

## SOME PROBLEMS CONCERNING ALGEBRAS OF HOLOMORPHIC FUNCTIONS

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In memory of Paweł Domański

**Abstract:** Let  $X$  be a complex Banach space with open unit ball  $B_X$ . We describe some recent work and a number of open problems related to the maximal ideal spaces of the Fréchet algebra of holomorphic functions of bounded type on  $X$  and the Banach algebra of bounded holomorphic functions on  $B_X$ .

**Keywords:** maximal ideal space,  $\mathcal{H}^\infty(B_X)$ ,  $\mathcal{A}_u(B_X)$ , cluster value theorem.

### 1. Introduction

This survey paper deals with problems that, by and large, involve two algebras of holomorphic functions on a complex Banach space  $X$  with open unit ball  $B_X$ . We first examine  $\mathcal{H}_b(X)$ , the algebra of holomorphic functions  $f : X \rightarrow \mathbb{C}$  such that  $\sup_{\|x\| \leq n} |f(x)| < \infty$  for all  $n \in \mathbb{N}$ . This is a *Fréchet* algebra when endowed with the usual topology generated by these sup-norms. The second algebra that we consider is  $\mathcal{H}^\infty(B_X)$ , the Banach algebra of bounded holomorphic functions  $f : B_X \rightarrow \mathbb{C}$  with the sup-norm on  $B_X$ .

The problems we describe here have been “percolating” for a while, in some cases for several decades. We must acknowledge, with great thanks, many conversations over the years with colleagues and friends about this general topic, out of which the present manuscript has emerged. Although we will now do so, it is risky to try to name the people who have helped me with this, since it is almost certain that someone will be inadvertently omitted: Daniel Carando, Brian Cole, Verónica Dimant, Javier Falcó, Pablo Galindo, Ted Gamelin, Domingo García, Alexander Izzo, Silvia Lassalle, M. Lilian Lourenço, O. Paques, Manuel Maestre, Luiza Amalia de Moraes, and Ignacio Zalduendo.

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Research supported in part by MINECO and FEDER Project MTM2017-83262-C2-1-P.

**2010 Mathematics Subject Classification:** primary: 46J15; secondary: 32A38, 42B30

After reviewing notation and some basic concepts from Banach algebras in this section, we will devote §2 to some background results and open problems related to  $\mathcal{H}_b(X)$ . In the third section, we will focus on  $\mathcal{H}^\infty(B_X)$ , where we will find problems even when  $X = (\mathbb{C}^2, \|\cdot\|_2)$ .

For the time being,  $\mathcal{A}$  denotes either of the two algebras  $\mathcal{H}_b(X)$  or  $\mathcal{H}^\infty(B_X)$ . As is customary, let  $\mathcal{M}(\mathcal{A})$  be the *spectrum*, or *maximal ideal space*, of  $\mathcal{A}$ . That is,  $\mathcal{M}(\mathcal{A})$  is the set

$$\{\varphi : \mathcal{A} \rightarrow \mathbb{C} \mid \varphi \text{ is a continuous homomorphism}\}.$$

Recall that a function  $\varphi : \mathcal{A} \rightarrow \mathbb{C}$  acting on an algebra  $\mathcal{A}$  is a *homomorphism* means that  $\varphi$  satisfies the two *algebraic* conditions:  $\varphi$  is a linear form which is also multiplicative. We exclude  $\varphi \equiv 0$  as such a homomorphism and consequently  $\varphi(1) = 1$ . (Note here that both of our algebras are unital with identity element being the constant function 1.) By a standard argument (see, e.g., [13], every homomorphism on a unital commutative Banach algebra is automatically continuous. On the other hand, if  $\mathcal{A}$  is a non-normed Fréchet algebra, then it is unknown if every complex homomorphism on  $\mathcal{A}$  is automatically continuous. This is known as the *Michael problem* (see, e.g., [8]). In order to make any progress at all in the case of  $\mathcal{H}_b(X)$ , we have added the word “continuous” to the definition of  $\mathcal{M}(\mathcal{A})$ .

In the case of Banach algebras, it is well-known ([13]) that every  $\varphi \in \mathcal{M}(\mathcal{A})$  has norm 1; that is  $\mathcal{M}(\mathcal{A})$  is a subset of the unit sphere of  $X$ ,  $S_{\mathcal{A}^*}$ , in this case. By the Alaoglu-Bourbaki theorem,  $B_{\mathcal{A}^*}$  is weak-star compact, and we conclude that  $\mathcal{M}(\mathcal{A})$  is compact for the induced topology. In our case, with  $\mathcal{A} = \mathcal{H}^\infty(B_X)$ , what this means is that any net  $(\varphi_\alpha)$  in  $\mathcal{M}(\mathcal{A})$  has a convergent subnet to some  $\varphi \in \mathcal{M}(\mathcal{A})$ ; i.e.  $\varphi_\alpha(f) \rightarrow \varphi(f)$  for each  $f \in \mathcal{A}$ . It is natural to ask for examples of elements of  $\mathcal{M}(\mathcal{H}^\infty(B_X))$ . Apart from the evaluation homomorphisms  $\delta_{x_0}$  for some fixed  $x_0 \in B_X$ , there are *no obvious* homomorphisms. However, as pointed out by Hoffman [9] in fact there is an *impressive* number of (non-obvious) homomorphisms. We will content ourselves with one more basic notion, which is a triviality in the case  $X = \mathbb{C}$ . For this, note the natural inclusion:  $X^* \subset \mathcal{H}^\infty(B_X)$ . Now, for any  $\varphi \in \mathcal{M}(\mathcal{H}^\infty(B_X))$ , define  $\Pi(\varphi) := \varphi|_{X^*}$ . Thus,  $\Pi : \mathcal{M}(\mathcal{H}^\infty(B_X)) \rightarrow \overline{B_{X^{**}}}$ . It is an easy exercise to show that  $\Pi$  is continuous when  $\mathcal{M}(\mathcal{H}^\infty(B_X))$  has its induced topology and  $\overline{B_{X^{**}}}$  has the weak-star topology. Thus the image of  $\Pi$  is compact. A computation confirms that  $\Pi \circ \delta_{x_0} = x_0$  for all  $x_0 \in B_X$ . Furthermore, by Goldstein’s theorem,  $B_X$  is a weak-star dense subset of  $\overline{B_{X^{**}}}$ , and we conclude that  $\Pi$  is a surjective mapping. It follows that for each  $x_0^{**} \in \overline{B_{X^{**}}}$ , the fiber  $\Pi^{-1}(x_0^{**}) \neq \emptyset$ . We will return to this in Section 3.

In the case of the Fréchet, not Banach, algebra  $\mathcal{A} = \mathcal{H}_b(X)$ , as already mentioned the Michael problem is essentially whether the word “continuous” can be removed from the definition of the spectrum of  $\mathcal{H}_b(X)$ ,

$$\mathcal{M}(\mathcal{H}_b(X)) = \{\varphi : \mathcal{H}_b(X) \rightarrow \mathbb{C} \mid \varphi \text{ is a continuous homomorphism}\}$$

(see, e.g., [7]). The set  $\mathcal{M}(\mathcal{H}_b(X))$  is not compact when endowed with the usual topology, even in the case when  $X = \mathbb{C}$ . (For, in this case,  $\mathcal{H}_b(\mathbb{C}) = \mathcal{H}(\mathbb{C})$  and  $\mathcal{M}(\mathcal{H}(\mathbb{C})) \sim \mathbb{C}$ .) We will examine this case in the next section.

**2. The maximal ideal space  $\mathcal{M}(\mathcal{H}_b(X))$**

Fix a complex Banach space  $X$  and an element  $\varphi \in \mathcal{M}(\mathcal{H}_b(X))$ . Since  $\varphi$  is continuous, from the definition of the topology of  $\mathcal{H}_b(X)$ , we see that there is  $R > 0$  such that

$$|\varphi(f)| \leq \sup_{x \in X, \|x\| \leq R} |f(x)| \tag{*}$$

Moreover, for each  $R$ , the set  $\{\varphi \in \mathcal{M}(\mathcal{H}_b(X)) \mid (*) \text{ holds}\}$  is compact.

Also, it is not difficult to see that there is a canonical extension mapping  $f \in \mathcal{H}_b(X) \rightsquigarrow \tilde{f} \in \mathcal{H}_b(X^{**})$  ([2]), so that of course  $\tilde{f}|_X = f$ . Moreover, an examination of the construction shows that  $f \rightarrow \tilde{f}$  is a homomorphism between the two Fréchet algebras (i.e. it is continuous, linear, and multiplicative). Consequently, for any fixed point  $z_0^{**} \in X^{**}$ , we have a continuous homomorphism  $\tilde{\delta}_{z_0^{**}} : \mathcal{H}_b(x) \rightarrow \mathbb{C}$ ,  $\tilde{\delta}_{z_0^{**}}(f) = \tilde{f}(z_0^{**})$ . Of course, when  $X$  is reflexive (and in particular when  $X$  is finite dimensional), this does not give anything new.

In fact, Davie and Gamelin provide an ingenious proof of the analogous result in the context of  $\mathcal{H}^\infty(B_X)$ . Specifically, for a fixed  $z_0^{**} \in B_X^{**}$ , there is an extension homomorphism  $\tilde{\delta}_{z_0^{**}} \in \mathcal{M}(\mathcal{H}^\infty(B_X))$ , given by  $\tilde{\delta}_{z_0^{**}}(f) = \tilde{f}(z_0^{**})$ . We will make use of this in the next section.

Let us turn to some examples before presenting some problems.

**Example 1.** We begin with what is an interesting special case,  $X = c_0$ . By [2], each point  $z_0^{**} \in \ell_\infty$  yields a homomorphism  $\tilde{\delta}_{z_0^{**}} \in \mathcal{M}(\mathcal{H}_b(c_0))$ . Thus, in the above sense, we have  $\ell_\infty \subseteq \mathcal{M}(\mathcal{H}_b(c_0))$ . What makes  $X = c_0$  special is the fact that the algebra generated by  $\{x^* \mid x^* \in c_0^*\}$  is dense in  $\mathcal{H}_b(c_0)$  ([14]). Thus, if  $\varphi$  is an arbitrary element of  $\mathcal{M}(\mathcal{H}_b(c_0))$ , using the fact that  $c_0^* \subset \mathcal{H}_b(c_0)$  along with the continuity of  $\varphi$ , we see first that the restriction of  $\varphi$  to  $c_0^*$  is equal to  $\tilde{\delta}_{z_0^{**}}$  for some  $z_0^{**} \in \ell_\infty$ , and thus  $\varphi = \tilde{\delta}_{z_0^{**}}$ .

The denseness of  $\text{alg}\{x^* \mid x^* \in c_0^*\}$  in  $\mathcal{H}_b(c_0)$  is unusual, and in fact  $\mathcal{M}(\mathcal{H}_b(X))$  is generally much larger than  $\{\tilde{\delta}_{z_0^{**}} \mid z_0^{**} \in X^{**}\}$ . Before discussing this further, let us recall the even more special case of  $X = T^*$ , the original Tsirelson space [16]. Here it turns out that  $\mathcal{H}_b(T^*)$  is nothing more than  $\{\delta_c \mid c \in T^*\}$ .

**Example 2.** Turning to what seems to be a more typical situation with  $X = \ell_2$ , let us take the sequence of homomorphisms  $\{\delta_{e_n} \mid n \in \mathbb{N}\}$ . Clearly, each  $\delta_{e_n} \in \{\varphi \in \mathcal{M}(\mathcal{H}_b(\ell_2)) \mid (*) \text{ holds with } R = 1\}$ . Hence, by compactness, there is a subnet  $\{\delta_{e_n}\}_n$  that converges to some  $\varphi$  in the same set. However,  $\varphi \neq \delta_x$  for any  $x \in \ell_2$ . To see this, observe that for each  $k \in \mathbb{N}$ , the function  $f_k(x) = 1 - \sum_{j=k}^\infty x_j^2 \in \mathcal{H}_b(\ell_2)$  satisfies  $f_k(e_n) = 0$  for all sufficiently large  $n \in \mathbb{N}$ . Therefore,  $\varphi(f_k) = 0$  for all  $k$ . Were it true that  $\varphi = \delta_x$  for some fixed  $x$ , then we would have that  $f_k(x) = 1 - \sum_{j=k}^\infty x_j^2 = 0$  for all  $k$ , i.e.  $\sum_{j=k}^\infty x_j^2 = 1$  for all  $k$ , which is impossible. Therefore  $\mathcal{M}(\mathcal{H}_b(\ell_2))$  contains homomorphisms which are not merely evaluations at points of  $\ell_2$ . This leads to some natural questions.

**Problem 1.** Describe the homomorphisms in  $\mathcal{M}(\mathcal{H}_b(\ell_2))$ .

**Problem 2.** Is the set of “obvious” homomorphisms  $\{\delta_x \mid x \in \ell_2\}$  dense in  $\mathcal{M}(\mathcal{H}_b(\ell_2))$ , endowed with the standard weak-star topology? (This is tantamount to asking for a version of the Corona Theorem.)

We now return to the case of a general complex Banach space  $X$ . Recalling the above discussion, there is a natural mapping which we now denote  $\iota_2, \iota_2 : X^{**} \simeq \{\tilde{\delta}_z \mid z^{**} \in X^{**}\} \rightarrow \mathcal{M}(\mathcal{H}_b(X))$ , via  $f \in \mathcal{H}_b(X) \rightsquigarrow \tilde{\delta}_z^{**}(f) = \tilde{f}(z^{**})$ . Let’s repeat this procedure: Fix  $p \in X^{iv}$ , where  $X^{iv}$  denotes the fourth dual of  $X$ . Define  $\tilde{\delta}_p : \mathcal{H}_b(X) \rightarrow \mathbb{C}$  by

$$\tilde{\delta}_p(f) = \tilde{f}(p).$$

(Here  $\tilde{f} \in \mathcal{H}_b(X^{iv})$ .) We thus get a mapping  $\iota_4$  taking  $X^{iv} \simeq \{\tilde{\delta}_p \mid p \in X^{iv}\}$  into  $\mathcal{M}(\mathcal{H}_b(X))$ . Of course, we can continue this procedure, getting a mapping  $\iota_{2n}$  for every  $n \in \mathbb{N}$ .

We have already seen that for  $X = \ell_2$ , these natural mappings are not surjections. In fact, it can happen that they are not necessarily injections either. If  $X$  is non-reflexive, then  $\iota_2$  is easily seen to be one-to-one. But what about  $\iota_{2n}$  for  $n \geq 2$ ? One example occurs by considering  $X = c_0$  for, in this case, although  $\iota_2 : \ell_\infty \rightarrow \mathcal{M}(\mathcal{H}_b(c_0))$  is an injection (in fact, it is a bijection), the map  $\iota_4$  from the fourth dual of  $c_0$  is not injective. A more interesting situation occurs with  $X = \ell_1$ . In this case, although  $\iota_2 : \ell_1^{**} \rightarrow \mathcal{M}(\mathcal{H}_b(\ell_1))$  is injective, it is not surjective. Moreover, there are points in  $p \in \ell_1^{iv} \setminus \ell_1^{**}$  such that  $\tilde{\delta}_p$  is a “new” homomorphism on  $\mathcal{H}_b(\ell_1)$  (i.e.  $\tilde{\delta}_p \neq \tilde{\delta}_{z^{**}}$  for any  $z^{**} \in \ell_1^{**}$ .) However, *this does not hold for every*  $p \in \ell_1^{iv}$ , and so  $\iota_4$  is not injective. The reason for this is simple, thanks to an observation by Joe Diestel: The cardinality of  $\ell_1^{iv}$  is strictly greater than the cardinality of  $\mathcal{M}(\mathcal{H}_b(\ell_1))$ .

**Problem 3.** Which points  $p \in \ell_1^{iv}$  really give “new” homomorphisms? That is, for which  $p$  is it the case that  $\tilde{\delta}_p \neq \tilde{\delta}_{z^{**}}$  for any  $z^{**} \in \ell_1^{**}$ ?

**Problem 4.** Are there points  $q \in \ell_1^{vi}$  such that  $\iota_6(q)$  is a “new” homomorphism?

(The reader is invited to search for papers in which the sixth, or even the fourth, dual of a Banach space is discussed.)

The above problems are closely related to what is known as *Arens regularity* (see, e.g., [1]), concerning automatic weak compactness of operators from a Banach space  $X$  to its dual  $X^*$ . In fact, what is involved here is what is known as *symmetric regularity*. The point is that not only is  $\ell_1$  not regular, it is also not symmetrically regular; that is, there is a symmetric continuous linear operator  $\ell_1 \rightarrow \ell_1^*$  which is not weakly compact (see, e.g., [3]).

### 3. The maximal ideal space of $H^\infty$ functions

We study here the maximal ideal space  $\mathcal{M}(\mathcal{H}^\infty(B_X))$ , where  $X$  is an arbitrary complex Banach space. Recall that since  $\mathcal{H}^\infty(B_X)$  is a Banach algebra, continuity

of the homomorphisms is automatic. Also, as mentioned in the Introduction, we have a natural mapping  $\Pi : \mathcal{M}(\mathcal{H}^\infty(B_X)) \rightarrow \overline{B_{X^{**}}}$ , where  $\Pi(\varphi)$  is the restriction of an element  $\varphi \in \mathcal{M}(\mathcal{H}^\infty(B_X))$  to  $X^* \subset \mathcal{H}^\infty(B_X)$ . We have already indicated that  $\Pi$  is continuous when the domain and range have the weak-star topology, that  $\Pi \circ \delta = id|_{B_X}$ , and that in fact  $\Pi$  is onto  $\overline{B_{X^{**}}}$  by Goldstein's theorem.

We will also need the notion of *fiber* over a point of  $B_{X^{**}}$ .

**Definition.** Let  $z^{**} \in B_{X^{**}}$ . The *fiber* over  $z^{**}$  is just  $\Pi^{-1}(z^{**})$ , which we have just observed to be non-empty for every  $z^{**}$ .

Finally, we will need one more important definition.

**Definition.** The *cluster set* of a function  $f \in \mathcal{H}^\infty(B_X)$  at the point  $z^{**} \in \overline{B_{X^{**}}}$  is the set of limits of values of  $f$  along nets in  $B_X$  that converge weak-star to  $z^{**}$ .

Let's restrict first to  $X = \mathbb{C}$ , so that we're merely dealing with the classical  $\mathcal{H}^\infty(\mathbb{D})$ . In this case, we recall the famous *Corona Theorem* of Lennart Carleson.

**Theorem (L. Carleson, [6]).** *The collection  $\delta(\mathbb{D})$  of point evaluations at points of the open unit disc is dense in the space  $\mathcal{M}(\mathcal{H}^\infty(\mathbb{D}))$ .*

Carleson's theorem appeared one year after a somewhat overlooked paper by I. J. Schark. In it, among other things, I. J. Schark proved the following *Cluster Valued Theorem*.

**Theorem (I. J. Schark, [15]).** *Fix  $f \in \mathcal{H}^\infty(\mathbb{D})$  and  $c \in \overline{\mathbb{D}}$ . Then the following sets are equal:*

$$\{w \in \mathbb{C} \mid \exists(x_\alpha) \subset \mathbb{D}, x_\alpha \rightarrow c \text{ and } f(x_\alpha) \rightarrow w\};$$

$$\{\varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(\mathbb{D})) \mid \Pi(\varphi) = c\}.$$

Several remarks comparing the two results above are in order.

**Remark 1.** For later use and comparison, we note that it is trivial that Schark's result is vacuous if the chosen point  $c \in \overline{\mathbb{D}}$  in fact lies in the open disc  $\mathbb{D}$ .

**Remark 2.** It is a straightforward exercise to verify that Carleson's theorem implies the result of I. J. Schark. The converse is false (and is not as straightforward).

**Remark 3.** It is not difficult to show one inclusion in J. Schark's theorem. Specifically, let  $w_0 \in \{w \in \mathbb{C} \mid \exists(x_\alpha) \subset \mathbb{D}, x_\alpha \rightarrow c \text{ and } f(x_\alpha) \rightarrow w\}$  so that there is a net  $x_\alpha \in \mathbb{C}, x_\alpha \rightarrow c$  and  $f(x_\alpha) \rightarrow w_0$ . Consider the ideal  $\mathcal{I} = \{g \in \mathcal{H}^\infty(\mathbb{D}) \mid g(x_\alpha) \rightarrow 0\}$ . Clearly, both the function  $g(z) = z - c$  and the function  $g(z) = f(z) - w_0$  belong to  $\mathcal{I}$ . Call  $\mathcal{M}$  any maximal ideal in  $\mathcal{H}^\infty(\mathbb{D})$  such that  $\mathcal{M} \supset \mathcal{I}$ . Since  $\mathcal{M} = \varphi^{-1}(0)$  for some homomorphism  $\varphi$ , it follows that  $\varphi(g) = 0$  for all  $g \in \mathcal{I}$ . From this it follows that  $\varphi(f) = w_0$  as required.

**Remark 4.** It is unknown if the analogous extension of Carleson’s theorem for higher dimensions, e.g. the ball of  $\mathbb{C}^2$  with the Euclidean or max norm, holds. Put briefly, for  $\dim X = 1$ , there are no known counterexamples to Carleson’s theorem, while for  $\dim X \geq 2$ , there are no known positive results. In light of this, it is evidently foolhardy to try to prove a version of Carleson’s theorem for infinite dimensional Banach spaces  $X$ . On the other hand, there is no known situation in which I. J. Scharck’s theorem is false.

From Remark 4 above, it is clearly sensible that rather than trying to extend the Corona Theorem, our focus should be on trying to extend the weaker form of the Corona theorem, as put forward in the case of  $\mathcal{H}^\infty(\mathbb{D})$  by I. J. Scharck. In short, the basic question we have is the following:

**Problem 5.** Let  $X$  be an infinite dimensional complex Banach space. Is there a version of the cluster value theorem that is valid in this context? Specifically, for a fixed  $f \in \mathcal{H}^\infty(B_X)$  and a fixed point  $z^{**} \in B_{X^{**}}$ , are the following two sets equal?

$$\{w \in \mathbb{C} \mid \exists \text{ a net } (x_\alpha)_\alpha \subset B_X, x_\alpha \rightarrow z^{**} \text{ weak - star and } f(x_\alpha) \rightarrow w\};$$

$$\{\varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(B_X)), \Pi(\varphi) = z^{**}\}.$$

When  $\dim X = \infty$ , an interesting feature of  $\mathcal{M}(\mathcal{H}^\infty(B_X))$  must be noted. Namely, the fiber over any, *even an interior*, point of  $B_{X^{**}}$  is rich. In particular, there is a homeomorphic copy of  $\beta\mathbb{N} \setminus \mathbb{N} \subset \Pi^{-1}(0)$  whenever  $X$  is infinite dimensional [5, §11]. (Compare with Remark 1 above.)

We are thus led to the following (possibly) easier problem:

**Problem 6.** Let  $X$  be an infinite dimensional complex Banach space, and let  $f \in \mathcal{H}^\infty(B_X)$  be a fixed function. Are the following two sets equal?

$$\{w \in \mathbb{C} \mid \exists \text{ a net } (x_\alpha)_\alpha \subset B_X, x_\alpha \rightarrow 0 \text{ weakly and } f(x_\alpha) \rightarrow w\};$$

$$\{\varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(B_X)), \Pi(\varphi) = 0\}.$$

For  $X = c_0$ , the answer to Problem 5 (and a fortiori Problem 6) is yes. Namely, we have the following result.

**Theorem ([4]).** Fix  $f \in \mathcal{H}^\infty(B_{c_0})$  and  $z^{**} \in \overline{B_{\ell_\infty}}$ . Then the two sets

$$\{w \in \mathbb{C} \mid \exists \text{ a net } (x_\alpha)_\alpha \in B_{c_0}, x_\alpha \rightarrow z^{**} \text{ weak - star and } f(x_\alpha) \rightarrow w\}$$

and

$$\{\varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(B_{\ell_\infty})), \Pi(\varphi) = z^{**}\}$$

are equal.

It is natural to think that the same result should be true for the apparently easier case of  $B_{\ell_2}$ , since here we don't have to concern ourselves with biduals. In fact, we don't know the answer to the (at least, theoretically simpler) Problem 6, for the following reason. The proof of the above cluster value theorem for  $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$  relies on the fact that for any  $n \in \mathbb{N}$ , any (continuous)  $n$ -homogeneous polynomial  $P : c_0 \rightarrow \mathbb{C}$  can be uniformly approximated on  $B_{c_0}$  by elements in the algebra generated by  $\ell_1 = c_0^*$ . We note that the 2-homogeneous polynomial  $P : \ell_2 \rightarrow \mathbb{C}$ ,  $P(x) = \sum_{j=1}^\infty x_j^2$ , shows that the same fact does not hold if  $c_0$  is replaced by  $\ell_2$ .

Using this property of polynomials on  $c_0$ , one can prove a key result needed in the proof of the above theorem. First, let us set some notation: For  $g \in \mathcal{H}^\infty(B_{c_0})$  and  $n \in \mathbb{N}$ , define  $g_n : B_{c_0} \rightarrow \mathbb{C}$  by  $g_n(x_1, \dots, x_n, x_{n+1}, x_{n+2}, \dots) = g(0, \dots, 0, x_{n+1}, x_{n+2}, \dots)$ . Clearly,  $g_n$  is also a bounded holomorphic function on  $B_{c_0}$ .

**Lemma.** Fix  $\varphi \in \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ . Suppose that  $\Pi(\varphi) = 0$  (as an element of  $\ell_\infty$ ). Then for any  $g \in \mathcal{H}^\infty(B_{c_0})$  and any  $n \in \mathbb{N}$ ,  $\varphi(g) = \varphi(g_n)$ .

This Lemma, which is crucial for the proof that Problem 5 holds for  $X = c_0$ , is false for  $\ell_2$ .

**Example 3.** Let  $(r_n)$  and  $(\epsilon_n) \subset \mathbb{R}$  be such that  $r_n \rightarrow 1^-$  and  $\epsilon_n \rightarrow 0^+$  very quickly. (How quickly will be explained below.) Consider the points  $p_n = \epsilon_n e_1 + r_n e_n$ , where  $r_n^2 + \epsilon_n^2 \rightarrow 1^-$ . Each such  $p_n$  provides a point evaluation homomorphism  $\delta_{p_n} \in \mathcal{M}(\mathcal{H}^\infty(B_{\ell_2}))$ . By compactness, the set  $\{\delta_{p_n}\}$  has a cluster point  $\varphi \in \mathcal{M}(\mathcal{H}^\infty(B_{\ell_2}))$ . For any  $k \in \mathbb{N}, k \geq 2$ , since  $\delta_{p_n}(e_k^*)$ , the  $k^{\text{th}}$  coordinate of  $p_n$ , is 0 for most  $n$ , it follows that  $\varphi(e_k^*) = 0$ . Furthermore, as  $n$  gets larger,  $\delta_{p_n}(e_1^*) = \epsilon_n \rightarrow 0$ . Consequently,  $\Pi(\varphi) = 0$ . Now, let  $g : B_{\ell_2} \rightarrow \mathbb{C}$  be defined by

$$g(x_1, x_2, \dots) = \frac{x_1}{[1 - \sum_{j=2}^\infty x_j^2]^{\frac{1}{2}}}$$

It is straightforward that  $g \in \mathcal{H}^\infty(B_{\ell_2})$ . Note that for all  $n$ ,  $\delta_{p_n}(g) = \frac{\epsilon_n}{[1 - r_n^2]^{\frac{1}{2}}}$  which we can arrange to be as close to 1 as we wish. Thus  $\varphi(g) = 1$ . On the other hand,  $g_1 \equiv 0$  so of course  $\varphi(g_1) = 0$ . Thus, the above Lemma is false for  $\mathcal{M}(\mathcal{H}^\infty(B_{\ell_2}))$ . Summarizing, we don't know if there is a cluster value theorem over  $0 \in B_{\ell_2}$ .

We conclude with several problems concerning individual fibers in  $\mathcal{M}(\mathcal{H}^\infty(B_X))$ . One basic general question is the following.

**Problem 7.** Fix a complex Banach space  $X$  and two points  $z^{**}$  and  $w^{**}$  in  $\overline{B_{X^{**}}}$ . What is the relation between the two fibers  $\Pi^{-1}(z^{**})$  and  $\Pi^{-1}(w^{**})$ ?

Note that the meaning of the word "relation" in Problem 7 must be specified. By "relation," does one mean homeomorphic, or is there a stronger, analytic sense in which the two fibers can be "equal?"

Of course, the question is trivial if  $X$  is finite dimensional with a “user-friendly” norm and both points have norm  $< 1$ . For, in this case, the fiber over each point is a singleton. However, in almost every other case, the question seems to be non-trivial and interesting. For instance, if  $X = \ell_2$  and  $\|z\| = \|w\| = 1$ , then  $\Pi^{-1}(z) \sim \Pi^{-1}(w)$ . The same result holds if  $\|z\|$  and  $\|w\|$  are both  $< 1$ . But what happens if  $\|z\| < 1 = \|w\|$ ? What happens if  $X$  is not Hilbert space?

Take  $X = c_0$ . If  $z, w \in B_{c_0}$ , then  $\Pi^{-1}(z) \sim \Pi^{-1}(w)$ . However, for  $\|z^{**}\| = \|w^{**}\| = 1$ ,  $z^{**}, w^{**} \in \overline{B_{\ell_\infty}}$ , the situation is murky. For instance, it seems unlikely that  $\Pi^{-1}(\frac{n}{n+1})$  should be in any way similar to  $\Pi^{-1}(1, 1, \dots, 1, \dots)$ , but we have no proof.

Even in the case of finite dimensional  $X$ , many questions remain. For instance, in the special cases  $\mathcal{H}^\infty(\mathbb{D})$  and  $\mathcal{H}^\infty(\mathbb{D}^2)$  what is known is that  $\Pi^{-1}(1) \sim \Pi^{-1}(a, b)$  if one of  $|a|, |b| = 1$  and the other is  $< 1$  [11]. Also,  $\Pi^{-1}(1)$  and  $\Pi^{-1}(1, 1)$  are not homeomorphic, but the argument really uses dimension 1 [10]. In fact, even if  $\dim X < \infty$  and even if  $\|z\|, \|w\| < 1$ , the problem of whether  $\Pi^{-1}(z)$  and  $\Pi^{-1}(w)$  are (somehow) the “same” is open in general. (This is related to what is known as Gleason’s problem [12].) Even more, it is apparently open whether, with  $\dim X < \infty$  and  $\|z\| < 1$ ,  $\Pi^{-1}(z) = \delta_z$ .

**Acknowledgement.** The author is, and in fact the reader should be, very grateful to the referee of this short work who managed to find and correct a number of confusing errors.

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**Received:** 26 January 2018; **revised:** 2 March 2018

