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ON BOHR RADII OF FINITE DIMENSIONAL COMPLEX BANACH SPACES

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To the memory of Paweł Domański

Abstract: We study the Bohr radius of the unit ball of a complex n-dimensional Banach space with an 1-unconditional basis in terms of its lattice convexity/concavity constants. As an application we give asymptotic estimates of the Bohr radius of the unit ball of the n-th section of Lorentz and Marcinkiewicz sequence spaces.

Keywords: Bohr radius, power series, polynomials, Banach sequence spaces.

1. Introduction

In [5] Harald Bohr proved that for every holomorphic function f on the unit disc

$$\sup_{|z|\leqslant\frac{1}{3}} \sum_{n=0}^{\infty} \Big| \frac{f^{(n)}(0)}{n!} z^n \Big| \leqslant \|f\|_\infty,$$

and the radius $r = \frac{1}{3}$ here is optimal. This result is usually referred to as Bohr's power series theorem, and it was discovered in the context of the study of Bohr's absolute convergence problem for Dirichlet series.

Let $X^n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space for which the unit vector basis $\{e_i\}_{i=1}^n$ is an 1-unconditional basis, i.e. given $x, y \in \mathbb{C}^n$ such that $|x_k| \leqslant |y_k|$ for every k, then $\|x\| \leqslant \|y\|$. Then, the Bohr radius $K(B_{X^n})$ of the open unit ball B_{X^n} in X^n is given by the supremum over all $0 < r < 1$ such that for all $f \in H_\infty(B_{X^n})$ (all bounded holomorphic functions on B_{X^n})

$$\sup_{z\in rB_{X^n}} \sum_{\alpha\in\mathbb{N}_0^n} \Big| \frac{\partial_\alpha f(0)}{\alpha!} z^\alpha \Big| \leqslant \|f\|_\infty.$$

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With this notion Bohr power series theorem reads $K(\mathbb{D}) = \frac{1}{3}$. For no other dimensions the precise value of the Bohr radius seems to be known, only estimates are available.

The following result, proved in [2], can be seen as a sort of highlight in this direction:

$$\lim_n \frac{K(B_{\ell_\infty^n})}{\sqrt{\frac{\log n}{n}}} = 1. \tag{1}$$

Several authors contributed with partial results and ideas to reach this up to now best result for $K(B_{\ell_\infty^n})$. Boas and Khavinson proved in [4] that the limit superior of the quotient in (1) is $\leqslant 1$ (using probabilistic methods through the Kahane-Salem-Zygmund inequality), and that the limit inferior of the quotient in (1) is $\geqslant \frac{1}{\sqrt{2}}$ was discovered in [9] as a consequence of the hypercontractivity of the Bohnenblust-Hille inequality.

The following simple observation will be useful (see [12, Lemma 2.5]).

Remark 1. If $X^n = (\mathbb{C}^n, \|\cdot\|_X)$ and $Y^n = (\mathbb{C}^n, \|\cdot\|_Y)$ are two Banach spaces such that $\{e_i\}_{i=1}^n$ is an 1-unconditional basis in both of them. Then

$$\psi(n)^{-1} K(B_{Y^n}) \leqslant K(B_{X^n}) \leqslant \psi(n) K(B_{Y^n}),$$

where $\psi(n) = \|\mathrm{id}\colon X^n \to Y^n\| \|\mathrm{id}\colon Y^n \to X^n\|$.

In order to describe some further essential results (which we also will apply in our study) we notice that throughout the paper we use the following notation: given two non-negative functions f and g defined on the same set A, we write $f \prec g$ if there is a constant $C > 0$ such that $f(t) \leqslant Cg(t)$ for all $t \in A$, while $f \asymp g$ means that $f \prec g$ and $g \prec f$ hold. Because of Bohr's power series Theorem the case $n = 1$ is perfectly known, and so in what follows, we are only interested in the case $n \geqslant 2$.

In the ℓ_r-case we have

$$K(B_{\ell_r^n}) \asymp \left(\frac{\log n}{n}\right)^{1-\frac{1}{\min\{r,2\}}}; \tag{2}$$

here the upper estimate was proved in [3] and the lower one in [10]. For the upper bound note that we have more precisely,

$$\limsup_n \frac{K(B_{\ell_r^n})}{\left(\frac{\log n}{n}\right)^{1-\frac{1}{\min\{r,2\}}}} \leqslant 1, \tag{3}$$

whereas in the case of the lower bound it is unknown whether

$$1 \leqslant \liminf_n \frac{K(B_{\ell_r^n})}{\left(\frac{\log n}{n}\right)^{1-\frac{1}{\min\{r,2\}}}}.$$

Our main interest in this contribution is to give several extensions of the asymptotics from (1), (2), and (3) replacing ℓ_r^n by some important classes of n-dimensional Banach space $X^n = (\mathbb{C}^n, \|\cdot\|)$, in particular n-dimensional Lorentz and Marcinkiewicz spaces.

2. Preliminaries

In what follows we will need some more notation. As usual a (complex) Banach sequence space is, by definition, a Banach lattice which is modelled on $\mathbb{N}$ and contains a sequence x with $\operatorname{supp} x = \mathbb{N}$. If X is a Banach sequence space, then for each $n \in \mathbb{N}$ by X^n we denote $\mathbb{C}^n$ equipped with the norm

$$\|z\|_{X^n} = \Big\| \sum_{i=1}^{n} z_i e_i \Big\|_X, \qquad z = (z_1, \dots, z_n) \in \mathbb{C}^n,$$

where $\{e_i\}$ denotes the standard unit vector basis in c_0. The fundamental function $\varphi_X \colon \mathbb{N} \to [0, \infty)$ is defined by

$$\varphi_X(n) = \Big\| \sum_{i=1}^{n} e_i \Big\|_X, \qquad n \in \mathbb{N}.$$

Recall that a Banach sequence space X is said to be symmetric whenever $x \in X$ we have that $x^* \in X$ and $\|x\| = \|x^*\|$. Here, as usual, the decreasing rearrangement $x^* = (x_k^*)_{k=1}^{\infty}$ of a complex sequence $x = (x_k)_{k=1}^{\infty}$ is given by

$$x_k^* := \inf \Big\{ \sup_{j \in \mathbb{N} \setminus J} |x_j|;\ J \subset \mathbb{N},\ \operatorname{card}(J) < k \Big\}, \qquad k \in \mathbb{N}.$$

Moreover, it is well-known that for every symmetric Banach sequence space X

$$\Big\| \sum_{k=1}^{n} e_k \Big\|_{X^n} \Big\| \sum_{k=1}^{n} e_k \Big\|_{(X')^n} = n,$$

where X' is the Köthe dual space of X (see, e.g., [18]). Then

$$\frac{n}{\varphi_X(n)} = \Big\| \sum_{k=1}^{n} e_k \Big\|_{(X')^n} = \|\mathrm{id} \colon X^n \to \ell_1^n\|. \tag{4}$$

We will use two more concepts that are crucial in the theory of Banach lattices (see [18]). A Banach function lattice X on a measure space (Ω, Σ, μ) is said to be p-convex, $1 \leqslant p \leqslant \infty$, respectively q-concave, $1 \leqslant q \leqslant \infty$, if there is a constant $C > 0$ such that for every choice of finitely many $x_1, \dots, x_N \in X$

$$\Big\| \Big(\sum_{k=1}^{N} |x_k|^p \Big)^{1/p} \Big\|_X \leqslant C \Big(\sum_{k=1}^{N} \|x_k\|_X^p \Big)^{1/p},$$

respectively,

$$\Big(\sum_{k=1}^{N} \|x_k\|_X^q \Big)^{1/q} \leqslant C \Big\| \Big(\sum_{k=1}^{N} |x_k|^q \Big)^{1/q} \Big\|_X$$

(with the usual modification whenever $p = \infty$). We define the p-convexity constant $M^{(p)}(X)$ (resp., q-concavity constant $M_{(q)}(X)$) to be the least constant C satisfying the above inequality. In case the Banach sequence space X is not p-convex (respectively not q-concave) we write $M^{(p)}(X) = \infty$ (respectively $M_{(q)}(X) = \infty$).

We notice that (see [18, Proposition 1.d.5]), if $r < p < s$ and the Banach sequence space X is p-convex (respectively p-concave), X is r-convex (respectively s-concave) with $M^{(r)}(X) \leqslant M^{(p)}(X)$ (respectively $M_{(s)}(X) \leqslant M_{(p)}(X)$).

Clearly every Banach lattice X is 1-convex and ∞-concave (with constants 1). The sequence space ℓ_p is p-convex as well as p-concave with $M^{(p)}(\ell_p) = M_{(p)}(\ell_p) = 1$ for all $1 \leqslant p \leqslant \infty$.

Given a Banach lattice X over a measure space (Ω, Σ, μ), we recall that its p-convexification, where $0 < p < \infty$, is the quasi-Banach lattice (respectively Banach lattice if $p \geqslant 1$)

$$X^{(p)} = \{x \in L^0(\mu) : \ |x|^p \in X\}$$

over (Ω, Σ, μ), equipped with the quasi-norm (respectively norm)

$$\|x\|_{X^{(p)}} := \||x|^p\|_X^{1/p}$$

(see [18, pp. 53-54] for the details). Here, as usual, $L^0(\mu)$ denotes the space of all (equivalence classes of μ-a.e. equal) measurable functions on Ω.

We note that $X^{(p)}$ is p-convex with $M^{(p)}(X^{(p)}) = 1$ for all $1 < p < \infty$.

3. Main results

In order to motivate our results we recall a few more abstract estimates on Bohr radii. Assume that $X^n = (\mathbb{C}^n, \|\cdot\|)$ is a Banach space for which the unit vector basis $\{e_i\}_{i=1}^n$ is 1-unconditional. We start with two lower estimates, and mention first a result from [4]

$$K(B_{\ell_\infty^n}) \leqslant K(B_{X^n}). \tag{5}$$

It was shown in [8] that

$$\frac{1}{4e}\frac{1}{d(X^n, \ell_1^n)} \leqslant K(B_{X^n}), \tag{6}$$

where $d(E, F)$ as usual denotes the Banach–Mazur distance between two finite dimensional Banach spaces E and F with $\dim(E) = \dim(F)$.

In general, $K(B_{X^n})$ and $d(X^n, \ell_1^n)$ are not asymptotically inverse to each other: Recall that $d(\ell_\infty^n, \ell_1^n) \asymp \sqrt{n}$ (see, e.g., [19, Proposition 37.6]), but from (1) it follows that $K(B_{\ell_\infty^n}) \asymp \sqrt{\log n/n}$. On the other hand the lower bound from (6) seems quite accurate in the sense that in all known cases there exists $r > 0$ which depends on X such that

$$K(B_{X^n}) \prec \frac{(\log n)^r}{d(X^n, \ell_1^n)}.$$

Typically, upper estimates of Bohr radii are of probabilistic nature – motivated by results from [11] (the case $r = 2$) Bayart proved in [1] that for every $1 \leqslant r \leqslant 2$ there is a constant $C = C(r) > 0$ such that for every Banach space X^n with a normalized 1-unconditional basis we have

$$K(B_{X^n}) \leqslant C(\log n)^{1-\frac{1}{r}} \frac{\|\mathrm{id}\colon X^n \to \ell_r^n\|}{\|\mathrm{id}\colon X^n \to \ell_1^n\|}; \tag{7}$$

if X^n is a section of a symmetric Banach sequence space X, then normalizing its basis and using Remark 1 shows that the preceding estimate holds with a constant $C = C(r, X)$.

Combining (1), (5) and [11, Proposition 4.6] we observe that for every symmetric 2-convex symmetric Banach sequence space X

$$K(B_{X^n}) \asymp \sqrt{\frac{\log n}{n}}. \tag{8}$$

Here the following remark seems interesting.

Remark 2. Assume that X is a 2-convex maximal (i.e., $X'' = X$ isometrically, where X'' denotes the Köthe bidual) or separable symmetric Banach sequence spaces such that

$$K(B_{X^n}) \prec \frac{1}{M_{(2)}(X^n)} \sqrt{\frac{\log n}{n}}$$

holds. Then $X = \ell_2$ within equivalence of norms.

Indeed, the assumption combined with (5) and (1) yields that there exists a constant $C > 0$ such that $M_{(2)}(X^n) \leqslant C$. Hence $\|\mathrm{id}\colon X^n \to \ell_2^n\| \leqslant C$, which gives that $X \hookrightarrow \ell_2$. To conclude it is enough to observe that 2-convexity of X implies that $\ell_2 \hookrightarrow X$.

Since for symmetric 2-convex symmetric Banach sequence space X we have $d(X^n, \ell_1^n) \asymp \sqrt{n}$ (see [19]), we can reformulate the asymptotic from (8) as follows

$$K(B_{X^n}) \asymp \frac{\sqrt{\log n}}{d(X^n, \ell_1^n)}. \tag{9}$$

For the scale of Banach sequence spaces ℓ_p, $1 \leqslant p \leqslant \infty$ the asymptotic from (2) as well as (3) distinguish the cases $1 \leqslant p \leqslant 2$ and $2 < p \leqslant \infty$. We extend these results to symmetric Banach sequence spaces – now distinguishing between the 2-convex case (Theorem 1) and r-concave case, $1 \leqslant r \leqslant 2$ (Theorem 2).

Theorem 1. *Let X be a symmetric Banach sequence space. Then*

$$\limsup_n \frac{K(B_{X^n})}{M^{(2)}(X^n)\sqrt{\frac{\log n}{n}}} \leqslant 1. \tag{10}$$

Moreover, if X is 2-convex, then

$$1 \leqslant \liminf_n \frac{K(B_{X^n})}{\sqrt{\frac{\log n}{n}}} \leqslant \limsup_n \frac{K(B_{X^n})}{\sqrt{\frac{\log n}{n}}} \leqslant M^{(2)}(X), \tag{11}$$

and consequently under the assumption $M^{(2)}(X) = 1$ we have

$$\lim_{n\to\infty} \frac{K(B_{X^n})}{\sqrt{\frac{\log n}{n}}} = 1.$$

Our second result is a sort of 'concave analog' of Theorem 1, and it obviously covers the case $1 \leqslant r \leqslant 2$ in (2).

Theorem 2. *Let X be a symmetric Banach sequence space, and $1 \leqslant r \leqslant 2$. Then*

$$\frac{1}{M_{(r)}(X^n)} \frac{(\log n)^{1-1/r}}{n^{1-1/r}} \prec K(B_{X^n}) \prec M_{(r)}(X^n) \frac{(\log n)^{1-1/r}}{n\varphi_X^{-1}(n)}. \tag{12}$$

In particular, if X is r-concave, then

$$\frac{(\log n)^{1-1/r}}{n^{1-1/r}} \prec K(B_{X^n}) \prec \frac{(\log n)^{1-1/r}}{n\varphi_X^{-1}(n)}.$$

All proofs will be given in the following two Sections 3.1 and 3.2. The proof of Theorem 2 needs the following independently interesting result which is a proper extension of (5) since $M_{(\infty)}(\ell_\infty^n) = 1$.

Proposition 3. *Let $X^n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space such that $\{e_i\}_{i=1}^n$ is an 1-unconditional basis, and $1 \leqslant r \leqslant \infty$. Then*

$$\frac{K(B_{\ell_r^n})}{M_{(r)}(X^n)} \leqslant K(B_{X^n}).$$

3.1. Proof of Theorem 1

We first prove that

$$\limsup_n \frac{K(B_{X^n})}{\sqrt{\frac{\log n}{n}}} \leqslant 1, \tag{13}$$

whenever we assume that $M^{(2)}(X^n) = 1$ for all n.

Denote by $\mathcal{P}_m(X^n)$ the linear space of all m-homogeneous polynomials $f(z) = \sum_{|\alpha|=m} c_\alpha(f) z^\alpha$, $z \in \mathbb{C}^n$, and endow it with the supremum norm given by $\|f\|_\infty = \sup_{z\in B_{X^n}} |f(z)|$. Obviously, all monomials $z^\alpha, \alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$ form a basis of this Banach space, and we denote its unconditional basis constant by

$\chi_{\mathrm{mon}}(\mathcal{P}_m(X^n))$, i.e., the best constant $C>0$ such that for all $f\in\mathcal{P}_m(X^n)$ and $\epsilon_\alpha\in\mathbb{D}$, $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=m$ we have

$$\Big\|\sum_{|\alpha|=m}\epsilon_\alpha c_\alpha(f)z^\alpha\Big\|_\infty\leqslant C\Big\|\sum_{|\alpha|=m}c_\alpha(f)z^\alpha\Big\|_\infty.$$

Moreover, denote by $K_m(B_{X^n})$ the so-called m-homogeneous Bohr radius given by the supremum over all $0<r<1$ such that for every m-homogeneous polynomial $f(z)=\sum_{|\alpha|=m}c_\alpha z^\alpha$, $z\in\mathbb{C}^n$

$$\sup_{z\in rB_{X^n}}\sum_{|\alpha|=m}|c_\alpha z^\alpha|\leqslant\|f\|_\infty.$$

Then a simple calculation (see [11, Lemma 2.1]) shows that

$$K_m(B_{X^n})=\frac{1}{\big(\chi_{\mathrm{mon}}(\mathcal{P}(^mX^n))\big)^{1/m}}.$$

The crucial tool we use is the following probabilistic estimate on unconditional basis constants of spaces of polynomials (see [1, Theorem 5.1] and also [11, Lemma 4.1]): For some constant C and all m,n

$$\frac{C}{m(\log n)^{\frac{3}{2}}(m!)^{\frac{1}{2}}}\left(\frac{\sup\limits_{\|z\|_{X^n}\leqslant 1}\sum\limits_{k=1}^n|z_k|}{\sup\limits_{\|z\|_{X^n}\leqslant 1}\Big(\sum\limits_{k=1}^n|z_k|^2\Big)^{\frac{1}{2}}}\right)^{m-1}\leqslant\chi_{\mathrm{mon}}(\mathcal{P}(^mX^n)).$$

Obviously, for all m

$$K(B_{X^n})\leqslant K_m(B_{X^n}),$$

and hence

$$K(B_{X^n})\leqslant\left(\frac{\sup\limits_{\|z\|_{X^n}\leqslant 1}\Big(\sum\limits_{k=1}^n|z_k|^2\Big)^{\frac{1}{2}}}{\sup\limits_{\|z\|_{X^n}\leqslant 1}\sum\limits_{k=1}^n|z_k|}\right)^{\frac{m-1}{m}}\sqrt[m]{\frac{m(\log n)^{\frac{3}{2}}(m!)^{\frac{1}{2}}}{C}}.$$

Since $M^{(2)}(X^n)=1$ and X is symmetric, we know from [13, Proposition 3.5] and (4) that

$$\sup_{\|z\|_{X^n}\leqslant 1}\Big(\sum_{k=1}^n|z_k|^2\Big)^{\frac{1}{2}}=\frac{\sup\limits_{\|z\|_{X^n}\leqslant 1}\sum\limits_{k=1}^n|z_k|}{\sqrt{n}}.$$

But then we deduce from Stirling's formula that for all m

$$\begin{aligned} K(B_{X^n}) &\leqslant n^{-\frac{1}{2}\frac{m-1}{m}} \cdot \sqrt[m]{\frac{m(\log n)^{\frac{3}{2}}\Big(2\sqrt{2\pi m}\big(\frac{m}{e}\big)^m\Big)^{\frac{1}{2}}}{C}} \\ &= \Big(\frac{m}{n}\Big)^{\frac{1}{2}} \cdot n^{\frac{1}{2m}}\frac{1}{\sqrt{e}} \cdot \sqrt[m]{\frac{m(\log n)^{\frac{3}{2}}\big(2\sqrt{2\pi m}\big)^{\frac{1}{2}}}{C}}. \end{aligned}$$

Hence, if we put $m = [\log n]$, then the second and third factor converge to 1 whenever $n \to \infty$, which is exactly what we need to finish the proof of (13).

In order to get rid of the assumption that $M^{(2)}(X^n) = 1$, $n \in \mathbb{N}$ in (13), we recall a well-known renorming construction from the theory of Banach function lattices X over the measure space (Ω, Σ, μ) (see again [18, pp. 53-54] for the details in the setting of abstract Banach lattices):

Let X be a p-convex Banach function lattice over the measure space (Ω, Σ, μ). Observe that for all $u \in X^{(1/p)}$ and $u_1,\dots,u_n \in X^{(1/p)}$ with $|u| \leqslant \sum_{k=1}^n |u_k|$, we have

$$\begin{aligned} \|u\|_{X^{1/p}} &\leqslant \Big\| \sum_{k=1}^n \big(|u_k|^{\frac{1}{p}}\big)^p \Big\|_{X^{(1/p)}} = \Big\| \Big(\sum_{k=1}^n \big(|u_k|^{\frac{1}{p}}\big)^p \Big)^{\frac{1}{p}} \Big\|_X^p \\ &\leqslant \big(M^{(p)}(X)\big)^p \sum_{k=1}^n \big\| |u_k|^{\frac{1}{p}} \big\|_X^p = \big(M^{(p)}(X)\big)^p \sum_{k=1}^n \|u_k\|_{X^{(1/p)}}. \end{aligned}$$

This shows that the functional given by

$$\|u\|_* = \inf\Big\{ \sum_{k=1}^n \|u_k\|_{X^{(1/p)}};\, n \in \mathbb{N}, u_1, \dots, u_n \in X^{(1/p)}, |u| \leqslant \sum_{k=1}^n |u_k| \Big\},$$

for all $u \in X^{(1/p)}$, defines an equivalent lattice norm on a quasi-Banach lattice $X^{(1/p)}$ with

$$\big(M^{(p)}(X)\big)^{-p} \|\cdot\|_{X^{(1/p)}} \leqslant \|\cdot\|_* \leqslant \|\cdot\|_{X^{(1/p)}}. \tag{14}$$

In consequence, we conclude that

$$\widetilde{X} := \big(X^{(1/p)}, \|\cdot\|_*\big)^{(p)}$$

is a Banach function lattice on (Ω, Σ, μ) with p-convexity constant equals 1, and $\widetilde{X} = X$ up to equivalence of norms (by (14)),

$$\big(M^{(p)}(X)\big)^{-1} \|x\|_X \leqslant \|x\|_{\widetilde{X}} \leqslant \|x\|_X, \qquad x \in X. \tag{15}$$

We are going to apply this construction to sections X^n of Banach sequence spaces X. It can be seen easily that $\widetilde{X^n}$ in this case is symmetric in the sense that $\|x\|_{\widetilde{X^n}} = \|x^*\|_{\widetilde{X^n}}$ for all $x \in \mathbb{C}^n$.

Finally, we are prepared to finish the proof of (10): With the notation we just established, we deduce from Remark 1 and the norm equivalence from (15) that

$$K(B_{X_n}) \leqslant M^{(2)}(X_n) K(B_{\widetilde{X_n}}).$$

So if we divide by $M^{(2)}(X_n)\sqrt{n/\log n}$, then the conclusion follows from (13).

For the proof of (11) notice that the upper estimate is immediate from (10). The lower estimate follows from (5) combined with (1).

3.2. Proof of Theorem 2

The proof of Proposition 3 is based on the following factorization lemma which has its origin in [16]; for a variant of it see also [17, Lemma 2.5].

Lemma 4. *Let $X^n = (\mathbb{C}^n, \|\cdot\|)$ be an n-dimensional Banach space such that $\{e_i\}_{i=1}^n$ forms an 1-unconditional basis. Then for each $u \in B_{X^n}$ there are $\xi, w \in \mathbb{C}^n$ with*

$$u = \xi w, \ \|w\|_{\ell_r^n} \leqslant M_{(r)}(X^n) \ \text{and} \ \|D_\xi : \ell_r^n \to X^n\| \leqslant 1.$$

Proof of Lemma 4. Let $x = (x(j))_{j=1}^n \in B_{X^n}$. Clearly, the multiplication operator $D_x : \ell_\infty^n \to X^n$ has norm 1 and we have for every $u_1, \ldots, u_N \in \ell_\infty^n$

$$\begin{aligned} \Big\| \Big(\sum_{k=1}^N |D_x u_k|^r \Big)^{\frac{1}{r}} \Big\|_{X^n} &= \Big\| \Big(|x(j)| \Big(\sum_{k=1}^N |u_k(j)|^r \Big)^{1/r} \Big)_j \Big\|_{X^n} \\ &\leqslant \|x\|_{X^n} \Big\| \Big(\Big(\sum_{k=1}^N |u_k(j)|^r \Big)^{1/r} \Big)_j \Big\|_{\ell_\infty^n} \\ &\leqslant \Big\| \Big(\sum_{k=1}^N |u_k|^r \Big)^{1/r} \Big\|_{\ell_\infty^n}. \end{aligned}$$

Then by Corollary 2 from [7] (note that the calculation above shows $M^{(r)}(D_x) \leqslant 1$), there is a factorization $D_x = D_\xi \circ R$ where $D_\xi \colon \ell_n^r \to X^n$ is a multiplication operator and $R \colon \ell_\infty^n \to \ell_n^r$ is some operator such that $\|R\| \, \|D_\xi\| \leqslant M^{(r)}(D_x) M_{(r)}(X^n) \leqslant M_{(r)}(X^n)$. Without loss of generality we may assume that $\|R\| \leqslant M_{(r)}(X^n)$ and $\|D_\xi\| \leqslant 1$. Let now $w = R\mathbf{1} \in \ell_r^n$, where $\mathbf{1}$ denotes the sequence constant to 1. Clearly, we obtain a factorization $x = \xi w$ with the desired properties. ■

We proceed with the

Proof of Proposition 3. We want to show that

$$\frac{K(B_{\ell_r^n})}{M_{(r)}(X^n)} \leqslant K(B_{X^n}).$$

We take $x \in X^n$ with $\|x\| < \frac{K(B_{\ell_r^n})}{M_{(r)}(X^n)}$ and prove that for each $f \in H_\infty(B_{X^n})$

$$\sum_\alpha |c_\alpha(f)x^\alpha| \leqslant \|f\|_{X^n}.$$

By Lemma 4, there is a factorization $B_{X^n} \ni \frac{M_{(r)}(X^n)}{K(B_{\ell_r^n})} x = \xi w$ with $\|w\|_{\ell_r^n} \leqslant M_{(r)}(X^n)$ and $\|D_\xi : \ell_r^n \to X^n\| \leqslant 1$. Setting $v := \frac{K(B_{\ell_r^n})}{M_{(r)}(X^n)} w$ we obtain a factorization $x = \xi v$ with $\|v\|_{\ell_r^n} \leqslant K(B_{\ell_r^n})$. Thus, by definition of $K(B_{\ell_r^n})$,

$$\begin{aligned}\sum_\alpha \left|c_\alpha(f)\, x^\alpha\right| &= \sum_\alpha \left|c_\alpha(f)\,(\xi v)^\alpha\right| \\ &= \sum_\alpha \left|c_\alpha(f \circ D_\xi)\, v^\alpha\right| \leqslant \|f \circ D_\xi\|_{B_{\ell_r^n}} \leqslant \|f\|_{B_{X^n}},\end{aligned}$$

which completes the proof. ∎

It remains to give the

Proof of Theorem 2. The lower estimate in (12) is a consequence of Proposition 3 and (2). For the upper estimate we again refer to estimate (7). We have that $\|\mathrm{id}\colon X^n \to \ell_r^n\| \leqslant M_{(r)}(X)$. Then the conclusion follows from (4). ∎

4. Concavity estimates and applications

In general it is a nontrivial problem to find good upper and lower estimates of r-concavity constants $M_{(r)}(X)$ of n-dimensional Banach lattices. We provide some general cases which reduce the problem to more easy functional expressions.

Analysis of the proof of Theorem 1.d.7 in [18] gives that for a given $1 \leqslant r < \infty$ there exists a positive constant $C = C(r) > 0$ such that for any finite dimensional Banach lattices X and Y with $\dim(X) = \dim(Y)$, we have

$$M^{(r)}(Y) \leqslant C\, d(X,Y) M^{(r)}(X), \qquad \text{for } 1 < r \leqslant 2,$$

and

$$M_{(r)}(Y) \leqslant C\, d(X,Y) M_{(r)}(X), \qquad \text{for } 2 \leqslant r < \infty.$$

For our applications we need estimates of this type, which are, however, not so subtle: If $X = (\mathbb{C}^n, \|\cdot\|_X)$ and $Y = (\mathbb{C}^n, \|\cdot\|_Y)$, then a straightforward calculation shows that for every $1 \leqslant p \leqslant \infty$,

$$M^{(p)}(X) \leqslant \|\mathrm{id} : X \to Y\| \|\mathrm{id} : Y \to X\| M^{(p)}(Y),$$

and analogously

$$M_{(p)}(X) \leqslant \|\mathrm{id} : X \to Y\| \|\mathrm{id} : Y \to X\| M_{(p)}(Y). \tag{16}$$

In view of this inequality, we note that if $X = (\mathbb{C}^n, \|\cdot\|_X)$ and $Y = (\mathbb{C}^n, \|\cdot\|_Y)$ are symmetric spaces with fundamental functions φ_X and φ_Y, then

$$\|\mathrm{id}\colon X \to Y\|\|\mathrm{id}\colon Y \to X\| \leqslant 2^4(1+\log n)^2 \mathcal{D}_1(X,Y),$$

where

$$\mathcal{D}_1(X,Y) := \max\left\{\varphi_X(k)\varphi_X^{-1}(m)\varphi_Y(k)\varphi_Y^{-1}(m);\ 1 \leqslant k,m \leqslant n\right\};$$

this follows from the proof of the main theorem in Gluskin's paper [14]. Combining the above estimates, we deduce that for any n-dimensional symmetric Banach space X and all $2 \leqslant r < \infty$, there exists $C = C(r)$ such that

$$\begin{aligned} M_{(r)}(X) &\leqslant C\,\|\mathrm{id}\colon X \to \ell_r^n\|\|\mathrm{id}\colon \ell_r^n \to X\| \\ &\leqslant C(1+\log n)^2 \mathcal{D}_1(X, \ell_r^n) \\ &\leqslant C(1+\log n)^2 \max_{1\leqslant k,m\leqslant n} \frac{\varphi_X(k)k^{1/r}}{\varphi_X(m)m^{1/r}}. \end{aligned}$$

The following result is now a consequence of Proposition 3 and (1).

Corollary 5. *Let $X^n = (\mathbb{C}^n, \|\cdot\|)$ be a symmetric Banach space and $1 \leqslant r \leqslant \infty$. Then*

$$\frac{1}{\sqrt{n}(1+\log n)^{\frac{3}{2}}} \min_{1\leqslant k,m\leqslant n} \frac{\varphi_{X^n}(m)\, m^{1/r}}{\varphi_{X^n}(k)\, k^{1/r}} \prec K(B_{X^n}).$$

Finally, we show applications to Lorentz and Marcinkiewicz symmetric sequence spaces. Let $w = (w_k)_{k=1}^\infty$ be a positive and non-increasing sequence.

We recall that, given $1 \leqslant p < \infty$, the Lorentz space $d(w,p)$ (associated with w) is defined to be the symmetric Banach sequence space of all sequences $x = (x_k)_{k=1}^\infty$ such that

$$\|x\|_{d(w,p)} := \Big(\sum_{k=1}^{\infty} (x_k^*)^p w_k\Big)^{1/p} < \infty.$$

In what follows, for every $1 \leqslant r < \infty$ and each $k \in \mathbb{N}$, we shall write

$$W_k := w_1 + \ldots + w_k \quad \text{and} \quad \rho_{r,k} = \Big(\frac{1}{k}(w_1^r + \ldots + w_k^r)\Big)^{1/r}.$$

It follows from [15, Proposition 2, Theorem 3] that in the case $1 < r \leqslant p < \infty$, we have

$$M_{(r)}(d^n(w,p)) = \frac{n^{1/r} w_1^{1/p}}{W_n^{1/p}}, \tag{17}$$

and in the case $r > p$,

$$M_{(r)}(d^n(w,p)) = \sup_{1\leqslant k\leqslant n} \left[\frac{\rho_{u,k}}{\rho_{1,k}}\right]^{1/p}, \tag{18}$$

where $u = r/(r-p)$.

We proceed with two corollaries on the asymptotic decay of Bohr radii in finite dimensional Lorentz spaces. Both statements of the following corollary are immediate consequences of (17), (18), Proposition 3 and (1).

Corollary 6. *For $1 < r \leqslant p < \infty$ we have*

$$\frac{W_n^{1/p}}{n^{1/r}}\Big(\frac{\log n}{n}\Big)^{1-\frac{1}{\min\{r,2\}}} \prec K(B_{d^n(w,p)}),$$

and for $r > p$ with $u = r/(r-p)$

$$\sup_{1\leqslant k\leqslant n}\Big[\frac{\rho_{1,k}}{\rho_{u,k}}\Big]^{1/p}\Big(\frac{\log n}{n}\Big)^{1-\frac{1}{\min\{r,2\}}} \prec K(B_{d^n(w,p)}).$$

Corollary 7. *Let $1 < r \leqslant 2$ and let $r \leqslant p < \infty$. Then we have*

$$\frac{W_n^{1/p}}{n^{1/r}}\Big(\frac{\log n}{n}\Big)^{1-\frac{1}{r}} \prec K(B_{d^n(w,p)}) \prec \Big(\frac{\log n}{n}\Big)^{1-\frac{1}{r}}.$$

Proof. The first estimate is included in the preceding corollary. For the upper estimate we combine the obvious estimate $\|x\|_r \leqslant n^{1/r-1/p}\|x\|_p$ with the inequality $\frac{W_n^{1/p}}{n^{1/p}}\|x\|_p \leqslant \|x\|_{d^n(w,p)}$ (see [15, Lemma 1]). This yields

$$\|x\|_r \leqslant \frac{n^{1/r}}{W_n^{1/p}}\|x\|_{d^n(w,p)}.$$

Since in this inequality we have equality for $x = \sum_{j=1}^n e_j$, it follows that for $1 \leqslant r \leqslant p < \infty$

$$\|\mathrm{id}\colon d^n(w,p) \to \ell_r^n\| = \frac{n^{1/r}}{W_n^{1/p}}.$$

To conclude it is enough to apply (7). ∎

Let us look at analog estimates for Bohr radii of finite dimensional Lorentz spaces. For $1 < r < \infty$ and $1 \leqslant s < \infty$ the (sequence) Lorentz space $\ell_{r,s}$ consists of all complex sequences $(x_k)_{k=1}^\infty$ such that

$$\|x\|_{r,s} := \Big(\sum_{k=1}^\infty (k^{1/r}x_k^*)^s\frac{1}{k}\Big)^{1/s} < \infty.$$

We notice that $\ell_{r,s}$ is a symmetric Banach space equipped with the norm $\|\cdot\|_{r,s}$ when $s \leqslant r$. If $s > r$, then $\|\cdot\|_{r,s}$ is only a quasi-norm, however, in this case $\|x\|_{r,s}^* := \|x^{**}\|_{r,s}$, where

$$x^{**} = \Big(\frac{1}{k}\sum_{j=1}^k x_j^*\Big)_{k=1}^\infty,$$

defines a symmetric norm on $\ell_{r,s}$.

The Marcinkiewicz space $\ell_{r,\infty}$ is defined to be the symmetric sequence space equipped with the norm

$$\|x\|_{r,\infty} = \sup_{k \geqslant 1} \frac{\sum_{j=1}^k x_j^*}{k^{1-1/r}}.$$

We will also use the well-known fact that the spaces $\ell_{p,q}$ are ordered lexicographically:

$$\begin{aligned} \ell_{p_1,q_1} &\hookrightarrow \ell_{p_2,q_2}, && \text{for } p_1 < p_2 \\ \ell_{p_1,q_1} &\hookrightarrow \ell_{p_2,q_2}, && \text{for } p_1 = p_2 \text{ and } q_1 < q_2. \end{aligned}$$

We need two technical lemmas.

Lemma 8. *If $1 < r \leqslant s < \infty$, then there exists a positive constant $C = C(r,s)$ such that for all n*

(i) $\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_r^n\| \leqslant C(1+\log n)^{\frac{1}{r}-\frac{1}{s}}$.
(ii) $M_{(r)}(\ell_{r,s}^n) \leqslant C(1+\log n)^{\frac{1}{r}-\frac{1}{s}}$.

Proof. (i) Let μ to be the measure defined on the power set of $[n] := \{1,\dots,n\}$ by $\mu(\{j\}) = 1/j$ for $j \in [n]$. Since for $1 \leqslant r \leqslant s < \infty$,

$$\begin{aligned} \left(\frac{1}{\mu([n])}\sum_{j=1}^n x_j^{*r}\right)^{1/r} &= \left(\frac{1}{\mu([n])}\sum_{j=1}^n \left(j^{1/r}x_j^*\right)^r \frac{1}{j}\right)^{1/r} \\ &\leqslant \left(\frac{1}{\mu([n])}\sum_{j=1}^n \left(j^{1/r}x_j^*\right)^s \frac{1}{j}\right)^{1/s}, \end{aligned}$$

and $\mu([n]) \leqslant 1 + \log n$, the required estimate follows.

(ii) Combining $\|\,\mathrm{id}\colon \ell_r^n \to \ell_{r,s}^n\| = 1$ with (i) and (16) yields

$$\begin{aligned} M_{(r)}(\ell_{r,s}^n) &\leqslant \|\mathrm{id}\colon \ell_{r,s}^n \to \ell_r^n\|\,\|\mathrm{id}\colon \ell_r^n \to \ell_{r,s}^n\| \\ &\leqslant C(1+\log n)^{\frac{1}{r}-\frac{1}{s}}. \end{aligned}$$ ■

The second lemma gives precise asymptotic estimates for the concavity constants of finite dimensional Lorentz space ((ii) will not be needed later – we only state it for the sake of completeness).

Lemma 9.

(i) *If $1 \leqslant s < r < \infty$, then for each $n \in \mathbb{N}$,*

$$M_{(r)}(\ell_{r,s}^n) \asymp (1+\log n)^{\frac{1}{s}-\frac{1}{r}},$$

where the constants of equivalence only depend on r, s.

(ii) *If $1 \leqslant r < s < \infty$, then for every $q \geqslant s$,*

$$\sup_{n \geqslant 1} M_{(q)}(\ell_{r,s}^n) < \infty.$$

Proof. (i) We observe that $\ell_{r,s} = d(w,s)$ holds isometrically, where $w = (w_j)$ with $w_j = j^{-\theta}$ for each $j \in \mathbb{N}$ and $\theta = 1 - \frac{s}{r}$. Since $0 < \theta < 1$, standard calculus gives

$$\frac{1}{1-\theta}(k^{1-\theta} - 1) \leqslant \sum_{j=1}^{k} j^{-\theta} \leqslant \frac{1}{1-\theta} k^{1-\theta}, \qquad k \in \mathbb{N}.$$

Hence $w_1 + \ldots + w_k \asymp \frac{r}{r-s} k^{\frac{s}{r}}$. This implies that for $u = r/(r-s)$ we obtain

$$\frac{\rho_{u,k}}{\rho_{1,k}} = \frac{\left(\frac{1}{k}\sum_{j=1}^{k} w_j^u\right)^{1/u}}{\frac{1}{k}\sum_{j=1}^{k} w_j} \asymp \Big(\sum_{j=1}^{k} \frac{1}{j}\Big)^{\frac{1}{u}} \asymp (1 + \log k)^{1-\frac{s}{r}}.$$

Thus it follows by the formula (17) shown above

$$M_{(r)}(\ell_{r,s}^n) = \sup_{1 \leqslant k \leqslant n} \left[\frac{\rho_{u,k}}{\rho_{1,k}}\right]^{1/s} \asymp (1 + \log n)^{\frac{1}{s} - \frac{1}{r}}$$

and so this completes the proof.

(ii) Let $w = (j^{\frac{s}{r}-1})_{j=1}^n$. It is well known and easily verified that for any $x = (x(j))_{j=1}^n \in \ell_{r,s}^n$,

$$\|x\|_{r,s} = \inf \Big(\sum_{j=1}^{n} |x(j)|^s w_{\sigma(j)}\Big)^{1/s},$$

where the infimum is taken over all permutations σ of $\{1, \ldots, n\}$. This implies that for any $x_1, \ldots, x_N$ in $\ell_{r,s}^n$, we get that

$$\begin{aligned}\Big(\sum_{k=1}^{N} \|x_k\|_{r,s}^s\Big)^{1/s} &= \Big(\sum_{k=1}^{N} \inf_{\sigma} \sum_{j=1}^{n} |x_k(j)|^s w_{\sigma(j)}\Big)^{1/s} \\ &\leqslant \Big(\inf_{\sigma} \sum_{j=1}^{n} \sum_{k=1}^{N} |x_k(j)|^s w_{\sigma(j)}\Big)^{1/s} \\ &= \Big\| \Big(\sum_{k=1}^{N} |x_k|^s\Big)^{1/s} \Big\|_{r,s}.\end{aligned}$$

This estimate together with $M_{(q)}(\ell_{r,s}^n) \leqslant M_{(s)}(\ell_{r,s}^n)$ for $q \geqslant s$ gives the assertion. ■

We finish collecting our knowledge on the asymptotic decay of $K(B_{\ell_{r,s}^n})$.

Corollary 10. *Let $1 < r < \infty$ and $1 \leqslant s \leqslant \infty$. Then the following statements are true for the sequence of n-dimensional Lorentz spaces $\ell_{r,s}^n$:*

(i) $\boldsymbol{2 < r < \infty}$ **:** *Then for any $s \in [1,2) \cup (r,\infty]$*

$$K(B_{\ell_{r,s}^n}) \asymp \sqrt{\frac{\log n}{n}},$$

for $2 \leqslant s \leqslant r$

$$\lim_n \frac{K(B_{\ell_{r,s}^n})}{\sqrt{\frac{\log n}{n}}} = 1.$$

(ii) $\boldsymbol{r = 2}$ **:** *For $1 \leqslant s \leqslant 2$*

$$K(B_{\ell_{2,s}^n}) \asymp \sqrt{\frac{\log n}{n}},$$

and for $2 < s \leqslant \infty$

$$\frac{\sqrt{\log n}}{\sqrt{n}} \prec K(B_{\ell_{2,s}^n}) \prec \frac{(\log n)^{1-1/s}}{\sqrt{n}}.$$

(iii) $\boldsymbol{1 < r < 2}$ **:** *For $1 \leqslant s \leqslant r$*

$$\frac{(\log n)^{1-\frac{1}{s}}}{n^{1-\frac{1}{r}}} \prec K(B_{\ell_{r,s}^n}) \prec \frac{(\log n)^{1-\frac{1}{r}}}{n^{1-\frac{1}{r}}}.$$

Moreover, for $r < s \leqslant \frac{r}{2-r}$ (i.e., $1 - \frac{2}{r} + \frac{1}{s} \geqslant 0$)

$$\frac{(\log n)^{1-\frac{2}{r}+\frac{1}{s}}}{n^{1-\frac{1}{r}}} \prec K(B_{\ell_{r,s}^n}) \prec \frac{(\log n)^{1-\frac{1}{s}}}{n^{1-\frac{1}{r}}}.$$

and for $\frac{r}{2-r} < s \leqslant \infty$

$$\frac{1}{n^{1-\frac{1}{r}}} \prec K(B_{\ell_{r,s}^n}) \prec \frac{(\log n)^{1-\frac{1}{s}}}{n^{1-\frac{1}{r}}}.$$

Proof. (i) For $1 \leqslant s < 2 < r$ the lower estimate follows from (5) and (1). For the upper estimate we use (7) which yields

$$K(B_{\ell_{r,s}^n}) \prec (\log n)^{\frac{1}{2}} \frac{\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_2^n\|}{\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_1^n\|}.$$

By (4) we have $\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_1^n\| = n^{1-1/r}$ and $\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_2^n\| \prec n^{1/2-1/r}$ (factorize through $\ell_{r,r}^n$ and use that $\ell_{r,s} \hookrightarrow \ell_{r,r}$ since $s \leqslant r$).

To prove the required estimate for $2 < r < s \leqslant \infty$, we first observe that it follows from formula (18) (see [15]) that in the case $r > p \geqslant 1$ and $w = (k^{-\alpha})$

with $0 < \alpha < 1$, the Lorentz space $d(w,p)$ is r-concave if $\alpha < 1/u$. We need here the well known special case of this result which states that the Lorentz space $\ell_{p,q}$ is 2-concave whenever $1 < q < p < 2$ (it should be pointed out that this result follows from [6, Theorem 3.5 (ii)] by the fact that a Banach lattice is 2-concave if and only it has Rademacher cotype 2). Now we apply the well known Köthe duality $(\ell_{r,s})' = \ell_{r',s'}$ (up to equivalence of norms), where $1/r + 1/r' = 1$ and $1/s + 1/s' = 1$. Since $1 < s' < r' < 2$, it follows from what we just explained, that $\ell_{r',s'}$ is 2-concave. Thus by the formula $M_{(2)}(X') = M^{(2)}(X)$ (true for all maximal Banach lattices), we deduce that $\ell_{r,s}$ is 2-convex and we can apply Theorem 1.

If $2 \leqslant s \leqslant r$, then the obvious isometric equality

$$\ell_{r,s} = (\ell_{r/s,1})^{(s)}$$

implies that $M^{(s)}(\ell_{r,s}) = 1$. Since $2 \leqslant s$, $M^{(2)}(\ell_{r,s}) \leqslant M^{(s)}(\ell_{r,s})$ and so $\ell_{r,s}$ is 2-convex with constant 1 and we again apply Theorem 1.

(ii) In both cases the lower estimate for the Bohr radius is a consequence of (5) and (1). In the first case the upper estimate follows from (7) combined with the estimates $\|\mathrm{id}\colon \ell_{2,s}^n \to \ell_2^n\| \leqslant 1$ and $\|\mathrm{id}\colon \ell_{2,s}^n \to \ell_1^n\| \asymp \sqrt{n}$ (by (4)):

$$K(B_{\ell_{2,s}^n}) \prec (\log n)^{\frac{1}{2}} \frac{\|\mathrm{id}\colon \ell_{2,s}^n \to \ell_2^n\|}{\|\mathrm{id}\colon \ell_{2,s}^n \to \ell_1^n\|} \prec \frac{\sqrt{\log n}}{\sqrt{n}}.$$

The second upper estimate follows the same way using Lemma 8.

(iii) For the proof of the first statement we note that by Proposition 3, (2) and Lemma 9 we have

$$\frac{(\log n)^{1-\frac{1}{s}}}{n^{1-\frac{1}{r}}} \prec \frac{K(B_{\ell_r^n})}{M_{(r)}(\ell_{r,s}^n)} \leqslant K(B_{\ell_{r,s}^n}).$$

On the other hand by (7)

$$K(B_{\ell_{r,s}^n}) \prec (\log n)^{1-\frac{1}{r}} \frac{\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_r^n\|}{\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_1^n\|},$$

which implies the upper estimate since $\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_r^n\| \leqslant 1$ as well as $\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_1^n\| \asymp n^{1-1/r}$. For the upper estimate in the second and third statement of (iii) use again (7), and combine it with the two facts $\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_1^n\| \asymp n^{1-1/r}$ (by (4)) and $\|\mathrm{id}\colon \ell_{r,s}^n \to \ell_r^n\| \prec (\log n)^{\frac{1}{r}-\frac{1}{s}}$ (by Lemma 8).

For the lower estimate in the second case we get from Proposition 3 and Lemma 8 (ii) that

$$K(\ell_{r,s}^n) \succ \Big(\frac{\log n}{n}\Big)^{1-\frac{1}{r}} \cdot \frac{1}{(\log n)^{\frac{1}{r}-\frac{1}{s}}} = \frac{(\log n)^{1-\frac{2}{r}+\frac{1}{s}}}{n^{1-\frac{1}{r}}}.$$

Finally, for the lower estimate in the third case use (6) and

$$d(\ell_{r,s}^n, \ell_1^n) \leqslant \|\mathrm{id}\colon \ell_{r,s}^n \to \ell_1^n\| \, \|\mathrm{id}\colon \ell_1^n \to \ell_{r,s}^n\| \leqslant n^{1-1/r}$$

(again (4)) to finish. ∎

References

[1] F. Bayart, *Maximum modulus of random polynomials*, Quart. J. Math. **63** (2012), 21–39.

[2] F. Bayart, D. Pellegrino and J. B. Seoane-Sepúlveda, *The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$*, Adv. Math. **264** (2014), 726–746.

[3] H. P. Boas, *Majorant series*, J. Korean Math. Soc. **37** (2000), no. 2, 321–337.

[4] H. Boas and D. Khavinson, *Bohr's power series theorem in several variables*, Proc. Amer. Math. Soc. **125** (1997), 2975–2979.

[5] H. Bohr, *A theorem concerning power series*, Proc. London Math. Soc. (2), **13** (1914), 1–5.

[6] J. Creekmore, *Type and cotype in Lorentz L_{pq} spaces*, Nederl. Akad. Wetensch. Indag. Math. **43** (1981), no. 2, 145–152.

[7] A. Defant, *Variants of the Maurey-Rosenthal theorem for quasi Köthe function spaces*, Positivity **5** (2001), 1572–9281

[8] A. Defant and L. Frerick, *Logarithmic lower bound for multi-dimensional Bohr radii*, Israel J. Math. **152** (2006), 17–28.

[9] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes and K. Seip, *The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive*, Ann. of Math. (2) **174** (2011), no. 1, 485–497.

[10] A. Defant and L. Frerick, *The Bohr radius of the unit ball of ℓ_n^p*, J. Reine Angew. Math. **660** (2011), 131–147.

[11] A. Defant, D. García and M. Maestre, *Bohr's power series theorem and local Banach space theory*, J. Reine Angew. Math. **557** (2003), 173–197.

[12] A. Defant, D. García and M. Maestre, *Estimates for the first and second Bohr radii of Reinhardt domains*, J. of Appr. Theory **128** (2004), 53–68.

[13] A. Defant, M. Mastyło and C. Michels, *Summing inclusion maps between symmetric sequence spaces*, a survey, in: Recent progress in functional analysis (Valencia, 2000), 43–60, North-Holland Math. Stud., **189**, North-Holland, Amsterdam, 2001.

[14] E. D. Gluskin, *Distances between some symmetric spaces.* (*Russian*) *Investigations on linear operators and the theory of functions*, XI. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **113** (1981), 218–224, 272; English transl. in Journal of Soviet Mathematics **22** (1983), no. 6, 1148–1846. DOI: 10.1007/BF01882586.

[15] G. J. O. Jameson, *The q-concavity constants of Lorentz sequence spaces and related inequalities*, Math. Z. **227** (1998), no. 1, 129–142.

[16] B. Maurey, *Théoremes de factorization pour les opérateurs linéaires a values dans les espaces L^p*, Astérisque **11** (1974), 135–154.

[17] P. Nilsson, *Interpolation of Banach lattices*, Studia Math. **82** (1985), no. 2, 135–154.

[18] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*, Function spaces. Springer-Verlag, Berlin-New York, 1979.

[19] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operators ideals*, Longman Scientific & Technical, 1989.

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