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HEIGHT REDUCING PROBLEM ON ALGEBRAIC INTEGERS

Shigeki Akiyama, Paulius Drungilas, Jonas Jankauskas

Abstract: Let α be an algebraic integer and assume that it is *expanding*, i.e., its all conjugates lie outside the unit circle. We show several results of the form $\mathbb{Z}[\alpha] = \mathcal{B}[\alpha]$ with a certain finite set $\mathcal{B} \subset \mathbb{Z}$. This property is called *height reducing property*, which attracted special interest in the self-affine tilings. Especially we show that if α is quadratic or cubic trinomial, then one can choose $\mathcal{B} = \{0, \pm 1, \ldots, \pm(|N(\alpha)| - 1)\}$, where $N(\alpha)$ stands for the absolute norm of α over $\mathbb{Q}$.

Keywords: expanding algebraic integer, height reducing property, canonical number system.

1. Introduction

Let α be an algebraic integer with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ lying outside the unit circle (including α itself). Such numbers are called *expanding* algebraic numbers. We are interested in the *height reducing property* of α, that is

$$\mathbb{Z}[\alpha] = \mathcal{B}[\alpha]$$

for a certain finite set $\mathcal{B} \subset \mathbb{Z}$. We note that

Lemma 1. *If an algebraic integer α, $|\alpha| > 1$, has height reducing property, then α is expanding.*

Proof. Suppose α has height reducing property with a finite set $\mathcal{B} \subset \mathbb{Z}$. First assume it has a conjugate β with $|\beta| < 1$. Set $B = \max_{b \in \mathcal{B}} |b|$ and take an integer $K > \frac{B}{1-|\beta|}$. Then K has an expression $K = \sum_{i=0}^{n} b_i \alpha^i$ for some integer n. Taking conjugate, we have

$$K < \sum_{i=0}^{\infty} B|\beta|^i$$

which gives a contradiction. Therefore all the conjugates of α must be not less than one in modulus. Assume that there is a conjugate β with $|\beta| = 1$. Then β must be a complex number and $\beta\beta' = 1$ where β' is a complex conjugate of β. By taking conjugate map which sends β to α, we get a contradiction. ■

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Note that roots of unity (with all their conjugates on the unit circle) also have height reducing property with a set $\mathcal{B} = \{-1, 0, 1\}$.

When α is expanding, it is of interest whether it has height reducing property, and how small the set $\mathcal{B}$ we can take. Denote by $N(\alpha)$ the absolute norm of α over $\mathbb{Q}$, i.e., $N(\alpha) = \alpha_1 \cdot \alpha_2 \ldots \alpha_d$. If we can choose $\mathcal{B} = \{0, 1, \ldots, |N(\alpha)| - 1\}$, we say $(\alpha, \mathcal{B})$ forms a *canonical number system* (CNS for short). The question of finding all α which gives CNS is studied by many authors. The early studies are found in [13, 14, 11]. Readers may consult [3, 2] for recent developments to solve the problem in a general frame work called *shift radix system*.

However not every expanding algebraic integer α generates a CNS. Indeed, if there is a positive conjugate β of α, one sees that -1 can not be in $\mathcal{B}[\alpha]$ which is shown by taking conjugate.

For the rest of the paper let $\mathcal{B} = \{0, \pm 1, \ldots, \pm(|N(\alpha)| - 1)\}$.

Kirat and Lau [16] introduced a slightly different height reducing property for expanding polynomials (all roots in $|z| > 1$, not necessarily irreducible) to consider the connectedness of a class of self-affine tiles. In our notation, they are interested in $N(\alpha) \in \mathcal{B}[\alpha]$ (see [17] for details).

In this paper we are mainly concerned with the following type of height reducing problem:

Question. Does the equality $\mathbb{Z}[\alpha] = \mathcal{B}[\alpha]$ hold for any expanding algebraic integer?

In the study of self-affine tilings, Lagarias and Wang [21] answered this question in affirmative manner using wavelet analysis by extending the result of [12]. To read this result out of their consecutive works, see Corollary 6.2 in [21] and Theorem 1.2 (ii) of [20]. However their proof is rather indirect and intricate, although the statement itself looks simple in nature. The first author [1] asked for a direct proof of $\mathbb{Z}[\alpha] = \mathcal{B}[\alpha]$ (see problem 12). In this paper we shall give several attempts to solve this question. For the moment, it is far from satisfactory but we hope this paper gives a starting point for other trials. First we show

Theorem 2. *For any expanding quadratic algebraic integer α the equality $\mathbb{Z}[\alpha] = \mathcal{B}[\alpha]$ holds.*

Theorem 2 is derived from Theorem 4. We obtain a similar result for expanding cubic trinomials.

Theorem 3. *Let α be an expanding cubic algebraic integer whose minimal polynomial is a trinomial (i.e., polynomial of the form $x^3 + ax^2 + c$ or $x^3 + bx + c$). Then $\mathbb{Z}[\alpha] = \mathcal{B}[\alpha]$.*

The set of expanding cubic trinomials splits into two disjoint subsets, say, A and B. For the trinomials from A we apply Theorem 4. The subset B consists of trinomials of the form $x^3 - cx \pm c$, $c \geqslant 2$, $c \neq 8$. Theorem 10 (see Section 3) shows that in case of a trinomial from B it is impossible to derive Theorem 3 from Theorem 4. Theorem 3 for trinomials from B is proved by constructing certain finite automaton, the so called counting automaton (see Section 5).

In general, we have the following result.

Theorem 4. *Suppose that an expanding algebraic integer α is a root of a polynomial*

$$P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$$

with

$$|a_0| \geqslant |a_1| + |a_2| + \ldots + |a_{d-1}| + 1.$$

Then $\mathbb{Z}[\alpha] = \tilde{\mathcal{B}}[\alpha]$ with $\tilde{\mathcal{B}} = \{0, \pm 1, \ldots, \pm(|a_0| - 1)\}$.

Theorem 4 follows from Proposition 3.1 of [9]. Nevertheless we present an alternative proof of Theorem 4 in Section 3.

Note that the strict inequality $|a_0| > |a_1| + |a_2| + \ldots + |a_{d-1}| + 1$ would imply that all the roots of $P(x)$ are expanding algebraic integers.

Unfortunately, not every expanding algebraic integer α possesses a polynomial $P(x)$ satisfying the conditions of the theorem with $P(0) = \pm N(\alpha)$. In the Note at the end of Section 3, we provide an infinite family of such algebraic numbers whose minimal polynomials over $\mathbb{Q}$ are certain cubic trinomials. Such examples are minimal in terms of degree and the number of non-zero coefficients.

The best result we could obtain using Theorem 4 for a general expanding algebraic integer is the following:

Theorem 5. *Let α be an expanding algebraic integer of degree d (over $\mathbb{Q}$). Suppose that α_1 is a conjugate of α of least modulus. Then for any integer $n \geqslant -\log(2^{1/d} - 1)/\log|\alpha_1|$ we have*

$$\mathbb{Z}[\alpha] = \mathcal{B}_n[\alpha]$$

with $\mathcal{B}_n = \{0, \pm 1, \ldots, \pm(|N(\alpha)|^n - 1)\}$.

The upper bound $|N(\alpha)|^n - 1$ for the size of digits in $\mathcal{B}_n$ is large. By using more sophisticated division procedure, we were able to prove the next result.

Theorem 6. *Let α be an expanding algebraic integer of degree d whose conjugates are $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. For any $\beta \in \mathbb{Z}[\alpha]$ there exists a nonzero polynomial $P(x) \in \mathbb{Z}[x]$ of height at most*

$$\max\left\{\frac{|N(\alpha)|}{2\sqrt{D(\alpha)}} \sum_{i=1}^{d} \frac{\sqrt{|\alpha_i|^2 - 1}}{(|\alpha_i| - 1)\sqrt{|\alpha_i|^{2d} - 1}} \prod_{j=1}^{d} \sqrt{\frac{|\alpha_j|^{2d} - 1}{|\alpha_j|^2 - 1}},\ |N(\alpha)|/2\right\}$$

such that $\beta = P(\alpha)$. Here $D(\alpha)$ stands for the discriminant of α.

The bound in our Theorem 6 seems to be much smaller than that of Theorem 5, however, there is no way of direct comparison. Nevertheless, in the division algorithm used in Theorem 6 we prove that in order to find the representations of elements of $\mathbb{Z}[\alpha]$ with smallest possible digits, it suffices to find the expansions of finitely many elements of $\mathbb{Z}[\alpha]$, whose conjugates in $\mathbb{Z}[\alpha_i]$ have absolute value less than or equal to $N(\alpha)/2(|\alpha_i| - 1)$.

2. Proofs of Theorem 4 and 5

Theorem 4 follows from the next lemma.

Lemma 7. *Suppose that an expanding algebraic integer α is a root of a polynomial $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ with*

$$|a_0| \geqslant |a_1| + |a_2| + \ldots + |a_{d-1}| + 1,$$

and $\tilde{\mathcal{B}} = \{0, \pm 1, \ldots, \pm(|a_0| - 1)\}$. Let $A_0, A_1, \ldots, A_{d-1}$ be integers with $A_0 \notin \tilde{\mathcal{B}}$. Then there exist integers $A'_0, A'_1, \ldots, A'_{d-1}$ and $c_0, c_1, \ldots, c_k \in \tilde{\mathcal{B}}$ such that

$$\begin{aligned} A_0 + A_1\alpha + \ldots + A_{d-1}\alpha^{d-1} = c_0 + c_1\alpha + \ldots + c_k\alpha^k \\ + \left(A'_0 + A'_1\alpha + \ldots + A'_{d-1}\alpha^{d-1}\right)\alpha^{k+1} \end{aligned}$$

and $|A'_0| + |A'_1| + \ldots + |A'_{d-1}| < |A_0| + |A_1| + \ldots + |A_{d-1}|$.

Proof of Lemma 7. If $A_0 + A_1\alpha + \ldots + A_{d-1}\alpha^{d-1} = 0$ then we can take $k = 0$, $c_0 = 0$ and $A'_i = 0$ for all $i = 0, 1, \ldots, d-1$.

Further, assume that $A_0 + A_1\alpha + \ldots + A_{d-1}\alpha^{d-1} \neq 0$.

Assume without loss of generality that $A_0 > 0$. Then $A_0 \notin \tilde{\mathcal{B}}$ implies $A_0 \geqslant |a_0|$. Divide A_0 by a_0:

$$A_0 = c_0 + qa_0, \qquad 0 \leqslant c_0 < |a_0|, \; q \neq 0.$$

(Note that $qa_0 > 0$.) Then $P(\alpha) = 0$ implies

$$a_0 = -a_1\alpha - a_2\alpha^2 - \ldots - a_{d-1}\alpha^{d-1} - \alpha^d$$

and

$$A_0 = c_0 + qa_0 = c_0 - qa_1\alpha - qa_2\alpha^2 - \ldots - qa_{d-1}\alpha^{d-1} - q\alpha^d.$$

Hence

$$\begin{aligned} &A_0 + A_1\alpha + \ldots + A_{d-1}\alpha^{d-1} \\ &\qquad = c_0 + (A_1 - qa_1)\alpha + \ldots + (A_{d-1} - qa_{d-1})\alpha^{d-1} - q\alpha^d \\ &\qquad = c_0 + \left(B_0 + B_1\alpha + \ldots + B_{d-1}\alpha^{d-1}\right)\alpha \end{aligned}$$

where $B_{d-1} = -q$ and $B_i = A_{i+1} - qa_{i+1}$, $i = 0, 1, \ldots, d-2$.

Further, $|a_0| \geqslant |a_1| + |a_2| + \ldots + |a_{d-1}| + 1$ implies

$$\begin{aligned} \sum_{i=0}^{d-1} |B_i| = \sum_{i=1}^{d-1} |A_i - qa_i| + |q| &\leqslant \sum_{i=1}^{d-1} |A_i| + |q| \left(\sum_{i=1}^{d-1} |a_i| + 1\right) \\ &\leqslant \sum_{i=1}^{d-1} |A_i| + |q||a_0| \leqslant \sum_{i=0}^{d-1} |A_i|. \end{aligned}$$

If $c_0 \neq 0$ then the last inequality is strict, since $A_0 = |c_0 + qa_0| > |q||a_0|$. On the other hand, if $\sum_{i=0}^{d-1} |B_i| < \sum_{i=0}^{d-1} |A_i|$ then we can take $k = 0$, $A_i' = B_i$, $i = 0, 1, \ldots, d-1$ and we are done.

Further, assume that $\sum_{i=0}^{d-1} |B_i| = \sum_{i=0}^{d-1} |A_i|$. (Then $c_0 = 0$.)

If $B_i \in \tilde{\mathcal{B}}$ for all $i = 0, 1, \ldots, d-1$ then we can take $k = d$, $c_j = B_{j-1}$, $j = 1, 2, \ldots, d$, $A_i' = 0$ for all $i = 0, 1, \ldots, d-1$ and we are done in this case.

Now suppose that $B_t \notin \tilde{\mathcal{B}}$ for some $t \in \{0, 1, \ldots, d-1\}$. Let $s \in \{0, 1, \ldots, d-1\}$ be the smallest integer for which $B_s \neq 0$. If $B_s \in \tilde{\mathcal{B}}$ (in that case $s < d-1$) then we can take $k = s+1$, $c_1 = \ldots = c_s = 0$, $c_{s+1} = B_s$ and $A_i' = B_{s+i+1}$, $i = 0, 1, \ldots, d-s-2$ and $A_i' = 0$ for $i > d-s-2$. Indeed,

$$\sum_{i=0}^{d-1} |A_i'| = \sum_{i=s+1}^{d-1} |B_i| < \sum_{i=s}^{d-1} |B_i| = \sum_{i=0}^{d-1} |A_i|.$$

Finally, if $B_s \notin \tilde{\mathcal{B}}$ then we can repeat the above procedure with the number $B_s + B_{s+1}\alpha + \ldots$. After a finite number of iterations we will receive the inequality $\sum_{i=0}^{d-1} |A_i'| < \sum_{i=0}^{d-1} |A_i|$. Otherwise the number

$$A_0 + A_1\alpha + \ldots + A_{d-1}\alpha^{d-1} \neq 0$$

would be divisible by α^n for every positive integer n, which is impossible, since α is expanding. ■

We will derive Theorem 5 from Theorem 4 using the following lemma.

Lemma 8. *Let $P(x) \in \mathbb{Z}[x]$ be a monic polynomial such that all roots of $P(x)$ are of modulus strictly greater than one. Then there exists a monic polynomial*

$$Q(x) = x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0 \in \mathbb{Z}[x]$$

which is a multiple of $P(x)$ and

$$|b_0| \geqslant |b_1| + |b_2| + \ldots + |b_{m-1}| + 1.$$

Moreover, for any integer $n \geqslant -\log(2^{1/d} - 1)/\log|\alpha_1|$ one can choose $Q(x)$ with $Q(0) = P(0)^n$, where d is the degree of $P(x)$ and α_1 is a root of $P(x)$ of least modulus.

Proof of Lemma 8. Let d be the degree of $P(x)$. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_d$ are all complex roots of $P(x)$ (not necessarily distinct). Assume without loss of generality that

$$1 < |\alpha_1| \leqslant |\alpha_2| \leqslant \ldots \leqslant |\alpha_d|.$$

Let n be a positive integer. Set

$$G(x) = \prod_{i=1}^{d} (x - \alpha_i^n) = x^d + g_{d-1}x^{d-1} + \ldots + g_1 x + g_0.$$

Clearly, all the coefficients g_i are integers. Now the inequality $1+|g_{d-1}|+\ldots+|g_1| \leqslant |g_0|$ is equivalent to

$$1+|g_{d-1}|+\cdots+|g_1|+|g_0| \leqslant 2|g_0|.$$

Dividing both sides by $|g_0|$ we obtain

$$\frac{1}{|g_0|}+\frac{|g_{d-1}|}{|g_0|}+\ldots+\frac{|g_1|}{|g_0|}+1 \leqslant 2.$$

Here the left hand side is

$$1+\left|\sum_{i=1}^{d} \alpha_i^{-n}\right|+\left|\sum_{i<j} \alpha_i^{-n}\alpha_j^{-n}\right|+\ldots+\left|\prod_{i=1}^{d} \alpha_i^{-n}\right| \leqslant \prod_{i=1}^{d}(1+|\alpha_i^{-n}|) \leqslant (1+|\alpha_1^{-n}|)^d.$$

Hence the inequality $1+|g_{d-1}|+\ldots+|g_1| \leqslant |g_0|$ holds provided $(1+|\alpha_1^{-n}|)^d \leqslant 2$ which is equivalent to $n \geqslant -\log(2^{1/d}-1)/\log|\alpha_1|$. Finally, note that the polynomial $Q(x)=G(x^n)=\prod_{i=1}^{d}(x^n-\alpha_i^n)$ is the required one. ∎

Remark 9. In Lemma 8 we get $g_0 = \pm P(0)$ provided the conjugates of α of degree d all lie in $|z| > (2^{1/d}-1)^{-1}$.

Proof of Theorem 5. Let α be an expanding algebraic integer whose minimal polynomial is $P(x)$. By Lemma 8 for any integer $n \geqslant -\log(2^{1/d}-1)/\log|\alpha_1|$ there is a monic polynomial $Q(x)$ with $Q(0) = P(0)^n$ which satisfies the condition of Theorem 4. Finally, note that $P(0) = \pm N(\alpha)$. ∎

Note. Suppose that α is an expanding algebraic integer. In order to prove the equality $\mathbb{Z}[\alpha] = \mathcal{B}[\alpha]$ using Theorem 4 one needs a polynomial $P(x)$ satisfying the conditions of Theorem 4 and $P(0) = \pm N(\alpha)$. Unfortunately, this is false in general. Consider an algebraic integer α which is the root of cubic trinomial $p(x) = x^3 - cx + c$, $c \geqslant 2, c \neq 8, c \in \mathbb{Z}$. If $p(x)$ is reducible in $\mathbb{Z}[x]$, then it has an integer root, say, m. The equation $m^3 = c(m-1)$ implies that $m-1$ divides m^3. Since $\gcd(m^3, m-1) = 1$ and $c > 0$, this implies $m-1 = 1$. Thus $m = 2, c = 8$. Hence the polynomial $p(x)$ is irreducible in $\mathbb{Z}[x]$ if $c \geqslant 2, c \neq 8$. By direct substitution one easily checks that $p(x)$ has three real roots in intervals $(-\sqrt{c}, -\sqrt{c}+1)$, $(1+1/c, 3/2)$ and $(\sqrt{c}-1, \sqrt{c})$ if $c \geqslant 7$, all of modulus strictly greater than one. For $c = 2,3,4,5,6$, the polynomial $p(x)$ has one real and two complex roots outside the unit circle, which can be verified by direct computation. Alternatively, use the Shur-Cohn criterion [10], [23]. Thus α is a cubic expanding algebraic integer. In Theorem 10 below, we prove that $\mathbb{Z}[\alpha] = \mathcal{B}[\alpha]$ in principle cannot be established by Theorem 4.

Theorem 10. *The polynomial $p(x) = x^3 - cx + c, c \in \mathbb{Z}, c \geqslant 2, c \neq 8$ does not divide any polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $|a_0| \geqslant |a_1| + |a_2| + \cdots + |a_n|$ and $a_0 = \pm c$.*

Proof of Theorem 10. Assume that there exists a polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ which is a multiple of $p(x)$ and satisfies $|a_0| \geqslant |a_1| + |a_2| + \cdots + |a_n|$ with $a_0 = \pm c$. Then $P(x) = p(x)q(x)$ for some non constant polynomial $q \in \mathbb{Z}[x]$. Since $a_0 = \pm c$, $q(0) = \pm 1$. Hence, any irreducible factor of $q(x)$ has a root of modulus less or equal to 1. Let ζ be one of such roots. Then $P(\zeta) = 0$ implies

$$-a_0 = a_1\zeta + a_2\zeta^2 + \ldots a_n\zeta^n. \tag{1}$$

This implies $\zeta^k = \pm 1$ for any coefficient $a_k \neq 0, k = 1 \ldots n$. Otherwise, by comparing the real parts of the complex numbers in both sides of (1), one has

$$|a_1| + |a_2| + \cdots + |a_n| > |\Re(a_1\zeta + a_2\zeta^2 + \ldots a_n\zeta^n)| = |a_0|,$$

which contradicts the assumption. This shows that ζ is a root of unity. Thus $q(x)$ is a product of cyclotomic polynomials and a constant $a \in \mathbb{Z}$. Since $q(0) = \pm 1$, $a = \pm 1$. We claim that

$$q(x) = \pm(x-1)^r(x+1)^s(x^2+1)^t(x^2+x+1)^u(x^2-x+1)^v, \tag{2}$$

with integer exponents $r, s, t, u, v \geqslant 0$. To prove this, it suffices to show that at least one coefficient a_1, a_2 or a_3 is not equal to 0, so we have $\zeta = \pm 1$, $\zeta^2 = \pm 1$ or $\zeta^3 = \pm 1$ in (1).

Assume that $a_1 = a_2 = a_3 = 0$. Let α be the root of polynomial $p(x) = x^3 - cx + c$. Then (1) with ζ replaced by α implies that α^4 divides $a_0 = \pm c$ in the ring R of algebraic integers of $\mathbb{Q}(\alpha)$. Note that $p(\alpha) = 0$ gives $\alpha^3 = c(\alpha - 1)$. Thus $\alpha^4 | c$ in R implies $\alpha^4 | \alpha^3$, so α is a unit in R. This is impossible, since $c \geqslant 2$ and $p(x)$ is irreducible if $c \neq 8$ so the claim (2) is proved.

Observe that $t \geqslant 1$ in (2) implies $2|k$ for every non zero coefficient a_k, $k = 1 \ldots n$ in (1), since $i^k = \pm 1$ if and only if $2|k$ (here, as usual, $i^2 = -1$). In this case, $P(x) = P(-x) = P_1(x^2)$ for some polynomial $P_1 \in \mathbb{Z}[x]$. This is impossible, since such a polynomial $P(x)$ would be divisible by $p(x)$ and $p(-x)$ so $p(0)^2 = c^2$ divides $a_0 = P(0) = \pm c$ contradicting $c \geqslant 2$.

Similarly, $3|k$ for any non-zero a_k in (1) provided $u \geqslant 1$ or $v \geqslant 1$, since $(\pm e^{\pm 2\pi i/3})^k = \pm 1$ if and only if $3|k$. In this case, $P(x) = P_1(x^3)$ for some $P_1 \in \mathbb{Z}[x]$. Set $\zeta = e^{2\pi i/3}$. Then $P(\alpha) = P(\zeta\alpha) = P_1(\alpha^3) = 0$ for any root α of $p(x)$. The polynomials $p(x)$ and $p(\zeta x)$ have no roots in common, since

$$p(\zeta\alpha) - p(\alpha) = (\zeta^3\alpha^3 - c\zeta\alpha + c) - (\alpha^3 - c\alpha + c) = c(1-\zeta)\alpha \neq 0.$$

This implies that $P(x)$ is a multiple of $p(x)p(\zeta x)$. Since all roots of P are of modulus greater or equal to one, one has $|P(0)| \geqslant |p(0)p(\zeta 0)| = |p(0)|^2 = c^2 > c = |a_0| = |P(0)|$, which again leads to the contradiction.

From the arguments given above, it follows that $t = u = v = 0$, thus $q(x) = (x-1)^r(x+1)^s$ is the only remaining possibility. Then

$$|P(i)|^2 = |p(i)q(i)|^2 = |(i^3 - ci + c)^2(i-1)^r(i+1)^s|^2 = ((1+c)^2 + c^2)2^{r+s}.$$

The inequality

$$|P(i)| \leqslant |a_n| + \cdots + |a_1| + |a_0| \leqslant 2|a_0| = 2c,$$

implies

$$((1+c)^2 + c^2)2^{r+s} \leqslant 4c^2$$

which is impossible unless $r = s = 0$. This contradicts the assumption that $q(x)$ is a non constant polynomial and concludes the proof of Theorem 10. ∎

3. Proof of Theorem 2

The following lemma provides a necessary condition for a quadratic algebraic integer to be expanding which will be used in the proof of Theorem 2.

Lemma 11. *Let α be an expanding quadratic algebraic integer with the minimal polynomial $x^2 + ax + b$. Then $|a| \leqslant |b|$. The equality $|a| = |b|$ holds if and only if $b = |a| \geqslant 2$ and $|a| \neq 4$.*

One could employ the necessary and sufficient conditions (see Corollary 2.1 of [4]) developed using the Schur-Cohn criterion [10], [23]. Nevertheless, we provide the proof of Lemma 11.

Proof of Lemma 11. We might assume that $a \geqslant 0$, since $a = -(\alpha + \alpha')$ and α is expanding if and only if $-\alpha$ is expanding. Here α' stands for the conjugate of α.

Suppose, contrary to our claim, that $a > |b|$. This implies the inequalities

$$(a-2)^2 \leqslant a^2 - 4b < (a+2)^2,$$

$$a - 2 \leqslant \sqrt{a^2 - 4b} < a + 2$$

and

$$\left| \frac{-a + \sqrt{a^2 - 4b}}{2} \right| \leqslant 1$$

which is a contradiction, since

$$\{\alpha, \alpha'\} = \left\{ \frac{-a \pm \sqrt{a^2 - 4b}}{2} \right\}.$$

Now, suppose that $|b| = a > 0$ and α is expanding. We claim that $b = a$. Indeed, $b = -a$ implies

$$0 < \frac{-a + \sqrt{a^2 + 4a}}{2} = \frac{2a}{a + \sqrt{a^2 + 4a}} < \frac{2a}{a + a} = 1$$

which again leads to the contradiction.

Thus $b = a > 0$. Assume that $b = a \geqslant 5$. Then

$$\begin{aligned}\min\{|\alpha|, |\alpha'|\} = \min\left\{\left|\frac{-a \pm \sqrt{a^2-4a}}{2}\right|\right\} &= \frac{a-\sqrt{a^2-4a}}{2}\\ = \frac{2a}{a+\sqrt{a^2-4a}} > \frac{2a}{a+a} &= 1\end{aligned}$$

which implies that α is expanding.

Finally, one easily checks that $b = a = 2$ or 3 implies that α is expanding, whereas $b = a = 1$ or 4 implies that α is not expanding quadratic algebraic integer. ■

Proof of Theorem 2. Let α be an expanding quadratic algebraic integer with the minimal polynomial $x^2 + ax + b$. Assume without loss of generality that $a \geqslant 0$. (Indeed, Theorem 2 holds for α if and only if it holds for $-\alpha$.) By Lemma 11, $0 \leqslant a \leqslant |b|$. If $a + 1 \leqslant |b|$ then the result follows from Theorem 4 with $P(x) = x^2 + ax + b$. Suppose that $a = |b|$. By Lemma 11 $b = a \geqslant 2$ and $a \neq 4$. Now the minimal polynomial of α is $x^2 + ax + a$ and we can apply Theorem 4 with $P(x) = (x-1)(x^2 + ax + a) = x^3 + (a-1)x^2 - a$. ■

4. Proof of Theorem 3

In the proof of Theorem 3 we will construct a finite automaton, which is called "transducer" (cf. [5], [8]). We follow the notations of [25].

Definition 12. *The 6-tuple $A = (Q, \Sigma, \Delta, q, q_0, \delta)$ is called a finite transducer automaton if*

- *Q, Σ and Δ are non empty, finite sets, and*
- *$q : Q \times \Sigma \to Q$ and $\delta : Q \times \Sigma \to \Delta$ are unique mappings.*

The sets Σ and Δ are called input and output alphabet, respectively. Q is the set of states and q_0 is the starting state. The mappings q and δ are called transformation and result function, respectively.

We will use the following characterization of expanding cubic polynomials.

Lemma 13. *The polynomial $p(x) = x^3 + ax^2 + bx + c$ with integer coefficients is expanding if and only if*

$$\begin{cases} |b - ac| < c^2 - 1, \\ |b + 1| < |a + c|. \end{cases} \tag{3}$$

Proof. This is Lemma 1 from Akiyama and Gjini [4]. ■

Proof of Theorem 3. Suppose that α is an expanding cubic algebraic integer whose minimal polynomial $p(x) = x^3 + ax^2 + bx + c$ is a trinomial. Then either $a = 0$ or $b = 0$. If $b = 0$ then the first inequality of (3) implies $|a||c| < c^2 - 1$ and $|a| < |c|$. Hence each expanding cubic trinomial x^3+ax^2+c satisfies $1+|a| \leqslant |c|$ and we can apply Theorem 4. Now suppose that $a = 0$. Then the second inequality of (3) implies $|b + 1| < |c|$. If $b \geqslant 0$ then $1 + |b| < |c|$ and again we can apply Theorem 4. Let $b < 0$. Then the inequality $|b + 1| < |c|$ implies $b \geqslant -|c|$. If $b \geqslant -|c| + 1$ then $1 + |b| \leqslant |c|$ and once again we can apply Theorem 4. Finally we are left with the trinomials $p_1(x) = x^3 - cx + c$, and $p_2(x) = x^3 - cx - c$, $c \geqslant 2$. Note that $p_2(-x) = -p_1(x)$. Hence it is enough to consider the trinomial $x^3 - cx + c$, $c \geqslant 2$. This trinomial is irreducible provided $c \neq 8$ (see the note before Theorem 10). However Theorem 10 shows that in this case it is impossible to apply Theorem 4. Instead we will construct a finite automaton for this trinomial.

Now we briefly discuss how to construct the counting automaton $A_0(1)$ which performs the addition of 1 in $\mathcal{B}[\alpha]$. We will follow the explanation presented in [25]. Denote $(\sigma_N, \dots, \sigma_0) = \sum_{j=0}^{N} \sigma_j \alpha^j$. We say that $(\sigma_N, \dots, \sigma_0)$ is an α-adic representation of $v \in \mathbb{Z}[\alpha]$ if $v = (\sigma_N, \dots, \sigma_0)$ and $\sigma_0, \dots, \sigma_N \in \mathcal{B}$. Suppose $v \in \mathbb{Z}[\alpha]$ has α-adic representation $v = (d_N(v), d_{N-1}(v), \dots, d_0(v))$. We want to add 1 to the α-adic representation of v, i.e., we want to construct the α-adic representation of $v+1 = (d_{N'}(v+1), d_{N'-1}(v+1), \dots, d_0(v+1))$, $d_j(v+1) \in \mathcal{B}$. We perform the addition digit wise, from right to left. First we add 1 to the first digit $d_0(v)$. The addition produces a carry $q_1 \in \mathbb{Z}[\alpha]$ obeying the scheme $d_0(v) + 1 = d_0(v+1) + \alpha q_1$. Note that in contrast to [25] our $d_0(v+1)$ and q_1 are not unique unless $d_0(v+1) = 0$. This reduces the problem of adding 1 to v to the problem of adding q_1 to $(d_N(v), d_{N-1}(v), \dots, d_1(v))$. Iterating this procedure yields the general scheme

$$d_j(v) + q_j = d_j(v+1) + \alpha q_{j+1}, \quad j \geqslant 0. \tag{4}$$

Since the division procedure (4) is not unique we restrict our iteration procedure to the following: for each pair $(q_j, d_j(v))$ we fix the pair $(q_{j+1}, d_j(v+1))$ satisfying (4), and each time the iteration starts with $(q_j, d_j(v))$ we will use the same pair $(q_{j+1}, d_j(v+1))$. Adopting the notation of Definition 12 we define the counting automaton $A_0(1)$ by setting

$Q =$ the set of possible carries,

$\Sigma = \Delta = \mathcal{B}$,

$q_0 = 1$,

$q : Q \times \Sigma \to Q : (q_j, d_j(v)) \mapsto q_{j+1}$ according to (4),

$\delta : Q \times \Sigma \to \Delta : (q_j, d_j(v)) \mapsto d_j(v+1)$ according to (4).

Now we explicitly construct the counting automaton $A_0(1)$ for α – a root of $x^3 - cx + c$, $c \geqslant 2$, $c \neq 8$. Consider the following table.

Number of carry	Carry	:	Input/Output	Next carry
0	0	:	$k\|k$	0
1	1	:	$k \leqslant c-2,\ k\|k+1$	0
		:	$c-1\|0$	2
2	$\overline{1}\,0\,c$	:	$k\|k$	3
3	$\overline{1}\,\overline{1}\,c$	:	$k\|k$	4
4	$\overline{1}\,\overline{1}\,c-1$	:	$k \leqslant 0,\ k\|k+c-1$	5
		:	$k \geqslant 1,\ k\|k-1$	4
5	$\overline{1}\,\overline{1}$	:	$k \geqslant \overline{c}+2,\ k\|k-1$	6
		:	$\overline{c}+1\|0$	7
6	$\overline{1}$	:	$k \geqslant \overline{c}+2,\ k\|k-1$	0
		:	$\overline{c}+1\|0$	8
7	$1\,0\,\overline{c}-1$	:	$k \geqslant \overline{c}+2,\ k\|k-1$	9
		:	$\overline{c}+1\|0$	10
8	$1\,0\,\overline{c}$	:	$k\|k$	9
9	$1\,1\,\overline{c}$	:	$k\|k$	11
10	$2\,1\,\overline{c}+\overline{c}$	:	$k\|k$	12
11	$1\,1\,\overline{c}+1$	:	$k \leqslant \overline{1},\ k\|k+1$	11
		:	$k \geqslant 0,\ k\|k-c+1$	13
12	$2\,2\,1+\overline{c}+\overline{c}$	:	$k \leqslant \overline{1},\ k\|k+1$	14
		:	$k \geqslant 0,\ k\|k-c+1$	15
13	$1\,1$	:	$k \leqslant c-2,\ k\|k+1$	1
		:	$c-1\|0$	16
14	$2\,2\,2+\overline{c}+\overline{c}$	:	$k \leqslant \overline{2},\ k\|k+2$	14
		:	$k \geqslant \overline{1},\ k\|k-c+2$	15
15	$1\,2\,\overline{c}+2$	:	$k \leqslant \overline{2},\ k\|k+2$	17
		:	$k \geqslant \overline{1},\ k\|k-c+2$	18
16	$\overline{1}\,0\,c+1$	:	$k \leqslant c-2,\ k\|k+1$	3
		:	$c-1\|0$	19
17	$1\,1\,\overline{c}+2$	:	$k \leqslant \overline{2},\ k\|k+2$	11
		:	$k \geqslant \overline{1},\ k\|k-c+2$	13
18	$1\,2$	:	$k \leqslant c-3,\ k\|k+2$	1
		:	$k \geqslant c-2,\ k\|k-c+2$	16
19	$\overline{2}\,\overline{1}\,c+c$	:	$k\|k$	20
20	$\overline{2}\,\overline{2}\,c+c+\overline{1}$	:	$k \leqslant 0,\ k\|k+c-1$	21
		:	$k \geqslant 1,\ k\|k-1$	22
21	$\overline{1}\,\overline{2}\,c-2$	:	$k \leqslant 1,\ k\|k+c-2$	23
		:	$k \geqslant 2,\ k\|k-2$	24
22	$\overline{2}\,\overline{2}\,\overline{2}+c+c$	:	$k \leqslant 1,\ k\|k+c-2$	21
		:	$k \geqslant 2,\ k\|k-2$	22
23	$\overline{1}\,\overline{2}$	:	$k \leqslant \overline{c}+2,\ k\|k+c-2$	7
		:	$k \geqslant \overline{c}+3,\ k\|k-2$	6
24	$\overline{1}\,\overline{1}\,c-2$	:	$k \leqslant 1,\ k\|k+c-2$	5
		:	$k \geqslant 2,\ k\|k-2$	4

Here $\overline{a}$ denotes $-a$. The second column "carry" indicates the carry. Carries are numbered in the first column "number of carry". The third column "input/output" defines the result function δ: $k \in \mathcal{B}$ denotes the input digit and $k|u(k)$ means that the corresponding output is $u(k) \in \mathcal{B}$. The fourth column "next carry" defines the transformation function q indicating the number of the next carry.

One can check that this counting automaton $A_0(1)$ has no "zero cycles," i.e., if we begin with any carry from the second column and start walking the zero path (each time taking input 0) eventually we will reach the sync point – carry 0. This means that we can add 1 to any α-adic representation $v \in \mathbb{Z}[\alpha]$ and obtain an α-adic representation of $v+1$.

If we run the counting automaton $A_0(1)$ starting with the carry no. 6 (i.e. $q_0 := \overline{1}$) this would produce the subtraction of 1. Now if we run $A_0(1)$ starting with the carry no. 13 this would produce addition of $1\,1 = \alpha + 1$. Then we take the resulting representation and subtract 1. This gives the addition of $1\,0 = \alpha$. Similarly running $A_0(1)$ with the starting carry no. 5 and then adding 1 we obtain the subtraction of α. If we run $A_0(1)$ starting with the carry no. 11, then subtract $1\,0 = \alpha$ and then for $c-1$ times add 1 we would get the addition of $1\,0\,0 = \alpha^2$. Finally running $A_0(1)$ with starting carry no. 4, then adding $1\,0 = \alpha$ and then for $c-1$ times subtracting 1 we obtain the subtraction of $1\,0\,0 = \alpha^2$. Hence starting with 0 and applying ± 1 or $\pm\alpha$ or $\pm\alpha^2$ we can find α-adic representation of any number lying in $\mathbb{Z}[\alpha] = \mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\alpha^2$. ■

Note. The polynomial $x^3 - cx + c$, $c \geqslant 2$, $c \neq 8$ is not a CNS polynomial (see Theorem 3 of [6]).

5. Proof of Theorem 6

Proof. Let $p(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0$ be the minimal polynomial of α . (Then $N(\alpha) = \alpha_1\alpha_2 \cdot \ldots \cdot \alpha_d = \pm a_0$.) Let $\gamma \in \mathbb{Z}[\alpha]$, $\gamma = C_0 + C_1\alpha + \ldots + C_{d-1}\alpha^{d-1}$, $C_j \in \mathbb{Z}$. Then the conjugates of γ are $\gamma_i = C_0 + C_1\alpha_i + \ldots + C_{d-1}\alpha_i^{d-1}$, $i = 1, 2, \ldots, d$. Consider the following division procedure. There are integers r and q such that $C_0 = r + a_0 q$ and $|r| \leqslant |a_0|/2$. The equality $p(\alpha_i) = 0$ implies

$$a_0 = -a_1\alpha_i - \ldots - a_{d-1}\alpha_i^{d-1} - \alpha_i^d.$$

Thus

$$C_0 = r - \alpha_i \left(a_1 q + a_2 q\alpha_i + \ldots + a_{d-1} q\alpha_i^{d-2} + q\alpha_i^{d-1}\right).$$

Denote

$$\gamma_i = r + \alpha_i \gamma_i'$$

where $\gamma_i' = C_0' + C_1'\alpha_i + \ldots + C_{d-1}'\alpha_i^{d-1}$ with integers $C_j' = C_{j+1} - a_{j+1}q$, $0 \leqslant j \leqslant d-2$, $C_{d-1}' = -q$. (Note that the numbers C_j' do not depend on the choice of conjugate γ_i.)

Now fix $i \in \{1, 2, \ldots, d\}$ and define the sequence $x_n^{(i)}$ as follows.

$$x_0^{(i)} = \beta_i = B_0 + B_1\alpha_i + \ldots + B_{d-1}\alpha_i^{d-1},$$

$B_j \in \mathbb{Z}$, $j = 0, 1, \ldots, d-1$, and $x_{n+1}^{(i)}$ is obtained from $x_n^{(i)}$ via the division procedure described above, i. e.,

$$x_n^{(i)} = r_n + \alpha_i x_{n+1}^{(i)}, \qquad |r_n| \leqslant |a_0|/2, \quad n \geqslant 0. \tag{5}$$

Then

$$\beta_i = r_0 + r_1\alpha_i + \ldots + r_{n-1}\alpha_i^{n-1} + \alpha_i^n x_n^{(i)} \tag{6}$$

and

$$\begin{aligned} |x_n^{(i)}| &= \left| \frac{\beta_i}{\alpha_i^n} - \frac{r_0}{\alpha_i^n} - \ldots - \frac{r_{n-1}}{\alpha_i} \right| \leqslant \frac{|\beta_i|}{|\alpha_i|^n} + \frac{|r_0|}{|\alpha_i|^n} + \ldots + \frac{|r_{n-1}|}{|\alpha_i|} \\ &\leqslant \frac{|\beta_i|}{|\alpha_i|^n} + \frac{|a_0|}{2}\left(\frac{1}{|\alpha_i|} + \frac{1}{|\alpha_i|^2} + \ldots\right) = \frac{|\beta_i|}{|\alpha_i|^n} + \frac{|a_0|}{2(|\alpha_i| - 1)}. \end{aligned}$$

Let $m = \min_{1 \leqslant i \leqslant d} |\alpha_i|$ and $M = \max_{1 \leqslant i \leqslant d} |\beta_i|$. Then the last inequality yields

$$|x_n^{(i)}| \leqslant \frac{M}{m^n} + \frac{|a_0|}{2(|\alpha_i| - 1)} \leqslant \frac{M}{m^n} + \frac{|a_0|}{2(m-1)}. \tag{7}$$

Thus the set $\{x_n^{(i)} \, : \, 1 \leqslant i \leqslant d, \; n \geqslant 0\}$ is finite, since it consists of algebraic integers of degree at most d whose conjugates are bounded. Now (5) implies that the sequence $x_n^{(i)}$ is periodic starting from certain $n \geqslant n_0$. (Note that n_0 does not depend on the choice of conjugate $x_n^{(i)}$.)

Further, take any $\delta_i \in \{x_n^{(i)} \, : \, n \geqslant n_0\}$. Since $\delta_i = x_n^{(i)}$ for infinitely many positive integers n, (7) shows that

$$|\delta_i| \leqslant \frac{|a_0|}{2(|\alpha_i| - 1)} = \frac{|N(\alpha)|}{2(|\alpha_i| - 1)} \tag{8}$$

for all $i = 1, 2, \ldots, d$. Since $\delta_i \in \mathbb{Z}[\alpha_i]$, there exist integers $A_0, A_1, \ldots, A_{d-1}$ such that

$$A_0 + A_1\alpha_i + \ldots + A_{d-1}\alpha_i^{d-1} = \delta_i, \qquad i = 1, 2, \ldots, d.$$

By Cramer's rule,

$$A_j = \frac{1}{\det(\alpha_i^r)} \begin{vmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{j-1} & \delta_1 & \alpha_1^{j+1} & \cdots & \alpha_1^{d-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{j-1} & \delta_2 & \alpha_2^{j+1} & \cdots & \alpha_2^{d-1} \\ & & & \cdots & \cdots & & & \\ 1 & \alpha_d & \cdots & \alpha_d^{j-1} & \delta_d & \alpha_d^{j+1} & \cdots & \alpha_d^{d-1} \end{vmatrix} \tag{9}$$

for $j = 0, 1, \ldots, d-1$. Denote by U_k, $1 \leqslant k \leqslant d$, the determinant obtained from the last determinant by omitting the $k-$th row and the $j+1-$th column. On applying Hadamard's inequality, one obtains

$$|U_k| \leqslant \prod_{r \neq k} \sqrt{\frac{|\alpha_r|^{2d} - 1}{|\alpha_r|^2 - 1}}. \tag{10}$$

It's well-known that $\det^2(\alpha_i^r) = D(\alpha)$, where $D(\alpha)$ stands for the discriminant of α (see, e. g., Chapter 2 of [24]). Then in view of (9), (8) and (10), we have

$$\begin{aligned}|A_j| &= \frac{1}{\sqrt{D(\alpha)}}\left|\sum_{k=1}^{d}\delta_k U_k\right| \leqslant \frac{1}{\sqrt{D(\alpha)}}\sum_{k=1}^{d}\frac{|N(\alpha)|}{2(|\alpha_k|-1)}\prod_{r\neq k}\sqrt{\frac{|\alpha_r|^{2d}-1}{|\alpha_r|^2-1}} \\ &= \frac{|N(\alpha)|}{2\sqrt{D(\alpha)}}\prod_{r=1}^{d}\sqrt{\frac{|\alpha_r|^{2d}-1}{|\alpha_r|^2-1}}\sum_{k=1}^{d}\frac{\sqrt{|\alpha_k|^2-1}}{(|\alpha_k|-1)\sqrt{|\alpha_k|^{2d}-1}}. \end{aligned} \tag{11}$$

Now, $\delta_i = x_n^{(i)}$ for certain n. Then in view of (6), we obtain

$$\beta = \beta_1 = r_0 + r_1\alpha + \ldots + r_{n-1}\alpha^{n-1} + \alpha^n\delta_1 =$$

$$r_0 + r_1\alpha + \ldots + r_{n-1}\alpha^{n-1} + A_0\alpha^n + A_1\alpha^{n+1} + \ldots + A_{d-1}\alpha^{n+d-1}.$$

Finally, in view of (11), the polynomial

$$P(x) = r_0 + r_1x + \ldots + r_{n-1}x^{n-1} + A_0x^n + A_1x^{n+1} + \ldots + A_{d-1}x^{n+d-1}$$

is the required one. ■

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Addresses: Shigeki Akiyama: Department of Mathematics, Faculty of Science, Niigata University, Ikarashi 2-8050, NIIGATA 950-2181, Japan;
Paulius Drungilas and Jonas Jankauskas: Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania.

E-mail: akiyama@math.sc.niigata-u.ac.jp, pdrungilas@gmail.com, jonas.jankauskas@gmail.com

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