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IRREDUCIBILITY OF GENERALIZED HERMITE-LAGUERRE POLYNOMIALS

Shanta Laishram, Tarlok N. Shorey

Abstract: For a rational $q = u + \frac{\alpha}{d}$ with $u, \alpha, d \in \mathbb{Z}$ with $u \geqslant 0, 1 \leqslant \alpha < d$, $\gcd(\alpha, d) = 1$, the *generalized Hermite-Laguerre polynomials* $G_q(x)$ are defined by

$$G_q(x) = a_n x^n + a_{n-1}(\alpha + (n-1+u)d)x^{n-1} + \cdots + a_1 \left(\prod_{i=1}^{n-1}(\alpha + (i+u)d)\right) x + a_0 \left(\prod_{i=0}^{n-1}(\alpha + (i+u)d)\right)$$

where $a_0, a_1, \cdots, a_n$ are arbitrary integers. We prove some irreducibility results of $G_q(x)$ when $q \in \{\frac{1}{3}, \frac{2}{3}\}$ and extend some of the earlier irreducibility results when q of the form $u + \frac{1}{2}$. We also prove a new improved lower bound for greatest prime factor of product of consecutive terms of an arithmetic progression whose common difference is 2 and 3.

Keywords: irreducibility, Hermite-Laguerre polynomials, arithmetic progressions, primes.

1. Introduction

Let n and $1 \leqslant \alpha < d$ be positive integers with $\gcd(\alpha, d) = 1$. Any positive rational q is of the form $q = u + \frac{\alpha}{d}$ where u is a non-negative integer. For integers $a_0, a_1, \cdots, a_n$, let

$$G(x) := G_q(x) = a_n x^n + a_{n-1}(\alpha + (n-1+u)d)x^{n-1} + \cdots + a_1 \left(\prod_{i=1}^{n-1}(\alpha + (i+u)d)\right) x + a_0 \left(\prod_{i=0}^{n-1}(\alpha + (i+u)d)\right).$$

This is an extension of Hermite polynomials and generalized Laguerre polynomials. Therefore we call $G(x)$ the generalized Hermite-Laguerre polynomial. For an integer $\nu > 1$, we denote by $P(\nu)$ the the greatest prime factor of ν and we put $P(1) = 1$. We prove

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Theorem 1. *Let* $P(a_0a_n) \leqslant 3$ *and suppose* $2 \nmid a_0a_n$ *if degree of* $G_{\frac{2}{3}}(x)$ *is* 43. *Then* $G_{\frac{1}{3}}$ *and* $G_{\frac{2}{3}}$ *are irreducible except possibly when* $1+3(n-1)$ *and* $2+3(n-1)$ *is a power of* 2*, respectively where it can be a product of a linear factor times a polynomial of degree* $n-1$.

Theorem 2. *Let* $1 \leqslant k < n$, $0 \leqslant u \leqslant k$ *and* $a_0a_n \in \{\pm 2^t : t \geqslant 0, t \in \mathbb{Z}\}$. *Then* $G_{u+\frac{1}{2}}$ *does not have a factor of degree* k *except possibly when* $k \in \{1, n-1\}, u \geqslant 1$.

Schur [Sch29] proved that $G_{\frac{1}{2}}(x^2)$ with $a_n = \pm 1$ and $a_0 = \pm 1$ are irreducible and this implies the irreducibility of H_{2n} where H_m is the $m-$th Hermite polynomial. Schur [Sch73] also established that Hermite polynomials H_{2n+1} are x times an irreducible polynomial by showing that $G_{\frac{3}{2}}(x^2)$ with $a_n = \pm 1$ and $a_0 = \pm 1$ is irreducible expect for some explicitly given finitely many values of n where it can have a quadratic factor. Further Allen and Filaseta [AlFi04] showed that $G_{\frac{1}{2}}(x^2)$ with $a_1 = \pm 1$ and $0 < |a_n| < 2n - 1$ is irreducible. Finch and Saradha [FiSa10] showed that $G_{u+\frac{1}{2}}$ with $0 \leqslant u \leqslant 13$ have no factor of degree $k \in [2, n-2]$ except for an explicitly given finite set of values of u where it may have a factor of degree 2.

From now onwards, we always assume $d \in \{2, 3\}$. A new ingredient in the proofs of Theorems 1 and 2 is the following result which we shall prove in Section 3.

Theorem 3. *Let* $k \geqslant 2$ *and* $d = 2, 3$. *Let* m *be a positive integer such that* $d \nmid m$ *and* $m > dk$. *Then*

$$P(m(m+d)\cdots(m+d(k-1))) > \begin{cases} 3.5k & \text{if } d=2 \text{ and } m \leqslant 2.5k \\ 4k & \text{if } d=2 \text{ and } m > 2.5k \\ 3k & \text{if } d=3 \end{cases} \tag{1}$$

unless $(m, k) \in \{(5, 2), (7, 2), (25, 2), (243, 2), (9, 4), (13, 5), (17, 6), (15, 7), (21, 8), (19, 9)\}$ *when* $d = 2$ *and* $(m, k) = (125, 2)$ *when* $d = 3$.

If $d = 2, 3$ and $m > dk$, this is an improvement of [LaSh06a].

In Section 4, we shall combine Theorem 3 with the irreducibility criterion from [ShTi10](see Lemma 4.1) to derive Theorems 1 and 2. This criterion come from Newton polygons. If p is a prime and m is a nonzero integer, we define $\nu(m) = \nu_p(m)$ to be the nonnegative integer such that $p^{\nu(m)}|m$ and $p^{\nu(m)+1} \nmid m$. We define $\nu(0) = +\infty$. Consider $f(x) = \sum_{j=0}^{n} a_jx^j \in \mathbb{Z}[x]$ with $a_0a_n \neq 0$ and let p be a prime. Let S be the following set of points in the extended plane:

$$S = \{(0, \nu(a_n)), (1, \nu(a_{n-1})), (2, \nu(a_{n-2})), \cdots, (n-1, \nu(a_1)), (n, \nu(a_0))\}$$

Consider the lower edges along the convex hull of these points. The left-most endpoint is $(0, \nu(a_n))$ and the right-most endpoint is $(n, \nu(a_0))$. The endpoints of each edge belong to S, and the slopes of the edges increase from left to right. When referring to the edges of a Newton polygon, we shall not allow two different edges to have the same slope. The polygonal path formed by these edges is called the Newton polygon of $f(x)$ with respect to the prime p. For the proof of Theorems 1 and 2, we use [ShTi10, Lemma 10.1] whose proof depends on Newton polygons.

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2. Preliminaries for Theorem 3

Let m and k be positive integers with $m > kd$ and $\gcd(m, d) = 1$. We write

$$\Delta(m, d, k) = m(m + d) \cdots (m + (k - 1)d).$$

For positive integers ν, μ and $1 \leqslant l < \mu$ with $\gcd(l, \mu) = 1$, we write

$$\pi(\nu, \mu, l) = \sum_{\substack{p \leqslant \nu \\ p \equiv l (\mathrm{mod}\ \mu)}} 1, \ \ \pi(\nu) = \pi(\nu, 1, 1)$$

$$\theta(\nu, \mu, l) = \sum_{\substack{p \leqslant \nu \\ p \equiv l (\mathrm{mod}\ \mu)}} \log p.$$

Let $p_{i,\mu,l}$ denote the ith prime congruent to l modulo μ. Let $\delta_\mu(i, l) = p_{i+1,\mu,l} - p_{i,\mu,l}$ and $W_\mu(i, l) = (p_{i,\mu,l}, p_{i+1,\mu,l})$. Let $M_0 = 1.92367 \times 10^{10}$.

We recall some well-known estimates on prime number theory.

Lemma 2.1. *We have*

(i) $\pi(\nu) \leqslant \frac{\nu}{\log \nu}\left(1 + \frac{1.2762}{\log \nu}\right)$ for $\nu > 1$

(ii) $\nu(1 - \frac{3.965}{\log^2 \nu}) \leqslant \theta(\nu) < 1.00008\nu$ for $\nu > 1$

(iii) $\sqrt{2\pi k}\ e^{-k} k^k e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi k}\ e^{-k} k^k e^{\frac{1}{12k}}$ for $k > 1$

(iv) $\mathrm{ord}_p(k!) \geqslant \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$ for $k > 1$ and $p < k$.

The estimates (i), (ii) are due to Dusart [Dus98, p.14], [Dus99]. The estimate (iii) is [Rob55, Theorem 6]. For a proof of (iv), see [LaSh04b, Lemma 2(i)]. ■

The following lemma is due to Ramaré and Rumely [RaRu96, Theorems 1, 2].

Lemma 2.2. *Let* $l \in \{1, 2\}$. *For* $\nu_0 \leqslant 10^{10}$, *we have*

$$\theta(\nu, 3, l) \geqslant \begin{cases} \frac{\nu}{2}(1 - 0.002238) & \text{for } \nu \geqslant 10^{10} \\ \frac{\nu}{2}\left(1 - \frac{2\times 1.798158}{\sqrt{\nu_0}}\right) & \text{for } 10^{10} > \nu \geqslant \nu_0 \end{cases} \tag{2}$$

and

$$\theta(\nu, 3, l) \leqslant \begin{cases} \frac{\nu}{2}(1 + 0.002238) & \text{for } \nu \geqslant 10^{10} \\ \frac{\nu}{2}\left(1 + \frac{2\times 1.798158}{\sqrt{\nu_0}}\right) & \text{for } 10^{10} > \nu \geqslant \nu_0 \end{cases}. \tag{3}$$

We derive from Lemmas 2.1 and 2.2 the following result.

Corollary 2.3. *Let* $M_0 < m \leqslant 131 \times 2k$ *if* $d = 2$ *and* $6450 \leqslant m \leqslant 10.6 \times 3k$ *if* $d = 3$. *Then* $P(\Delta(m,d,k)) \geqslant m$.

Proof. Let $M_0 < m \leqslant 131 \times 2k$ if $d = 2$ and $6450 \leqslant m \leqslant 10.6 \times 3k$ if $d = 3$. Then $k \geqslant k_1$ where $k_1 = 7.34 \times 10^7, 203$ when $d = 2, 3$, respectively. Let $1 \leqslant l < d$ and assume $m \equiv l (\mathrm{mod}\ d)$. We observe that $P(\Delta(m,d,k) \geqslant m$ holds if

$$\theta(m + d(k-1), d, l) - \theta(m-1, d, l) = \sum_{\substack{m \leqslant p \leqslant m+(k-1)d \\ p \equiv l(d)}} \log p > 0.$$

Now from Lemmas 2.1 and 2.2, we have

$$\frac{\theta(m-1,d,l)}{\frac{m-1}{\phi(d)}} < \theta_1 := \begin{cases} 1.00008 & \text{if } d = 2 \\ 1 + \frac{2 \times 1.798158}{\sqrt{6450}} & \text{if } d = 3 \end{cases}$$

and

$$\frac{\theta(m+(k-1)d,d,l)}{\frac{m+(k-1)d}{\phi(d)}} > \theta_2 := \begin{cases} 1 - \frac{3.965}{\log^2(10^{10})} & \text{if } d = 2 \\ 1 - \frac{2 \times 1.798158}{\sqrt{6450}} & \text{if } d = 3. \end{cases}$$

Thus $P(\Delta(m,d,k) \geqslant m$ holds if

$$\theta_2(m + d(k-1)) > \theta_1 m$$

i.e., if

$$\frac{d(k-1)}{m} > \frac{\theta_1}{\theta_2} - 1.$$

This is true since for $k \geqslant k_1$, we have

$$\frac{dk(1-\frac{1}{k})}{\frac{\theta_1}{\theta_2} - 1} \geqslant \frac{dk(1-\frac{1}{k_1})}{\frac{\theta_1}{\theta_2} - 1} > (dk) \begin{cases} 131.3 & \text{if } d = 2 \\ 10.6 & \text{if } d = 3 \end{cases}$$

and m is less than the last expression. Hence the assertion. ∎

Now we give some results for $d = 2$. The next result follows from Lemma 2.1 (ii).

Corollary 2.4. *Let* $d = 2, k > 1$ *and* $2k < m < 4k$. *Then*

$$P(\Delta(m,d,k)) > \begin{cases} 3.5k & \text{if } m \leqslant 2.5k \\ 4k & \text{if } m > 2.5k \end{cases} \tag{4}$$

unless $(m,k) \in \{(5,2),(7,2),(9,4),(13,5),(17,6),(15,7),(21,8),(19,9)\}$.

Proof. We observe that the set $\{m, m+2, \ldots, m+2(k-1)\}$ contains all primes between $3.5k$ and $4k$ if $m \leqslant 2.5k$ and all primes between $4k$ and $4.5k$ if $2.5k < m < 4k$. Therefore (4) holds if

$$\theta(4k) > \theta(3.5k) \text{ and } \theta(4.5k) > \theta(4k).$$

Let $(r, s) = (3.5, 4)$ or $(4, 4.5)$. Then from Lemma 2.1, we see that $\theta(sk) > \theta(rk)$ if

$$sk(1 - \frac{3.965}{\log^2(sk)}) > 1.00008 \times rk$$

or

$$\frac{s - 1.00008r}{1.00008r} > \frac{s}{1.00008r} \frac{3.965}{\log^2(sk)}$$

or

$$k > \frac{1}{s} \exp\left(\sqrt{\frac{3.965s}{s - 1.00008r}}\right).$$

This is true for $k \geqslant 88$. Thus $k \leqslant 87$. For $10 \leqslant k \leqslant 87$, we check that there is always a prime in the intervals $(3.5k, 4k)$ and $(4k, 4.5k)$ and hence (4) follows in this case. For $2 \leqslant k \leqslant 9$, the assertion follows by computing $P(\Delta(m, 2, k))$ for each $2k < m < 4k$. ■

The following result concerns Grimm's Conjecture, [LaSh06b, Theorem 1].

Lemma 2.5. *Let $m \leqslant M_0$ and l be such that $m+1, m+2, \cdots, m+l$ are all composite numbers. Then there are distinct primes P_i such that $P_i|(m+i)$ for each $1 \leqslant i \leqslant l$.*

As a consequence, we have

Corollary 2.6. *Let $4k < m \leqslant M_0$. Then either $P(\Delta(m, 2, k)) > 4k$ or $P(\Delta(m, 2, k)) \geqslant p_{k+1}$.*

Proof. If $m+2i$ is prime for some i with $0 \leqslant i < k$, then the assertion holds clearly since $P(\Delta(m, 2, k)) \geqslant m+2i > 4k$. Thus we suppose that $m+2i$ is composite for all $0 \leqslant i < k$. Since m is odd, we obtain that $m+2i+1$ with $0 \leqslant i < k$ are all even and hence composite. Therefore $m, m+1, m+2, \cdots, m+2k-1$ are all composite and hence, by Lemma 2.5, there are distinct primes P_j with $P_j|(m-1+j)$ for each $1 \leqslant j \leqslant 2k$. Therefore $\omega(\Delta(m, 2, k)) \geqslant k$ implying $P(\Delta(m, 2, k)) \geqslant p_{k+1}$. ■

Corollary 2.7. *Let $d = 2$ and $4k < m \leqslant M_0$. Then $P(\Delta(m, 2, k)) > 4k$ for $k \geqslant 30$.*

Proof. By Corollary 2.6, we may assume that $P(\Delta(m, 2, k)) \geqslant p_{k+1}$. By Lemma 2.1, we get $p_{k+1} \geqslant k \log k$ which is $> 4k$ for $k \geqslant 60$. For $30 \leqslant k < 60$, we check that $p_{k+1} > 4k$. Hence the assertion follows. ■

The following result follows from [Leh64, Tables IIA, IIIA].

Lemma 2.8. *Let $d = 2$, $m > 4k$ and $2 \leqslant k \leqslant 37, k \neq 35$. Then $P(\Delta(m, 2, k)) > 4k$.*

Proof. The case $k = 2$ is immediate from [Leh64, Table IIA]. Let $k \geqslant 3$ and $m \geqslant 4k$. For m and $1 \leqslant i < k$ such that $m + 2i = N$ with N given in [Leh64, Tables IIA, IIIA], we check that $P(\Delta(m, 2, k)) > 4k$. Hence assume that $m + 2i$ with $1 \leqslant i < k$ is different from those N given in [Leh64, Tables IIA, IIIA].

For every prime $31 < p \leqslant 4k$, we delete a term in $\{m, m+2, \cdots, m+2(k-1)\}$ divisible by p. Let $i_1 < i_2 < \ldots < i_l$ be such that $m + 2i_j$ is in the remaining set where $l \geqslant k - (\pi(4k) - \pi(31))$. From [Leh64, Tables IIA, IIIA], we observe that $i_{j+1} - i_j \geqslant 3$ implying $k-1 \geqslant i_l - i_1 \geqslant 3(l-1) \geqslant 3(k - \pi(4k) + 10)$. However we find that the inequality $k - 1 \geqslant 3(k - \pi(4k) + 10)$ is not valid except when $k = 28, 29$. Hence the assertion of the Lemma is valid except possibly for $k = 28, 29$.

Therefore we may assume that $k = 28, 29$. Further we suppose that $l = k - (\pi(4k) - \pi(31)) = 10$ otherwise $3(l - 1) \geqslant 30 > k - 1$, a contradiction. Thus we have either $i_{10} - i_1 = 27$ implying $i_1 = 0, i_{j+1} = i_j + 3 = 3j$ for $1 \leqslant j \leqslant 9$ or $i_1 = 1, i_{j+1} = i_j + 3 = 3j + 1$ for $1 \leqslant j \leqslant 9$ or $i_{10} - i_1 = 28$ implying

$$i_1 = 0, \qquad i_{j+1} = \begin{cases} 3j & \text{if } 1 \leqslant j \leqslant r \\ 3j + 1 & \text{if } r < j \leqslant 9 \end{cases} \qquad \text{for some } r \geqslant 1.$$

Let $X = m + 2i_1 - 6$. Note that X is odd since m is odd. Also $X \geqslant 4k + 1 - 6 \geqslant 107$. We have either

$$P((X + 6) \cdots (X + 54)(X + 60)) \leqslant 31 \tag{5}$$

or there is some $r \geqslant 1$ for which

$$P((X + 6) \cdots (X + 6r)(X + 6(r + 1) + 2) \cdots (X + 60 + 2)) \leqslant 31. \tag{6}$$

Note that (5) is the only possibility when $k = 28$. Now we consider (5). Suppose $3|X$. Then putting $Y = \frac{X}{3}$, we get $P((Y + 2) \cdots (Y + 18)(Y + 20)) \leqslant 31$ which implies $Y + 2 < 20$ by Corollary 2.4 and Lemma 2.8 with $k = 10$. Since $X + 6 \geqslant m \geqslant 113$, we get a contradiction. Hence we may assume that $3 \nmid X$. Then $3 \nmid (X + 6) \cdots (X + 54)(X + 60)$. After deleting terms $X + 6i$ divisible by primes $11 \leqslant p \leqslant 31$, we are left with three terms divisible by primes 5 and 7 and hence $m \leqslant X + 6 \leqslant 35$ which is again a contradiction. Therefore (5) is not possible.

Now we consider (6) which is possible only when $k = 29$. Since $X + 6 = m > 4k = 116$, we have $X > 110$. Suppose $r = 1, 9$. Then we have $P((X + 12 + 2) \cdots (X + 54 + 2)(X + 60 + 2)) \leqslant 31$ if $r = 1$ and $P((X + 6) \cdots (X + 54)) \leqslant 31$ if $r = 9$. Putting $Y = X + 8$ in the first case and $Y = X$ in the latter, we get $P((Y + 6) \cdots (Y + 54)) \leqslant 31$. Suppose $3|Y$. Then putting $Z = \frac{Y}{3}$, we get $P((Z + 2) \cdots (Z + 18)) \leqslant 31$ which implies $Z + 2 \leqslant 18$ by Corollary 2.4 and Lemma 2.8 with $k = 9$. Since $Z + 2 \geqslant \frac{X}{3} > \frac{110}{3}$, we get a contradiction. Hence we may assume that $3 \nmid Y$. Then $3 \nmid (Y + 6) \cdots (Y + 54)$. After deleting terms

$Y+6i$ divisible by primes $11 \leqslant p \leqslant 31$, we are left with two terms divisible by primes 5 and 7 only. Let $Y+6i = 5^{a_1}7^{b_1}$ and $Y+6j = 5^{a_2}7^{b_2}$ where $b_1 \leqslant 1 < b_2$ and $a_2 \leqslant 1 < a_1$. Since $|i-j| \leqslant 8$, the equality $6(i-j) = 5^{a_1}7^{b_1} - 5^{a_2}7^{b_2}$ implies $5^a - 7^b = \pm 6, \pm 12, \pm 18, \pm 24, \pm 36, \pm 48$. By taking modulo 6, we get $(-1)^a \equiv 1$ modulo 6 implying a is even. Taking modulo 8 again, we get either

$$b \text{ is even}, \qquad 5^a - 7^b = (5^{\frac{a}{2}} - 7^{\frac{b}{2}})(5^{\frac{a}{2}} + 7^{\frac{b}{2}}) = \pm 24, \pm 48$$

giving

$$5^a = 25, 7^b = 49 \tag{7}$$

or

$$b \text{ is odd}, \qquad 5^a - 7^b = -6, 18.$$

Let $5^a - 7^b = -6$. Considering modulo 5, we get $2^b \equiv 1$ implying $4|b$, a contradiction. Let $5^a - 7^b = 18$. By considering modulo 7 and modulo 9 and since a is even, we get $3|(a-2)$ and $3|(b-1)$ implying $(5^{\frac{a+1}{3}})^3 + 35(-7^{\frac{b-1}{3}})^3 = 90$. Solving the Thue equation $x^3 + 35y^3 = 90$ gives $x = 5, y = -1$ or $25 - 7 = 18$ is the only solution. Hence $6 \cdot 3 = 25 - 7 = X + 6i - (X + 6j)$. Also the solution (7) implies $-6 \cdot 4 = 25 - 49 = X + 6i - (X + 6j)$. Thus $X \leqslant 25$ which is not possible.

Assume now that $2 \leqslant r \leqslant 8$. Then $P((X+6)(X+12)(X+56)(X+62)) \leqslant 31$. Suppose $3|X(X+2)$. Putting $Y = \frac{X+6}{3}$ if $3|X$ and $Y = \frac{X+56}{3}$ if $3|(X+2)$, we get either $P(Y(Y+2)(3Y+50)(6Y+56)) \leqslant 31$ or $P(Y(Y+2)(3Y-50)(3Y-44)) \leqslant 31$. In particular $P(Y(Y+2)) \leqslant 31$. For $Y = N-2$ given by [Leh64, Table IIA] such that $P(Y(Y+2)) \leqslant 31$, we check that $P((3Y+50)(3Y+56)) > 31$ and $P((3Y-50)(3Y-44)) > 31$ except when $Y \in \{55, 145, 297, 1573\}$. This gives $m = X + 6 = 3Y - 50$ and then we further check that $P(\Delta(m, 2, k)) > 116$. Hence we suppose $3 \nmid X(X+2)$. Then $3 \nmid (X+6)\cdots(X+6r)(X+6(r+1)+2)\cdots(X+60+2)$. If a prime power p^a divides two terms of the product, then $p^a|(X+6j), p^a|(X+6i)$ or $p^a|(X+6j+2), p^a|(X+6i+2)$ or $p^a|(X+6j), p^a|(X+6i+2)$ for some i, j. Hence $p^a|6(i-j)$ or $p^a|6(i-j)+2$. Since $1 \leqslant j < i \leqslant 10$, we get $p^a \in \{5, 7, 11, 13, 19, 25\}$. After deleting terms divisible by primes $5 \leqslant p \leqslant 31$ to their highest powers, we are left with two terms such that their product divides $25 \cdot 7 \cdot 11 \cdot 13 \cdot 19$ and hence $X+6 \leqslant \sqrt{25 \cdot 7 \cdot 11 \cdot 13 \cdot 19}$ or $X+6 \leqslant 689$. We check that $P((X+6)(X+12)(X+56)(X+62)) > 31$ for $110 \leqslant X \leqslant 683$ except when $X \in \{113, 379\}$. Further we check that $P(\Delta(m, 2, k)) > 116$ for $m = X + 6$. Hence the result. ∎

The remaining results in this section deal with the case $d = 3$. The first one is a computational result.

Lemma 2.9. *Let $l \in \{1, 2\}$. If $p_{i,3,l} \leqslant 6450$, then $\delta_3(i, l) \leqslant 60$.*

As a consequence, we obtain

Corollary 2.10. *Let $d = 3$ and $3k < m \leqslant 6450$ with $gcd(m, 3) = 1$. Then* (1) *holds unless $(m, k) = (125, 2)$.*

Proof. For $k \leqslant 20$, it follows by direct computation. For $k > 20$, (1) follows as $3(k-1) \geqslant 60$ and, by Lemma 2.9, the set $\{m+3i : 0 \leqslant i < k\}$ contains a prime. ■

We shall also need the following result of Nagell [Nag58](see [Cao99]) on diophantine equations.

Lemma 2.11. *Let $a, b, c \in \{2, 3, 5\}$ and $a < b$. Then the solutions of*

$$a^x + b^y = c^z \qquad \text{in integers } \ x > 0,\ y > 0,\ z > 0$$

are given by

$$(a^x, b^y, c^z) \in \{(2,3,5), (2^4, 3^2, 5^2), (2, 5^2, 3^3), (2^2, 5, 3^2), (3, 5, 2^3), (3^3, 5, 2^5), (3, 5^3, 2^7)\}.$$

As a corollary, we have

Corollary 2.12. *Let $X > 80, 3 \nmid X$ and $1 \leqslant i \leqslant 7$. Then the solutions of*

$$P(X(X+3i)) = 5 \qquad \text{and} \qquad 2|X(X+3i)$$

are given by

$$(i, X) \in \{(1,125), (2,250), (4,500), (5,625)\}.$$

Proof. Let $1 \leqslant i \leqslant 7$. We observe that $2|X, 2|(X+3i)$ only if X and i are both even and $5|X, 5|(X+3i)$ only if $i = 5$. Let the positive integers r, s and $\delta = \text{ord}_2(i) \in \{0,1,2\}$ be given by

$$X = 2^{r+\delta},\ \ X+3i = 2^\delta 5^s \qquad \text{or} \qquad X = 2^\delta 5^s,\ \ X+3i = 2^{r+\delta} \ \text{ if } i \neq 5 \tag{8}$$

and

$$X = 5^{s+1},\ \ X+3i = 5 \times 2^r \qquad \text{or} \qquad X = 5 \times 2^r,\ \ X+3i = 5^{s+1} \ \text{ if } i = 5, \tag{9}$$

where $r + 2 \geqslant r + \delta \geqslant 7$ and $s \geqslant 2$ since $X > 80$. Hence we have

$$2^r - 5^s = \pm\left(\frac{X+3i}{2^{\text{ord}_2(i)} \cdot 5^{\text{ord}_5(i)}} - \frac{X}{2^{\text{ord}_2(i)} \cdot 5^{\text{ord}_5(i)}}\right) = \pm 3 \times \frac{i}{2^{\text{ord}_5(i)} \cdot 5^{\text{ord}_5(i)}}. \tag{10}$$

Let $i \in \{1,2,4,5\}$. Then $2^r - 5^s = \pm 3$. By Lemma 2.11, we have $2^r = 2^7$, $5^s = 5^3$ and $2^7 - 5^3 = 3$ implying $X = 2^{\text{ord}_2(i)} \cdot 5^{3+\text{ord}_5(i)}$ and $X + 3i = 2^{7+\delta} \cdot 5^{\text{ord}_5(i)}$. These give the solutions stated in the Corollary.

Let $i \in \{3, 6\}$. Then $2^r - 5^s = \pm 9 = \pm 3^2$. Since $\min(2^r, 5^s) > 16$, we observe from Lemma 2.11 that there is no solution.

Let $i = 7$. Then $2^r - 5^s = \pm 21$. Let s be even. Since $2^r > 16$, taking modulo 8, we find that $-1 \equiv \pm 21($ modulo 8) which is not possible. Hence s is odd. Then $2^r - 5^s \equiv 2^r + 2^s \equiv 0$ modulo 7. Since $2^r, 2^s \equiv 1, 2, 4$ modulo 7, we get a contradiction. ■

3. Proof of Theorem 3

Let $D = 4, 3$ according as $d = 2, 3$, respectively. Let $v = \frac{m}{dk}$. Assume that

$$P(\Delta(m, d, k)) = P(m(m+d)\cdots(m+(k-1)d) < Dk. \tag{11}$$

Then

$$\omega(\Delta(m, d, k)) \leqslant \pi(Dk) - 1. \tag{12}$$

For every prime $p \leqslant Dk$ dividing Δ, we delete a term $m + i_p d$ such that $\mathrm{ord}_p(m + i_p d)$ is maximal. Note that $p|(m+id)$ for at most one i if $p \geqslant k$. Then we are left with a set T with $1 + t := |T| \geqslant k - \pi(Dk) + 1 := 1 + t_0$. Let $t_0 \geqslant 0$ which we assume in this section to ensure that T is non-empty. We arrange the elements of T as $m + i'_0 d < m + i'_1 d < \cdots < m + i'_{t_0} d < .. < m + i'_t d$. Let

$$\mathfrak{P} := \prod_{\nu=0}^{t_0}(m + i'_\nu d) \geqslant d^{k-\pi(Dk)+1} \prod_{i=0}^{k-\pi(Dk)} (vk+i). \tag{13}$$

We now apply [LaSh04b, Lemma 2.1, (14)] to get

$$\mathfrak{P} \leqslant (k-1)! d^{-\mathrm{ord}_d (k-1)!}.$$

Comparing the upper and lower bounds of $\mathfrak{P}$, we have

$$d^{\pi(Dk)} \geqslant \frac{d^{k+1} \prod_{i=0}^{k-\pi(Dk)} (vk+i)}{(k-1)! d^{-\mathrm{ord}_d (k-1)!}}$$

which imply

$$d^{\pi(Dk)} \geqslant \frac{d^{k+1} d^{\mathrm{ord}_d (k-1)!} (vk)^{k+1-\pi(Dk)}}{(k-1)!}. \tag{14}$$

By using the estimates for $\mathrm{ord}_d((k-1)!)$ and $(k-1)!$ given in Lemma 2.1, we obtain

$$\begin{aligned}(vdk)^{\pi(Dk)} &> \frac{(vdk)^{k+1} d^{(k-d)/(d-1)} (k-1)^{-1}}{\sqrt{2(k-1)\pi}(\frac{k-1}{e})^{k-1} \exp\left(\frac{1}{12(k-1)}\right)} \\ &= \left(evd^{\frac{d}{d-1}} \frac{k}{k-1}\right)^k \frac{v\sqrt{k}}{ed^{1/(d-1)}\sqrt{2\pi}} \sqrt{\frac{k}{k-1}} \exp\left(-\frac{1}{12(k-1)}\right)\end{aligned}$$

implying

$$\pi(Dk) > \frac{k \log(evd^{\frac{d}{d-1}}) + (k+\frac{1}{2})\log(\frac{k}{k-1}) - \frac{1}{12(k-1)} + \frac{1}{2}\log\frac{v^2 k}{2\pi e^2 d^{\frac{2}{d-1}}}}{\log(vdk)}. \tag{15}$$

Again by using the estimates for $\pi(\nu)$ given in Lemma 2.1 and $\frac{\log(vdk)}{\log(Dk)} = 1+\frac{\log\frac{vd}{D}}{\log(Dk)}$, we derive

$$\begin{aligned} 0 >& \frac{1}{2}\log\frac{v^2k}{2\pi e^2 d^{\frac{2}{d-1}}} - \frac{1}{12(k-1)} \\ &+ k\left(\log\left(evd^{\frac{d}{d-1}}\right) - D\left(1+\frac{\log\frac{vd}{D}}{\log(Dk)}\right)\left(1+\frac{1.2762}{\log(Dk)}\right)\right). \end{aligned} \tag{16}$$

Let v be fixed with $vd \geqslant D$. Then expression

$$F(k,v) := \log\left(evd^{\frac{d}{d-1}}\right) - D\left(1+\frac{\log\frac{vd}{D}}{\log(Dk)}\right)\left(1+\frac{1.2762}{\log(Dk)}\right)$$

is an increasing function of k. Let $k_1 := k_1(v)$ be such that $F(k,v) > 0$ for all $k \geqslant k_1$. Then we observe that the right hand side of (16) is an increasing function for $k \geqslant k_1$. Let $k_0 := k_0(v) \geqslant k_1$ be such that the right hand side of (16) is positive. Then (16) is not valid for all $k \geqslant k_0$ implying (15) and hence (14) are not valid for all $k \geqslant k_0$.

Also for a fixed k, if (16) is not valid at some $v = v_0$, then (14) is also not valid at $v = v_0$. Observe that for a fixed k, if (14) is not valid at some $v = v_0$, then (14) is also not valid when $v \geqslant v_0$.

Therefore for a given $v = v_0$ with $v_0 d \geqslant D$, the inequality (14) is not valid for all $k \geqslant k_0(v_0)$ and $v \geqslant v_0$.

3(a). Proof of Theorem 3 for the case $d = 3$

Let $d = 3$ and let the assumptions of Theorem 3 be satisfied. Let $2 \leqslant k \leqslant 11$ and $m > 3k$. Observe that $k - \pi(3k) + 1 = 0$ for $k \leqslant 8$ and $k - \pi(3k) + 1 = 1$ for $9 \leqslant k \leqslant 11$. If $T \neq \phi$, then $m \leqslant 2^3 \times 5 \times 7 = 280$.

By Corollary 2.10, we may assume that $2 \leqslant k \leqslant 8$, $m \geqslant 6450$ and $T = \phi$. Further i_p exists for each prime $p \leqslant 3k$, $p \neq 3$ and $i_p \neq i_q$ for $p \neq q$ otherwise $|T| \geqslant k - \pi(3k) + 1 + 1 > 0$. Also $pq \nmid (m + id)$ for any i whenever $p, q \geqslant k$ otherwise $T \neq \phi$. Thus $P((m+3i_2)(m+3i_5)) = 5$ if $k < 8$. For $k = 8$, we get $P((m+3i_2)(m+3i_5)) \leqslant 7$ with $P((m+3i_2)(m+3i_5)) = 7$ only if $7|m$ and $\{i_2, i_5\} \cap \{0, 7\} \neq \phi$.

Let $k \leqslant 7$ or $k = 8$ with $P((m+3i_2)(m+3i_5)) = 5$. Let $j_0 = \min(i_2, i_5)$, $X = m + 3j_0$ and $i = |i_2 - i_5|$. Then $X \geqslant 6450$ and this is excluded by Corollary 2.12.

Let $k = 8$ and $P((m+3i_2)(m+3i_5)) = 7$. Then $7|m$ and $\{i_2, i_5\} \cap \{0, 7\} \neq \phi$. Hence $i_7 = 0$ or 7 and $7 \in \{i_2, i_5\}$ if $i_7 = 0$ and $0 \in \{i_2, i_5\}$ if $i_7 = 7$. If $5 \nmid m(m+21)$, then $\{i_2, i_7\} = \{0, 7\}$ and either

$$m = 7 \times 2^r, \quad m + 21 = 7^{1+s} \qquad \text{or} \qquad m = 7^{1+s}, \quad m + 21 = 7 \times 2^r$$

implying $2^r - 7^s = \pm 3$. Since $2^r \geqslant \frac{m}{7} > 40$, we get by taking modulo 8 that $(-1)^{s+1} \equiv \pm 3$ which is a contradiction. Thus $5|m(m+21)$ implying $2 \times 5 \times$

$7|m(m+21)$. By taking the prime factorization, we obtain

$$m = 2^{a_0}5^{b_0}7^{c_0}, \qquad m+21 = 2^{a_1}5^{b_1}7^{c_1}$$

with $\min(a_0, a_1) = \min(b_0, b_1) = 0$, $\min(c_0, c_1) = 1$ and further $b_0 + b_1 = 1$ if $i_2 \in \{0, 7\}$ and $a_0 + a_1 \leqslant 2$ if $i_5 \in \{0, 7\}$. From the identity $\frac{m+21}{7} - \frac{m}{7} = 3$, we obtain one of

(i) $2^a - 5 \cdot 7^c = \pm 3$ or
(ii) $5 \cdot 2^a - 7^c = \pm 3$ or
(iii) $5^b - 2^\delta \cdot 7^c = \pm 3$ or
(iv) $2^\delta \cdot 5^b - 7^c = \pm 3$

with $\delta \in \{1, 2\}$. Further from $m \geqslant 6450$, we obtain $c \geqslant 3$ and

$$a \geqslant 9, \qquad a \geqslant 7, \qquad b \geqslant 4, \qquad b \geqslant 3 \tag{17}$$

according as (i), (ii), (iii), (iv) hold, respectively. These equations give rise to a Thue equation

$$X^3 + AY^3 = B \tag{18}$$

with integers $X, Y, A > 0, B > 0$ given by

	$c \pmod 3$	Equation	A	B	X	Y
(i)	$0,1$	$2^a - 5 \cdot 7^c = \pm 3$	$5 \cdot 2^{a'} \cdot 7^{c'}$	$3 \cdot 2^{a'}$	$\pm 2^{\frac{a+a'}{3}}$	$\pm 7^{\frac{c-c'}{3}}$
(ii)	$0,1$	$5 \cdot 2^a - 7^c = \pm 3$	$25 \cdot 2^{a'} \cdot 7^{c'}$	$75 \cdot 2^{a'}$	$\pm 5 \cdot 2^{\frac{a+a'}{3}}$	$\pm 7^{\frac{c-c'}{3}}$
(iii)	$0,1$	$5^b - 2^\delta \cdot 7^c = \pm 3$	$2^\delta \cdot 5^{b'} \cdot 7^{c'}$	$3 \cdot 5^{b'}$	$\pm 5^{\frac{b+b'}{3}}$	$\pm 7^{\frac{c-c'}{3}}$
(iv)	$0,1$	$2^\delta \cdot 5^b - 7^c = \pm 3$	$2^{3-\delta} \cdot 5^{b'} \cdot 7^{c'}$	$2^{3-\delta} \cdot 5^{b'} \cdot 3$	$\pm 2 \cdot 5^{\frac{b+b'}{3}}$	$\pm 7^{\frac{c-c'}{3}}$
(v)	2	$2^a - 5 \cdot 7^c = \pm 3$	$175 \cdot 2^{a'}$	525	$\pm 5 \cdot 7^{\frac{c+1}{3}}$	$\pm 2^{\frac{a-a'}{3}}$
(vi)	2	$5 \cdot 2^a - 7^c = \pm 3$	$35 \cdot 2^{a'}$	21	$\pm 7^{\frac{c+1}{3}}$	$\pm 2^{\frac{a-a'}{3}}$
(vii)	2	$5^b - 2^\delta \cdot 7^c = \pm 3$	$2^{3-\delta} \cdot 5^{b'} \cdot 7$	$21 \cdot 2^{3-\delta}$	$\pm 2 \cdot 7^{\frac{c+1}{3}}$	$\pm 5^{\frac{b-b'}{3}}$
(viii)	2	$2^\delta \cdot 5^b - 7^c = \pm 3$	$2^\delta \cdot 5^{b'} \cdot 7$	21	$\pm 7^{\frac{c+1}{3}}$	$\pm 5^{\frac{b-b'}{3}}$

where $0 \leqslant a', b' < 3$ are such that X, Y are integers and $c' = 0, 1$ according as $c \pmod 3) = 0, 1$, respectively. For example, $2^a - 5 \cdot 7^c = \pm 3$ with $c \equiv 0, 1 \pmod 3$ implies $(\pm 2^{\frac{a+a'}{3}})^3 + 5 \cdot 2^{a'} 7^{c'} (\pm 7^{\frac{c-c'}{3}})^3 = 3 \cdot 2^{a'}$ where a' is such that $3|(a + a')$. This give a Thue equation (18) with $A = 5 \cdot 2^{a'} 7^{c'}$ and $B = 3 \cdot 2^{a'}$.

By using (17), we see that at least two of

$$\mathrm{ord}_2(XY) \geqslant 2 \qquad \text{or} \qquad \mathrm{ord}_5(XY) \geqslant 1 \qquad \text{or} \qquad \mathrm{ord}_7(XY) \geqslant 1 \tag{19}$$

hold except for (*vi*) and (*viii*) where $\mathrm{ord}_2(XY) \geqslant 1$, $\mathrm{ord}_7(XY) \geqslant 1$ in case of (*vi*) and $\mathrm{ord}_2(XY) = 0$, $\mathrm{ord}_7(XY) \geqslant 1$ in case of (*viii*). Using the command

T:=Thue($X^3 + A$); Solutions(T, B);

in *Kash*, we compute all the solutions in integers X, Y of the above Thue equations. We find that none of solutions of Thue equations satisfy (19).

Hence we have $k \geqslant 12$. For the proof of Theorem 3, we may suppose from Corollaries 2.10 and 2.3 that

$$m \geqslant \max(6450, 10.6 \times 3k). \tag{20}$$

Let $12 \leqslant k \leqslant 19$. Since $t_0 \geqslant 1, 2$ for $12 \leqslant k \leqslant 16$ and $17 \leqslant k \leqslant 19$, respectively, we have

$$m \leqslant \sqrt{\mathfrak{P}} \leqslant \sqrt{4 \times 8 \times 5^2 \times 7^2 \times 11 \times 13} < 6450 \qquad \text{if } 12 \leqslant k \leqslant 16$$
$$m \leqslant \sqrt[3]{\mathfrak{P}} \leqslant \sqrt[3]{4 \times 8 \times 16 \times 5^3 \times 7^2 \times 11 \times 13 \times 17} < 6450 \qquad \text{if } 17 \leqslant k \leqslant 19.$$

This is not possible by (20).

Thus $k \geqslant 20$. Then $m \geqslant 6450$ and $v \geqslant 10.6$ by (20) satisfying $v_0 d \geqslant D = d = 3$. Now we check that $k_0 \leqslant 180$ for $v = 10.6$. Therefore (14) is not valid for $k \geqslant 180$ and $v \geqslant 10.6$. Thus $k < 180$. Further we check that (15) is not valid for $20 \leqslant k < 180$ at $v = \frac{6450}{3k}$ except when $k \in \{21, 25, 28, 37, 38\}$. Hence (14) is not valid for $20 \leqslant k < 180$ when $v \geqslant \frac{6450}{3k}$ except when $k \in \{21, 25, 28, 37, 38\}$. Thus it suffices to consider $k \in \{21, 25, 28, 37\}$ where we check that (14) is not valid at $v = \frac{6450}{3k}$ and hence it is not valid for all $v \geqslant \frac{6450}{3k}$. Finally we consider $k = 38$ where we find that (14) is not valid at $v = \frac{8000}{3k}$. Thus $m < 8000$. For $l \in \{1, 2\}$ and $p_{i,3,l} \leqslant 8000$, we find that $\delta_3(i, 3, l) < 90$ implying the set $\{m, m+3, \ldots, m+3(38-1)\}$ contains a prime. Hence the assertion follows since $m > 3k$. ■

3(b). Proof of Theorem 3 for $d = 2$

Let $d = 2$ and let the assumptions of Theorem 3 be satisfied. The assertion for Theorem 3 with $k \geqslant 2$ and $m \leqslant 4k$ follows from Corollary 2.4. Thus $m > 4k$. For $2 \leqslant k \leqslant 37$, $k \neq 35$, Lemma 2.8 gives the result. Hence for the proof of Theorem 3, we may suppose that $k = 35$ or $k \geqslant 38$. Further from Corollaries 2.3 and 2.7, we may assume that

$$m \geqslant \max(M_0, 131 \times 2k). \tag{21}$$

Let $k = 35, 38$. Then $t_0 = 1, 2$ for $k = 35, 38$, respectively and we have

$$m \leqslant \sqrt{\mathfrak{P}} \leqslant \sqrt{27 \cdot 9 \cdot 25 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31}$$
$$< 10^{10} \qquad \text{if } k = 35$$
$$m \leqslant \sqrt[3]{\mathfrak{P}} \leqslant \sqrt[3]{27 \cdot 9^2 \cdot 25 \cdot 5^2 \cdot 7^3 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37}$$
$$< 10^{10} \qquad \text{if } k = 38.$$

This is not possible by (21).

Thus we assume that $k \geqslant 39$. Let $v \geqslant 131$ and we check that $k_0 \leqslant 500$ for $v = 131$. Therefore (14) is not valid for $k \geqslant 500$ and $v \geqslant 131$. Hence from (21), we get $k < 500$. Further $v \geqslant \frac{M_0}{2\times 500} \geqslant 10^7$. We check that $k_0 \leqslant 70$ at $v = 10^7$ implying (14) is not valid for $k \geqslant 70$ and $v \geqslant 10^7$. Thus $k < 70$. For each $39 \leqslant k < 70$, we find that (14) is not valid at $v = \frac{M_0}{2k}$ and hence for all $v \geqslant \frac{M_0}{2k}$. This is a contradiction. ■

4. Proof of Theorems 1 and 2

Recall that $q = u + \frac{\alpha}{d}$ with $1 \leqslant \alpha < d$. We observe that if $G(x)$ has a factor of degree k, then it has a cofactor of degree $n-k$. Hence we may assume from now on that if $G(x)$ has a factor of degree k, then $k \leqslant \frac{n}{2}$. The following result is [ShTi10, Lemma 10.1].

Lemma 4.1. *Let* $1 \leqslant k \leqslant \frac{n}{2}$ *and*

$$d \leqslant 2\alpha + 2 \qquad \text{if} \quad (k,u) = (1,0).$$

If there is a prime p *with*

$$p|(\alpha + (n+u-k)d)\cdots(\alpha + (n+u-1)d), \qquad p \nmid a_0 a_n.$$

such that

$$p \geqslant \begin{cases} (k+u-1)d + \alpha + 1 & \text{if } u > 0 \\ (k+u-1)d + \alpha + 2 & \text{if } u = 0 \end{cases}$$

Then $G(x)$ *has no factor of degree* k.

Let $d = 3$. By putting $m = \alpha + 3(n-k)$ and taking $p = P(\Delta(m,3,k))$, we find from Lemma 4.1 and Theorem 3 that $G_{\frac{1}{3}}$ and $G_{\frac{2}{3}}$ does not have a factor of degree $k \geqslant 2$ except possibly when $k = 2, \alpha = 2, m = 2 + 3(n-2) = 125$. This gives $n = 43$ and we use [ShTi10, Lemma 2.13] with $p = 2, r = 2$ to show that $G_{\frac{2}{3}}$ do not have a factor of degree 2. Further except possibly when $m = \alpha + 3(n-1) = 2^l$ for positive integers l, $G_{\frac{1}{3}}$ and $G_{\frac{2}{3}}$ do not have a linear factor. This proves Theorem 1.

Let $d = 2$. Let $k = 1, u = 0$. We have $P(1 + 2(n-1)) \geqslant 3$ and hence taking $p = P(1 + 2(n-1))$ in Lemma 4.1, we find that $G_{\frac{1}{2}}$ does not have a factor of degree 1. Hence from now on, we may suppose that $k \geqslant 2$ and $0 \leqslant u \leqslant k$. For $(m,k) \in \{((5,2),(7,2),(9,4),(13,5),(17,6),(15,7),(21,8),(19,9)\}$, we check that $P(\Delta(m,2,k)) \geqslant m$. For $0 \leqslant u \leqslant k$, by putting $m = 1 + 2(n+u-k)$, we find from $n \geqslant 2k$ and Theorem 3 that

$$P(\Delta(m,2,k)) > 2(k+u) = \begin{cases} \min(2(k+u), 3.5k) & \text{if } u \leqslant 0.5k \\ \min(2(k+u), 4k) & \text{if } 0.5k < u \leqslant k \end{cases}$$

except when $k = 2, (u,m) \in \{(1,25),(2,25),(2,243)\}$. Observe that if $p > 2(k+u)$, then $p \geqslant 2(k+u)+1$. Now we take $p = P(\Delta(m,2,k))$ in Lemma 4.1 to obtain that $G_{u+\frac{1}{2}}$ do not have a factor of degree k with $k \geqslant 2$ except possibly when $k = 2, u = 1, n = 13$ or $k = 2, u = 2, n \in \{12, 121\}$. We use [ShTi10, Lemma 2.13] with $(p,r) = (3,1),(7,1)$ to show that $G_{u+\frac{1}{2}}$ do not have a factor of degree 2 when $(u,n) = (1,13),(2,12)$ and $(u,n) = (2,121)$, respectively. ■

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Addresses: Shanta Laishram: Stat-Math Unit, Indian Statistical Institute, 7 S.J.S. Sansanwal Marg, New Delhi, 110016, India;
Tarlok N. Shorey: Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India.

E-mail: shanta@isid.ac.in, shorey@math.iitb.ac.in

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