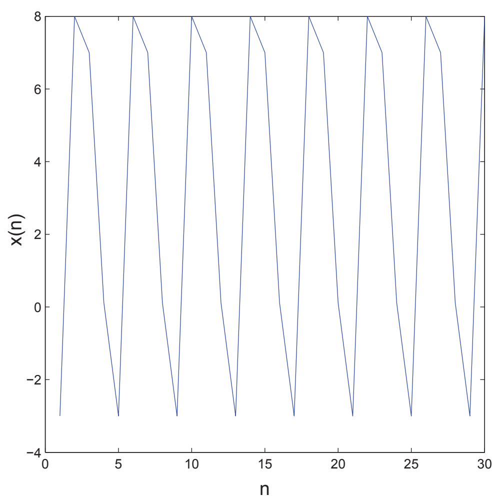

Figure 4: Plot of $x_{n+1} = (x_n x_{n-2})/(x_{n-1}(-1 + x_n x_{n-2}))$

4. Solution of the difference equation $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(-1-x_n x_{n-2})}$

In this section we obtain the form of the solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(-1 - x_n x_{n-2})}, \qquad n = 0, 1, ..., \tag{5}$$

where the initial conditions are arbitrary nonzero positive real numbers and $x_{-2}x_0 \neq -1$.

Theorem 3. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(5). Then Eq.(5) has a periodic solutions with period four and for $n = 0, 1, ...$*

$$x_{4n-2} = r, \qquad x_{4n-1} = k, \qquad x_{4n} = h, \qquad x_{4n+1} = \frac{hr}{k(-1 - hr)},$$

where $x_{-2} = r,\ x_{-1} = k,\ x_0 = h.$

Proof. As the proof of Theorem 2 and will be omitted. ■

Numerical examples

Example 5. Consider $x_{-2} = 11,\ x_{-1} = -6,\ x_0 = -9$. See Fig. 5.

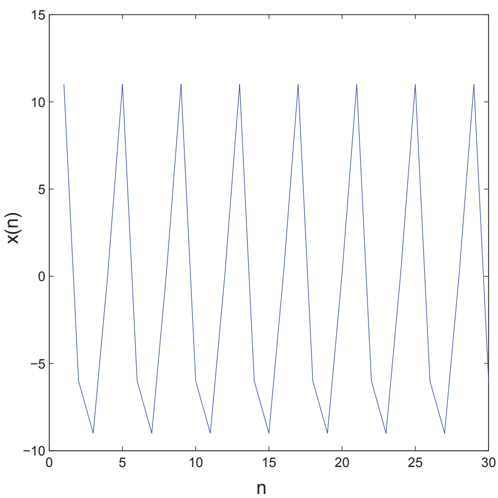

Figure 5: Plot of $x_{n+1} = (x_n x_{n-2})/(x_{n-1}(-1 - x_n x_{n-2}))$

Example 6. See Fig. 6, since $x_{-2} = 4,\ x_{-1} = 2,\ x_0 = 7$.

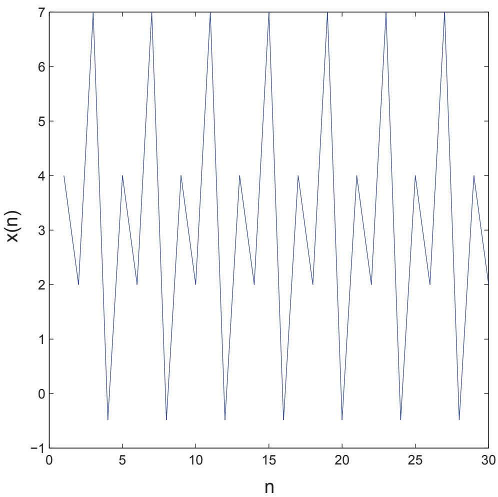

Figure 6: Plot of $x_{n+1} = (x_n x_{n-2})/(x_{n-1}(-1 - x_n x_{n-2}))$

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