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HOW SLOWLY CAN A BOUNDED SEQUENCE CLUSTER?

John Bentin

Abstract: We propose a simple measure of how slowly a bounded real sequence clusters. This measure, called *separation*, is the infimum, over all finite segments of the sequence with at least two terms, of the ratio of the least distance between the terms in the segment to the general supremum of such a distance for a segment of that length. An example of a highly separated sequence is given. To create a more separated sequence, we modify van der Corput's construction, replacing the powers of a base by the even-numbered terms of the Fibonacci sequence. The result coincides initially with the sequence built stepwise by maximizing separation for each extra term. We conjecture that these sequences are the same and of maximal separation.

Keywords: bounded real sequence, clustering, separation.

1. Introduction

The Bolzano–Weierstrass theorem tells us that a bounded real infinite sequence has a cluster point. Clustering is unavoidable in a bounded infinite sequence, but some sequences cluster more slowly than others. We try to capture this idea in the concept of *separation*. Roughly speaking, a highly separated sequence is one where consecutive points and, to a gently diminishing extent, sequentially more distant points are widely spaced. Separation (which will be defined in the next section) and the established concept of *dispersion* [1] have some common features, but generally the two measures are quite different in their properties.

2. A highly separated sequence

Let $x_0, x_1, \dots$ be the distinct terms of a bounded real sequence with infimum a and supremum b. Consider any run of consecutive terms—say $n+1$ of them. The least distance between any two of the terms in this run can then be at most $(b-a)/n$. If x_i and x_j are such a closest pair of terms, then $|x_i - x_j| \leqslant (b-a)/n \leqslant (b-a)/|i-j|$. It follows that the quantity

$$|i-j||x_i - x_j| \qquad (i, j = 0, 1, \dots;\ i \neq j)$$

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is bounded above by $b-a$ whenever x_i and x_j are a closest pair in a run. This bound is attained or asymptotic in some cases and so cannot generally be reduced. To create a "most separated" sequence, we try to construct the sequence so that $|i-j||x_i-x_j|$ is bounded *below*—for any such pair and hence for any unequal i and j whatsoever—by a constant c that is as large as possible in relation to the *span* of the sequence, $b-a$. Thus $c/(b-a)$ is our measure of separation that we want to maximize.

To standardize, we could take $b-a=1$ and maximize c. For the present, it is convenient to set $c=1$ and try to minimize $b-a$. So our target is to solve the following problem with a sequence that has the smallest possible span.

Problem 1. Find a bounded infinite sequence of real numbers $x_0, x_1, \ldots$ such that

$$|i-j||x_i-x_j| \geqslant 1 \qquad \text{for all unequal } i,j \in \mathbb{N}.$$

This problem has a solution $x_n = \alpha_n$ defined by

$$\alpha_n \equiv n \pmod{\alpha} \qquad \text{with } 0 \leqslant \alpha_n < \alpha \text{ for } n \in \mathbb{N},$$

where $\alpha^2 - 3\alpha + 1 = 0$ with $\alpha > 2$. We will need to recall that the roots α and $\bar{\alpha}$ of $x^2 - 3x + 1 = 0$ are irrational.

Theorem 1. $|i-j|\,|\alpha_i - \alpha_j| \geqslant 1$ *for all distinct* $i,j \in \mathbb{N}$.

Proof. Without loss of generality, take $n = i-j > 0$. Let $k \in \mathbb{N}$ with $k \leqslant n+1$. Then $n - k\bar{\alpha} > 0$ since $\bar{\alpha} < \frac{1}{2}$. Also $n - k\alpha \neq 0$ since α is irrational. So the product $(n-k\bar{\alpha})(n-k\alpha)$, which equals the integer $n^2 - 3nk + k^2$, is nonzero. Hence $1 \leqslant (n-k\bar{\alpha})|n-k\alpha|$ and therefore

$$1 \leqslant n|n-k\alpha|. \tag{1}$$

Let $\alpha_n = n - b_n\alpha$, where $b_n \in \mathbb{N}$ is uniquely determined, for each $n = 1, 2, \ldots$, by $0 \leqslant n - b_n\alpha < \alpha$. Note that $b_n \leqslant n$. We consider the two cases $\alpha_i > \alpha_j$ and $\alpha_i < \alpha_j$ separately.

If $\alpha_i > \alpha_j$, then $|\alpha_i - \alpha_j| = \alpha_i - \alpha_j = i - j - (b_i - b_j)\alpha = \alpha_{i-j}$, since $0 \leqslant \alpha_i - \alpha_j < \alpha$. Now choose $k = b_n$. Then $k \leqslant n+1$, and inequality (1) yields $1 \leqslant n|n-k\alpha| = n|n-b_n\alpha| = n|\alpha_n| = |i-j||\alpha_i - \alpha_j|$.

In the case when $\alpha_i < \alpha_j$, we have $\alpha - (\alpha_j - \alpha_i) = i - j - (b_i - b_j - 1)\alpha$, which is in $[0\,,\alpha)$ and so is equal to α_{i-j}. In this case, choose $k = b_n + 1$. Then inequality (1) gives $1 \leqslant n|n-k\alpha| = n|n-(b_n+1)\alpha| = n|\alpha - (n - b_n\alpha)| = n|\alpha - \alpha_n| = (i-j)|\alpha - \alpha_{i-j}| = |i-j||\alpha_i - \alpha_j|$. ∎

3. A more separated sequence

Suppose that a real bounded sequence $\boldsymbol{x} = (\,x_n : n \in \mathbb{N}\,)$ is nonconstant, and let $a = \inf\{\,x_n : n \in \mathbb{N}\,\}$ and $b = \sup\{\,x_n : n \in \mathbb{N}\,\}$. Our measure of separation may be stated as

$$\operatorname{sep} \boldsymbol{x} = \frac{1}{b-a} \inf\{\,|i-j||x_i - x_j| : i,j \in \mathbb{N} \text{ with } i \neq j\,\}. \tag{2}$$

For completeness, define the separation of a constant sequence to be zero. Clearly, separation is invariant under shifting and scaling of the sequence by constants.

Our solution sequence of integers modulo α, which has infimum 0 and supremum α, has separation $1/\alpha = \bar{\alpha} = 0.381966\ldots$. However, there is a sequence, also a solution to Problem 1, with higher separation: approximately 0.3944196702. To define this sequence, we start with the sequence of integers $\mathrm{a}_1, \mathrm{a}_2, \ldots$, which begins 1, 3, 8, 21, 55, 144, 377, ..., defined by the recurrence relation $\mathrm{a}_{i-1} - 3\mathrm{a}_i + \mathrm{a}_{i+1} = 0$ $(i = 1, 2, \ldots)$ with $\mathrm{a}_0 = 0$ and $\mathrm{a}_1 = 1$. Every nonnegative integer can be expressed as a sum of these a_i using the largest of them possible. For example, $100 = 3+21+21+55$ and $2012 = 1+8+8+21+987+987$. We can thus code 100 as 01021; and 2012 codes as 10210002 in this way (the 0s correspond to missing sequence terms). This code can be constructed, in reverse order, in a systematic way similar to that of changing the base of expression for an integer—say from base 10 to base 3. To code a positive integer n, divide it by the largest sequence term a_r not exceeding n, and record the result $c_r \in \{1, 2\}$; thus $n = c_r\,\mathrm{a}_r + d_r$, where $0 \leqslant d_r < \mathrm{a}_r$. Then divide the remainder d_r by the next-largest sequence term a_{r-1} and record that result $c_{r-1} \in \{0, 1, 2\}$; so $d_r = c_{r-1}\,\mathrm{a}_{r-1} + d_{r-1}$, where $0 \leqslant d_{r-1} < \mathrm{a}_{r-1}$; and so on until all the digits $c_1, \ldots, c_r$ have been found. The code looks similar to ternary (or base 3) notation although, for convenience, we write our code in the natural (ascending) order of the a_i. Zero, corresponding to an empty sum, codes as an empty string (not the string 0, which is of length 1). The code-based solution to Problem 1 is:

$$x_n = \sum_{i=1}^{r} \frac{c_i}{\mathrm{a}_i} \qquad \text{when } n \text{ is coded as } c_1 \ldots c_r.$$

Thus the solution sequence begins $x_0 = 0$ and continues 1, 2, $\frac{1}{3}$, $1\frac{1}{3}$, $2\frac{1}{3}$, $\frac{2}{3}$, $1\frac{2}{3}$, $\frac{1}{8}, \ldots$. The above examples yield $x_{100} = \frac{1}{3} + \frac{2}{21} + \frac{1}{55}$ and $x_{2012} = 1 + \frac{2}{8} + \frac{1}{21} + \frac{2}{987}$. We remark here that this way of building a sequence is structurally similar to van der Corput's [1], but using alternate terms of the Fibonacci sequence instead of powers of a base integer, with some entailed restrictions.

To show that this claimed solution really works needs a more formal approach and some labour, beginning with a definition. We define a *tern* as a finite—possibly empty—string of digits, each 0, 1, or 2, such that any two occurrences of 2 have a zero between them; the latter condition will be referred to as “the rule of 2s”. The positions, or places, of the digits are enumerated $1, 2, \ldots$ from the left. A tern is considered to end at its last nonzero digit. However, it is convenient to assign the digit 0 to virtual places beyond that position if the need arises. We mark terns with a prefix: ${}^{\mathsf{T}}c_1 \ldots c_r$ denotes a tern of length r. In notating strings, 0^k means a string of k 0s, while 1^k is k 1s (k is allowed to be zero); and $c_m \ldots c_n$ is a string of length $n - m + 1$ (if $n = m - 1$, then the string is empty).

For any tern, there is a corresponding integer:

$${}^{\mathsf{T}}c_1 \ldots c_r \longmapsto \sum_{i=1}^{r} c_i \mathrm{a}_i.$$

We will establish that this map is one-to-one from the set $\mathcal{T}$ of terns onto $\mathbb{N}$. A lemma is needed first.

Lemma 1. $\mathrm{a}_j + \sum_{i=j}^{k} \mathrm{a}_i + \mathrm{a}_k = \mathrm{a}_{j-1} + \mathrm{a}_{k+1}$ $(j = 1, 2, \ldots;\ k = j, j+1, \ldots)$.

Proof. The result for $k = j$ is the recurrence relation for the a_i. Suppose that the result has been established for $k = j + l$:

$$\mathrm{a}_j + \sum_{i=j}^{j+l} \mathrm{a}_i + \mathrm{a}_{j+l} = \mathrm{a}_{j-1} + \mathrm{a}_{j+l+1}.$$

Adding $2\mathrm{a}_{j+l+1} - \mathrm{a}_{j+l}$ to each side gives

$$\mathrm{a}_j + \sum_{i=j}^{j+l+1} \mathrm{a}_i + \mathrm{a}_{j+l+1} = \mathrm{a}_{j-1} + 3\mathrm{a}_{j+l+1} - \mathrm{a}_{j+l} = \mathrm{a}_{j-1} + \mathrm{a}_{j+l+2},$$

which is the result for $k = j + l + 1$. By induction, the result holds for all $k = j, j+1, \ldots$. ■

By setting $j = 1$ in Lemma 1, we can easily get the following result.

Corollary 1. $(\mathrm{a}_i : i = 1, 2, \ldots)$ *is a strictly increasing sequence of positive terms satisfying* $\mathrm{a}_{i+1} > 2\mathrm{a}_i$ $(i = 1, 2, \ldots)$.

From the recurrence relation for the a_i and their nonnegativity, it follows that $\mathrm{a}_{i+1} \leqslant 3\mathrm{a}_i$ $(i = 1, 2, \ldots)$. Therefore, in forming the code string (in reverse) for a positive integer through division by the largest a_i that does not exceed it, and successively for each resulting remainder, only the digits 0, 1, and 2 can appear. Moreover, a block of the form 22, 212, ..., $21^n2, \ldots$ cannot arise. To see this, suppose contrarily that there were such a block, spanning (say) the jth to the kth digits, where $j < k$. From Lemma 1, we have $2\mathrm{a}_j + \sum_{i=j+1}^{k-1} \mathrm{a}_i + 2\mathrm{a}_k \geqslant \mathrm{a}_{k+1}$. But the left-hand side here is at most the remainder when the division is by a_{k+1}, and this must be less than a_{k+1}: a contradiction. It follows that the coding function we described takes its values in $\mathcal{T}$, the set of terns:

$$\mathrm{t} : \mathbb{N} \longrightarrow \mathcal{T} : n \longmapsto \mathrm{t}n = {}^{\mathsf{T}}c_1 \ldots c_r, \qquad n = \sum_{i=1}^{r} c_i \mathrm{a}_i,$$

where the largest possible a_i and corresponding smallest possible $c_i \in \{0, 1, 2\}$ are chosen. We will also write

$$\mathrm{n} : \mathcal{T} \longrightarrow \mathbb{N} : {}^{\mathsf{T}}c_1 \ldots c_r \longmapsto \mathrm{n}c_1 \ldots c_r = \sum_{i=1}^{r} c_i \mathrm{a}_i. \tag{3}$$

It is clear that t and n are bijective and inverse to each other. We will use this fact implicitly from now on. The proposed solution to Problem 1 can now be stated formally.

Theorem 2. *The sequence* $(\mathrm{f}_n : n \in \mathbb{N})$, *defined by*

$$\mathrm{f}_n = \sum_{i=1}^{r} \frac{c_i}{\mathrm{a}_i} \qquad \textit{iff} \qquad n = \mathrm{n}c_1 \dots c_r,$$

satisfies the condition $|p-q||\mathrm{f}_p - \mathrm{f}_q| \geqslant 1$ *for all unequal* $p, q \in \mathbb{N}$.

The proof of Theorem 2 will be deferred until we have gathered some useful techniques and results. The first step is to extend the set $\mathcal{T}$ of terns by allowing negative digits. For convenience, we write $\bar{1}$ for -1 and $\bar{2}$ for -2. A *subtern* ${}^{\mathsf{S}}e_1 \dots e_t$ is a string of digits from $\{\bar{2}, \bar{1}, 0, 1, 2\}$ such that $e_i = c_i - d_i$ for $i = 1, \dots, t$, where ${}^{\mathsf{T}}c_1 \dots c_r$ and ${}^{\mathsf{T}}d_1 \dots d_s$ are terns and $t = \max\{\, i : c_i - d_i \neq 0 \,\}$; here we extend a tern by appending 0s as far as necessary to allow the placewise subtraction.

A block of the form 21^k2, and similarly $\bar{2}\bar{1}^k\bar{2}$, cannot arise in a subtern. To see this for 21^k2 (the argument for $\bar{2}\bar{1}^k\bar{2}$ is similar), suppose otherwise and let these digits be the mth to the $(m+k+1)$th digits of the subtern ${}^{\mathsf{S}}e_1 \dots e_t$ formed from the terns ${}^{\mathsf{T}}c_1 \dots c_r$ and ${}^{\mathsf{T}}d_1 \dots d_s$ as $e_i = c_i - d_i$ $(i = 1, \dots, t)$. Then $c_m - d_m = 2$, $c_{m+i} - d_{m+i} = 1$ $(i = 1, \dots, k)$, and $c_{m+k+1} - d_{m+k+1} = 2$. It follows that $c_m = 2$, $c_{m+i} \geqslant 1$ $(i = 1, \dots, k)$, and $c_{m+k+1} = 2$. But this contradicts the rule of 2s applied to ${}^{\mathsf{T}}c_1 \dots c_r$. Conversely, any finite string $e_1 \dots e_t$ of digits from $\{\bar{2}, \bar{1}, 0, 1, 2\}$ in which blocks of the form 21^k2 and $\bar{2}\bar{1}^k\bar{2}$ are absent is the difference of terns ${}^{\mathsf{T}}c_1 \dots c_r$ and ${}^{\mathsf{T}}d_1 \dots d_s$: for example, $e_i = c_i - d_i$, where

$$c_i = \begin{cases} 0 & \text{if } e_i \leqslant 0 \\ e_i & \text{if } e_i > 0 \end{cases} \qquad \text{and} \qquad d_i = \begin{cases} -e_i & \text{if } e_i \leqslant 0 \\ 0 & \text{if } e_i > 0 \end{cases} \qquad (i = 1, \dots, t),$$

with $r = \max\{\, i \in \mathbb{N} : c_i > 0 \,\}$ and $s = \max\{\, i \in \mathbb{N} : d_i > 0 \,\}$.

Let $\mathcal{S}$ denote the set of subterns. The map n in eqn (3) naturally extends to a map z defined on $\mathcal{S}$ by

$$\mathrm{z} : \mathcal{S} \longrightarrow \mathbb{Z} : {}^{\mathsf{S}}c_1 \dots c_r \longmapsto \mathrm{z}c_1 \dots c_r = \sum_{i=1}^{r} c_i \mathrm{a}_i.$$

We also define a fraction-valued map

$$\mathrm{f} : \mathcal{S} \longrightarrow \mathbb{Q} : {}^{\mathsf{S}}c_1 \dots c_r \longmapsto \mathrm{f}c_1 \dots c_r = \sum_{i=1}^{r} \frac{c_i}{\mathrm{a}_i}.$$

Suppose that we operate on a subtern σ by the addition of the digits of a string of the form $0^i\bar{1}3\bar{1}$, called a *basic patch* or a *patch of order zero*, or of the form $0^i\bar{1}21^j2\bar{1}$, called a *patch of order* $j+1$, to the placewise corresponding digits of the subtern. Then a consequence of Lemma 1 is, as long as the result σ' of the operation is still a subtern, that the z-value is unaffected: $\mathrm{z}\sigma' = \mathrm{z}\sigma$. The recursive application of patches to a subtern is called *patching*. We will patch strictly according to the following rules:

1. A patch is applied only to a subtern whose first nonzero digit is positive.
2. The initial $\bar{1}$ of a patch must be added to a positive digit that is not preceded by any negative digit.
3. The final $\bar{1}$ of a patch must be added either (a) to a positive digit or (b) to the virtual zero beyond the end of the subtern, thus creating an extra digit $\bar{1}$.
4. The intermediate block of positive digits of a patch, namely, 3, 22, 212, or generally 21^k2, must be added only to a corresponding block of nonpositive digits that includes at least one negative digit.
5. The patching process stops as soon as either (a) all the digits are nonnegative (i.e. the subtern is a tern) or (b) the final digit of the subtern is $\bar{1}$ and all the preceding digits are nonnegative.

A subtern satisfying either condition of rule 5 is said to be *fully patched*. (Later we will relax rule 5b to allow the extension of subterns.) To illustrate the patching process, an example is shown below. At each stage, the digits to be patched are distinguished, and the patch to be applied is shown on the right.

$$\begin{array}{cccccccccccccccccl}
0 & \mathbf{1} & \mathbf{\bar{1}} & \mathbf{1} & \bar{1} & \bar{2} & \bar{1} & \bar{1} & 0 & \bar{1} & 1 & 0 & 0 & 0 & \bar{2} & & \quad & (0\bar{1}3\bar{1}) \\
0 & 0 & \mathbf{2} & \mathbf{0} & \mathbf{\bar{1}} & \mathbf{\bar{2}} & \mathbf{\bar{1}} & \mathbf{\bar{1}} & \mathbf{0} & \mathbf{\bar{1}} & \mathbf{1} & 0 & 0 & 0 & \bar{2} & & & (0^2\bar{1}21^52\bar{1}) \\
0 & 0 & 1 & \mathbf{2} & \mathbf{0} & \mathbf{\bar{1}} & \mathbf{0} & \mathbf{0} & \mathbf{1} & 1 & 0 & 0 & 0 & 0 & \bar{2} & & & (0^3\bar{1}21^22\bar{1}) \\
0 & 0 & 1 & 1 & 2 & 0 & 1 & 2 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\bar{2}} & \mathbf{0} & & (0^9\bar{1}21^32\bar{1}) \\
0 & 0 & 1 & 1 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & \bar{1} & & (\text{fully patched}).
\end{array}$$

Suppose that we are given an initial subtern whose first nonzero digit is positive. To see that the patching process always stops, observe that a patch of order n (i.e. of length $n+3$) raises the sum of digits by $n+1$ for $n \in \mathbb{N}$. If the final digit of the original subtern is positive, then patching does not extend its length r (the length may be reduced, since a created terminal 0 is ignored). So the patching must stop before the sum of digits exceeds the sum of r 2s (otherwise the rule of 2s would be violated). Now consider the case when the final digit is negative (as in the example above). Suppose that patching has gone as far as possible without extending the subtern. Then the subtern at this stage will comprise an initial block of nonnegative digits ending in a positive digit appended by a block of nonpositive digits ending in a negative digit: say ${}^{\mathsf{S}}c_1 \dots c_m c_{m+1} \dots c_{m+k}$, where $c_m > 0$, $c_{m+i} \leqslant 0$ ($i = 1, \dots, k-1$), and $c_{m+k} < 0$. If the subtern is not yet fully patched, apply a patch $0^{m-1}\bar{1}3\bar{1}$ (if $k = 1$) or $0^{m-1}\bar{1}21^{k-2}2\bar{1}$ (if $k \geqslant 2$) to extend the subtern by one digit (a $\bar{1}$) and result in a subtern that has no digit $\bar{2}$. If the subtern is still not fully patched, then continue the patching process as far as possible without extending the subtern. The result will be, as before, a subtern comprising a block of nonnegative digits ending in a positive digit appended by a block of nonpositive digits, ending in a $\bar{1}$, that now features no digit $\bar{2}$: say ${}^{\mathsf{S}}c_1 \dots c_l c_{l+1} \dots c_{l+j}$, where $c_l > 0$, $c_{l+i} \in \{0, \bar{1}\}$ ($i = 1, \dots, j-1$), and $c_{l+j} = \bar{1}$. This time around, a patch $0^{l-1}\bar{1}3\bar{1}$ (if $j = 1$) or $0^{l-1}\bar{1}21^{j-2}2\bar{1}$ (if $j \geqslant 2$) clears any remaining negative digits, except for the newly introduced terminal $\bar{1}$, and so the subtern is fully patched.

The reader will perhaps be glad to learn at this stage that only one further, comparatively simple, recursive operation on subterns is required: *shunting*. Initially we restrict shunting to terns. A shunt is performed when a positive digit is reduced by 1 while a higher-placed digit less than 2 is increased by 1. The result of a shunt must still be a tern—that is, the rule of 2s must not be violated—and shunting must not go beyond the end of a tern. A tern that cannot be shunted (further) is *fully shunted*. Examples of fully shunted terns are the empty string, 1, 2, 01, 02, 12, …, 00001, …, 0001112, …. A nonempty fully shunted tern must be of the form 0^m1 or 0^m1^n2, where $m, n \in \mathbb{N}$.

As remarked in Corollary 1, the inequalities $0 < \mathrm{a}_1 < \mathrm{a}_2 < \cdots$ hold. From this it is easy to see that shunting increases the n-value, or equivalently the z-value, of terns. In a parallel way, shunting decreases the f-value.

The shunting process may be simply extended to subterns of the form $\tau\bar{1}$, where τ is a nonempty tern, by restricting the shunting to the initial block of nonnegative digits, i.e. to τ. If τ is of length r, and τ' is the result of fully shunting τ ($\tau' = \tau$ if τ is already fully shunted), then $\mathrm{z}\tau \leqslant \mathrm{z}\tau' \leqslant \mathrm{z}1^{r-1}2 = \mathrm{z}0^r1 - 1$, the latter equality holding by Lemma 1. Therefore $\mathrm{z}\tau\bar{1} = \mathrm{z}\tau - \mathrm{z}0^r1$ is negative. It follows that shunting a subtern of the form $\tau\bar{1}$ (τ a tern) increases its z-value while maintaining the negativity of the z-value. Hence shunting such subterns reduces the absolute z-value: $|\mathrm{z}\tau'\bar{1}| < |\mathrm{z}\tau\bar{1}|$, where τ' derives from τ by shunting. In a similar way it is easily seen that the fractional value $\mathrm{f}\tau'\bar{1}$ of a shunted subtern $\tau'\bar{1}$, while remaining positive, is less than the fractional value $\mathrm{f}\tau\bar{1}$ of the subtern from which it derives.

After recording the following three useful lemmas, we will be ready to embark on the proof of Theorem 2.

Lemma 2. $\mathrm{a}_n^2 - \mathrm{a}_{n-1}\,\mathrm{a}_{n+1} = 1 \quad (n = 1, 2, \dots)$.

Proof. The result clearly holds for $n = 1$. Suppose it true for $n = k$. Then

$$\begin{aligned} 1 = \mathrm{a}_k^2 - \mathrm{a}_{k-1}\,\mathrm{a}_{k+1} &= \mathrm{a}_k^2 - (3\mathrm{a}_k - \mathrm{a}_{k+1})\mathrm{a}_{k+1} = \mathrm{a}_{k+1}^2 - \mathrm{a}_k(3\mathrm{a}_{k+1} - \mathrm{a}_k) \\ &= \mathrm{a}_{k+1}^2 - \mathrm{a}_k\,\mathrm{a}_{k+2}, \end{aligned}$$

establishing the result for $n = k + 1$ and, by induction, for all $n = 1, 2, \dots$. ■

Lemma 3. $\dfrac{1}{\mathrm{a}_{n-1}} - \dfrac{3}{\mathrm{a}_n} + \dfrac{1}{\mathrm{a}_{n+1}} = \dfrac{3}{\mathrm{a}_{n-1}\,\mathrm{a}_n\,\mathrm{a}_{n+1}} \quad (n = 2, 3, \dots)$.

Proof.

$$\begin{aligned} \frac{1}{\mathrm{a}_{n-1}} - \frac{3}{\mathrm{a}_n} + \frac{1}{\mathrm{a}_{n+1}} &= \frac{\mathrm{a}_n\,\mathrm{a}_{n+1} - 3\mathrm{a}_{n-1}\,\mathrm{a}_{n+1} + \mathrm{a}_{n-1}\,\mathrm{a}_n}{\mathrm{a}_{n-1}\,\mathrm{a}_n\,\mathrm{a}_{n+1}} \\ &= \frac{\mathrm{a}_n\,\mathrm{a}_{n+1} - 3(\mathrm{a}_n^2 - 1) + (3\mathrm{a}_n - \mathrm{a}_{n+1})\mathrm{a}_n}{\mathrm{a}_{n-1}\,\mathrm{a}_n\,\mathrm{a}_{n+1}} \quad \text{(by Lemma 2)} \\ &= \frac{3}{\mathrm{a}_{n-1}\,\mathrm{a}_n\,\mathrm{a}_{n+1}}. \end{aligned}$$

■

Lemma 4. $\mathrm{a}_m\,\mathrm{a}_{m+n-1} - \mathrm{a}_{m-1}\,\mathrm{a}_{m+n} = \mathrm{a}_n$ $(m = 1, 2, \ldots; n = 0, 1, \ldots)$.

Proof. The result is obvious for $n = 0$, and Lemma 2 is the result for $n = 1$. Suppose the result to hold for $n = 0, \ldots, k$ and all $m = 1, 2, \ldots,$ where $k \geqslant 1$. Taking $n = k$ and $m = l \geqslant 2$ gives

$$\mathrm{a}_l\,\mathrm{a}_{l+k-1} - \mathrm{a}_{l-1}\,\mathrm{a}_{l+k} = \mathrm{a}_k \qquad (l = 2, 3, \ldots). \tag{4}$$

Also, taking $n = k - 1$ and $m = l + 1$ gives

$$\mathrm{a}_{l+1}\,\mathrm{a}_{l+k-1} - \mathrm{a}_l\,\mathrm{a}_{l+k} = \mathrm{a}_{k-1} \qquad (l = 2, 3, \ldots). \tag{5}$$

Multiplying eqn (4) through by 3 and subtracting eqn (5) yields

$$(3\mathrm{a}_l - \mathrm{a}_{l+1})\mathrm{a}_{l+k-1} - (3\mathrm{a}_{l-1} - \mathrm{a}_l)\mathrm{a}_{l+k} = 3\mathrm{a}_k - \mathrm{a}_{k-1},$$

or

$$\mathrm{a}_{l-1}\,\mathrm{a}_{l+k-1} - \mathrm{a}_{l-2}\,\mathrm{a}_{l+k} = \mathrm{a}_{k+1} \qquad (l = 2, 3, \ldots).$$

That is, $\mathrm{a}_m\,\mathrm{a}_{m+k} - \mathrm{a}_{m-1}\,\mathrm{a}_{m+k+1} = \mathrm{a}_{k+1}$ $(m = 1, 2, \ldots)$, which is the result for $n = k + 1$. By induction, the general result follows. ■

Proof of Theorem 2. Let $g = |p - q|\,|\mathrm{f}_p - \mathrm{f}_q|$, where $p,q \in \mathbb{N}$ with $p \neq q$. Our task is to show that $g \geqslant 1$. Without loss of generality, we may assume $\mathrm{f}_p \geqslant \mathrm{f}_q$. Let $p = \mathrm{n}c_1 \ldots c_r$ and $q = \mathrm{n}d_1 \ldots d_s$, and let σ be the subtern ${}^{\mathsf{S}}e_1 \ldots e_t$, where $e_i = c_i - d_i$ for $i = 1, \ldots, t$, with $t = \max\{\, i : c_i - d_i \neq 0 \,\}$. Then $g = |\mathrm{z}e_1 \ldots e_t|\mathrm{f}e_1 \ldots e_t$. Let $\tilde\sigma = {}^{\mathsf{S}}\tilde e_1 \ldots \tilde e_{\tilde t}$ be the result of fully patching σ. Then either (A) $p > q$ and $\tilde e_i \geqslant 0$ for $i = 1, \ldots, \tilde t$, or (B) $p < q$, $\tilde e_i \geqslant 0$ for $i = 1, \ldots, \tilde t - 1$, and $\tilde e_{\tilde t} = \bar 1$.

In case A, because $\mathrm{z}\tilde\sigma = \mathrm{z}\sigma$ and $\mathrm{f}\tilde\sigma \leqslant \mathrm{f}\sigma$, we have

$$g \geqslant \sum_{i=1}^{\tilde t} \tilde e_i\,\mathrm{a}_i \sum_{i=1}^{\tilde t} \frac{\tilde e_i}{\mathrm{a}_i} \geqslant \sum_{i=1}^{\tilde t} \tilde e_i^2 \geqslant 1$$

since all $\tilde e_i$ are nonnegative and not all are 0.

In case B, we shunt $\tilde\sigma$ to a subtern of the form (B1) $0^{m-1}1\bar 1$ or (B2) $0^{m-1}1^n 2\bar 1$, where n and $m - 1$ are in $\mathbb{N}$.

In case B1:

$$\begin{aligned} g - 1 \geqslant |\mathrm{z}\tilde\sigma|\mathrm{f}\tilde\sigma - 1 &\geqslant |\mathrm{z}0^{m-1}1\bar 1|\mathrm{f}0^{m-1}1\bar 1 - 1 \\ &= (\mathrm{a}_{m+1} - \mathrm{a}_m)\left(\frac{1}{\mathrm{a}_m} - \frac{1}{\mathrm{a}_{m+1}}\right) - 1 = \frac{\mathrm{a}_m^2 - 3\mathrm{a}_m\,\mathrm{a}_{m+1} + \mathrm{a}_{m+1}^2}{\mathrm{a}_m\,\mathrm{a}_{m+1}}. \end{aligned}$$

The numerator of this last expression is

$$\mathrm{a}_m^2 - \mathrm{a}_{m+1}(3\mathrm{a}_m - \mathrm{a}_{m+1}) = \mathrm{a}_m^2 - \mathrm{a}_{m-1}\,\mathrm{a}_{m+1},$$

which is positive by Lemma 2. It follows that $g > 1$.

In case B2, we now relax rule 5b to continue patching indefinitely. Observe that applying the basic (order 0) patch $0^{m-1}\bar{1}3\bar{1}$ to $0^{m-1}2\bar{1}$ produces $0^{m-1}12\bar{1}$, while applying the multiple (order $k+1$) patch $0^{m-1}\bar{1}21^k2\bar{1}$ produces $0^{m-1}1^{k+2}2\bar{1}$ ($k \in \mathbb{N}$). Thus, by Lemma 1, all subterns $0^{m-1}1^j2\bar{1}$ ($j \in \mathbb{N}$) have the same z-value,

$$\mathrm{z}0^{m-1}1^j2\bar{1} = \mathrm{z}0^{m-1}2\bar{1} \qquad (j \in \mathbb{N}), \tag{6}$$

for a given m. At the same time, by Lemma 3, the same patching reduces the f-values according to the order, or multiplicity, of the patch. That is, $(\,\mathrm{f}0^{m-1}1^i2\bar{1} : i \in \mathbb{N}\,)$ is a strictly decreasing sequence of positive terms which therefore has a limit $\sum_{i=m}^{\infty} 1/\mathrm{a}_i$. It follows that

$$\mathrm{f}0^{m-1}1^n2\bar{1} > \sum_{i=m}^{\infty} \frac{1}{\mathrm{a}_i}. \tag{7}$$

Since patching preserves z-values and reduces f-values,

$$\begin{aligned}
g &\geqslant |\mathrm{z}0^{m-1}1^n2\bar{1}|\,\mathrm{f}0^{m-1}1^n2\bar{1} \\
&= |\mathrm{z}0^{m-1}2\bar{1}|\,\mathrm{f}0^{m-1}1^n2\bar{1} \quad \text{(by eqn (6))} \\
&> |\mathrm{z}0^{m-1}2\bar{1}| \sum_{i=m}^{\infty} \frac{1}{\mathrm{a}_i} \quad \text{(by eqn (7))} \\
&= (\mathrm{a}_{m+1} - 2\mathrm{a}_m) \sum_{i=m}^{\infty} \frac{1}{\mathrm{a}_i} \quad \text{(by Corollary 1)} \\
&= (\mathrm{a}_m - \mathrm{a}_{m-1}) \sum_{i=m}^{\infty} \frac{1}{\mathrm{a}_i} \\
&= 1 + \lim_{k\to\infty} \left[\sum_{i=m}^{m+k} \left(\frac{\mathrm{a}_m}{\mathrm{a}_{i+1}} - \frac{\mathrm{a}_{m-1}}{\mathrm{a}_i} \right) - \frac{\mathrm{a}_m}{\mathrm{a}_{m+k+1}} \right] \\
&= 1 + \sum_{i=m}^{\infty} \frac{\mathrm{a}_m\,\mathrm{a}_i - \mathrm{a}_{m-1}\,\mathrm{a}_{i+1}}{\mathrm{a}_i\,\mathrm{a}_{i+1}} \quad \text{(by Corollary 1)} \\
&= 1 + \sum_{i=m}^{\infty} \frac{\mathrm{a}_{i-m+1}}{\mathrm{a}_i\,\mathrm{a}_{i+1}} \quad \text{(by Lemma 4)} \\
&> 1.
\end{aligned}$$

■

4. Further remarks and questions

Let $\beta_0 = 0$, and define β_n ($n = 1, 2, \ldots$) to be the least positive real number x that satisfies $(n-k)|x - \beta_k| \geqslant 1$ for $k = 0, \ldots, n-1$. We conjecture that $(\,\beta_n : n \in \mathbb{N}\,)$ and $(\,\mathrm{f}_n : n \in \mathbb{N}\,)$ are equal and of maximal separation. However, without a proof, the question in the title remains. Is there a sequence that clusters more slowly than $(\mathrm{f}_n : n \in \mathbb{N})$, namely is more separated in the sense defined in eqn (2)?[1] More fundamentally, is there a "measure of reluctance to cluster" that is more elementary, natural, or canonical than that of separation presented here? And what would be an example of a "most slowly clustering" sequence under such a definition?

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References

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Address: John Bentin: 22 Lynn Close, Oxford OX3 0JH, UK.

E-mail: johnbentin@yahoo.co.uk

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[1] The author has shown that the f-sequence is of maximal separation, in a later paper to appear in the present journal.