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ON COMPACTNESS OF TOEPLITZ OPERATORS IN BERGMAN SPACES

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To the memory of Paweł Domański

Abstract: In this paper we consider Toepliz operators with (locally) integrable symbols acting on Bergman spaces A^p $(1 < p < \infty)$ of the open unit disc of the complex plane. We give a characterization of compact Toeplitz operators with symbols in L^1 under a mild additional condition. Our result is new even in the Hilbert space setting of A^2, where it extends the well-known characterization of compact Toeplitz operators with bounded symbols by Stroethoff and Zheng.

Keywords: Toeplitz operator, Bergman space, compact operator.

1. Introduction and notation.

Consider the Banach space $L^p := (L^p(\mathbb{D}, dA), \|\cdot\|_p)$, where $1 < p < \infty$ and dA is the normalized area measure on the unit disc $\mathbb{D}$ of the complex plane, and the *Bergman space* A^p, which is the closed subspace of L^p consisting of analytic functions. The *Bergman projection* P is the orthogonal projection of L^2 onto A^2, and it has the integral representation

$$Pf(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta).$$

It is also known to be a bounded projection of L^p onto A^p for every $1 < p < \infty$. For an integrable function $a : \mathbb{D} \to \mathbb{C}$ and, say, bounded analytic functions f, the *Toeplitz operator* T_a with *symbol* a is defined by

$$T_a f = P(af) = \int_{\mathbb{D}} \frac{a(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta). \tag{1.1}$$

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In [8] and [9], we have given a generalized definition of Toeplitz operators, which we denote here by $\mathbf{T}_a$. In particular, we extended the definition to locally integrable symbols and showed that the generalized Toeplitz operator is bounded under a weak "averaging" condition (see (1.3) below). Since the generalized definition coincides with (1.1), whenever the latter makes sense, our condition is the weakest known sufficient condition for the boundedness of a Toeplitz operator. We recall this result in Theorem 1.1. The question of whether this condition is also necessary for boundedness remains open.

Our main interest here is on *compactness* of Toeplitz operators. Unlike in Hardy spaces, nontrivial Toeplitz operators may well be compact when acting on Bergman spaces even with unbounded symbols. For $a \in C(\overline{\mathbb{D}})$, T_a is compact if and only if $a(z) = 0$ for all $z \in \partial\mathbb{D}$; see [4, 10]. For more general symbols, characterizations are often given in terms of the Berezin transform: For any compact operator T on A^p, the Berezin transform of T vanishes on $\partial\mathbb{D}$. This is also sufficient for compactness of operators in the Toeplitz algebra generated by bounded symbols. However, there are compact Toeplitz operators with unbounded symbols whose Berezin transforms do not vanish. For further details on compactness and the Berezin transform, see [1, 5] for operators on $A^p(\mathbb{D})$, and [2, 3] and the references therein for more general Bergman spaces. A different type of characterization, involving the Möbius functions, was found by Stroethoff and Zheng [6, 7]. Their approach was based on the use the reproducing kernel functions and other Hilbert space techniques.

In Section 3 we shall apply our methods to generalize the results of Stroethoff and Zheng [7], which concern the characterization of compact Toeplitz operators with only bounded symbols: we shall relax the boundedness assumption on the symbol and extend their result from the Hilbert-space case to all Bergman spaces A^p with $1 < p < \infty$.

The following notation will be used throughout the paper. For all $z, \lambda \in \mathbb{D}$ we write

$$\begin{gathered}\varphi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}, \qquad W(z) = 1 - |z|^2, \qquad K_\lambda(z) = \frac{1}{(1 - \bar{\lambda} z)^2}, \\ k_\lambda(z) = W(\lambda) K_\lambda(z) = \frac{1 - |\lambda|^2}{(1 - \bar{\lambda} z)^2}.\end{gathered} \tag{1.2}$$

Given $z \in \mathbb{D}$ and $S > 0$, we write $B(z, S)$ for the Euclidean disc with center z and radius S and $D_h(z, S)$ for the hyperbolic disc with center z and radius S; the latter is the same as the hyperbolic disc $D(z, S)$ in [11], Proposition 4.4. By C, C', c etc. we denote positive constants, the exact value of which may vary from place to place but not in the same chain of inequalities. If the constant depends on some parameter or function, say, n or g, this is denoted by C_n or C_g etc. All function spaces consist of functions on the disc $\mathbb{D}$, unless otherwise stated. In particular, the space of bounded analytic functions on the disc is denoted by H^∞ and the space of locally integrable functions on $\mathbb{D}$ is denoted by L^1_{loc}. For clarity, we write $C^k(\mathbb{D})$

for the space of k times continuously differentiable functions in $\mathbb{D}$. For $0 < \rho < 1$ we set $\mathbb{D}_\rho := \{z \in \mathbb{D} : |z| \leqslant \rho\}$. Given $a \in L^1_{\text{loc}}$ we denote by a_ρ the function, which coincides with a on $\mathbb{D}_\rho$ and equals 0 elsewhere.

Given a continuously differentiable function f of a variable $z = x + iy = re^{i\theta} \in \mathbb{D}$, we denote $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$, $\partial_r = \partial/\partial r$ and $\partial_\theta = \partial/\partial\theta$.

We let $\mathcal{D}$ be a family of the sets $D := D(r,\theta) \subset \mathbb{D}$, where

$$D = \{\rho e^{i\phi} | r \leqslant \rho \leqslant 1 - \frac{1}{2}(1-r),\ \theta \leqslant \phi \leqslant \theta + \pi(1-r)\}$$

for all $0 < r < 1$, $\theta \in [0, 2\pi]$. We denote $|D| := \int_D dA$ and, for $\zeta = \rho e^{i\phi} \in D(r,\theta)$,

$$\hat{a}_D(\zeta) := \frac{1}{|D|} \int_r^\rho \int_\theta^\phi a(\varrho e^{i\varphi}) \varrho d\varphi d\varrho.$$

In the sequel we will consider functions $a \in L^1$ (or even $a \in L^1_{\text{loc}}$) such that there exists a constant $C > 0$ such that

$$|\hat{a}_D(\zeta)| \leqslant C \tag{1.3}$$

for all $D \in \mathcal{D}$ and all $\zeta \in D$. The following result is contained in Theorem 2.1 of [9].

Theorem 1.1. *Let $1 < p < \infty$. If a symbol $a \in L^1$ satisfies the condition* (1.3)*, then the limit*

$$\mathbf{T}_a f = \lim_{\rho\to 1} T_{a_\rho} f, \qquad \text{where } T_{a_\rho} f(z) = \int_{\mathbb{D}_\rho} \frac{a(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta), \quad f \in A^p, \tag{1.4}$$

converges in the strong operator topology and defines a bounded operator $\mathbf{T}_a : A^p \to A^p$ for all $1 < p < \infty$.

Moreover, the transposed operator $\mathbf{T}_a^* : A^q \to A^q$ can be written as

$$\mathbf{T}_a^* f(z) = \lim_{\rho\to 1} \int_{\mathbb{D}_\rho} \frac{\overline{a}(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta) \tag{1.5}$$

for $f \in A^q$ and this limit also converges in the strong operator topology.

The theorem actually holds as such for symbols $a \in L^1_{\text{loc}}$. Notice that for a fixed $0 < \rho < 1$ the restriction of any $f \in A^p$ to $\mathbb{D}_\rho$ is a bounded function and the operator T_{a_ρ} is bounded in A^p. Formula (1.4) allows us to define the Toeplitz operator even in many cases, where the defining integral of the conventional formula (1.1) does not converge, and it is used throughout this paper. This generalization of the definition of Toeplitz operators is a most natural one, it coincides with the conventional definition whenever the integral formula (1.1) makes sense, and it is considered from many points of view in [9].

2. Preliminary results.

Theorem 1.1 is based on the following lemma, the proof of which is included in the calculations (3.6)–(3.13) of the citation [8].

Lemma 2.1. *Assume that the symbol $a \in L^1$ satisfies the condition* (1.3)*, let $1 < p < \infty$ and let $f \in A^p$. Then, there exists a constant $C_p > 0$ such that*

$$\Big| \lim_{\rho\to 1} \int\limits_{\mathbb{D}_\rho} \frac{a(\zeta)f(\zeta)}{(1-z\bar{\zeta})^2} dA(\zeta) \Big| \leqslant C_p \int\limits_{\mathbb{D}} \frac{|f(\zeta)| + |f'(\zeta)| W(\zeta) + |f''(\zeta)| W(\zeta)^2}{|1-z\bar{\zeta}|^2} dA(\zeta).$$

Notice that the limit on the left-hand side converges for every $z \in \mathbb{D}$ as a consequence of Theorem 1.1.

The lemma and the following proposition hold even for $a \in L^1_{\mathrm{loc}}$. Instead of going into the details of the proof Lemma 2.1 we prove a technical generalization of it, which is essential for the subsequent considerations. For the formulation of this result we fix $1 < p < \infty$ and let $g \in L^\infty(\mathbb{D}) \cap C^2(\mathbb{D})$ be a function such that given $S > 0$, there exists a constant $C_S > 0$ such that for every $z \in \mathbb{D}$,

$$|(\partial_r^j \partial_\theta^k g)(z)| \leqslant C_S \inf_{\zeta \in D_h(z,S)} |(\partial_r^j \partial_\theta^k g)(\zeta)|, \qquad j,k = 0,1, \tag{2.1}$$

and such that for some $C_g > 0$

$$\|g\|_\infty + \max_{j=1,2} \|W \partial_j g\|_\infty + \max_{j,k=1,2} \|W^2 \partial_j \partial_k g\|_\infty \leqslant C_g < \infty. \tag{2.2}$$

Moreover, let $h \in C^2(\mathbb{D})$, $h(\mathbb{D}) \subset \mathbb{D}$, be a function, which also satisfies (2.2) and in addition, for some $C_h > 0$,

$$|1 - z\overline{h(\zeta)}| \geqslant C_h |1 - z\bar{\zeta}| \tag{2.3}$$

for every $z, \zeta \in \mathbb{D}$. For example, h could be a Möbius transform.

Proposition 2.2. *Let p, a and f be as in Lemma 2.1, and let the functions g and h be as in* (2.1)*–*(2.3)*. Then, the limit*

$$\lim_{\rho\to 1} \int\limits_{\mathbb{D}_\rho} \frac{a(\zeta)f(\zeta)g(\zeta)}{(1-z\overline{h(\zeta)})^2} dA(\zeta) \tag{2.4}$$

converges in A^p and there exists a constant $C_{p,g,h} > 0$ such that

$$\Big| \lim_{\rho\to 1} \int\limits_{\mathbb{D}_\rho} \frac{a(\zeta)f(\zeta)g(\zeta)}{(1-z\overline{h(\zeta)})^2} dA(\zeta) \Big| \leqslant C_{p,g,h} \int\limits_{\mathbb{D}} \frac{|f(\zeta)| + |f'(\zeta)| W(\zeta) + |f''(\zeta)| W(\zeta)^2}{|1-z\bar{\zeta}|^2} dA(\zeta). \tag{2.5}$$

Proof. The result follows from the proof of Theorem 2.1. of [9], but some changes are needed. We present them as briefly as we can; for more details the reader is asked to see the citation. Following the notation of the citation, we define the countably many sets $D\big(1-2^{-m+1}, 2\pi(\mu-1)2^{-m}\big) \in \mathcal{D}$, where $m \in \mathbb{N}, \mu = 1, \dots, 2^{-m}$, which form a decomposition of the disc $\mathbb{D}$. These are indexed in some order into a family $(D_n)_{n=1}^{\infty}$, so that every D_n is of the form

$$D_n = \{z = re^{i\theta} \,|\, r_n < r \leqslant r_n',\ \theta_n < \theta \leqslant \theta_n'\}$$

where, for some m and μ,

$$r_n = 1-2^{-m+1}, \quad r_n' := 1-2^{-m}, \qquad \theta_n = \pi(\mu-1)2^{-m+1}, \quad \theta_n' := \pi\mu 2^{-m+1}.$$

Given $f \in A^p$ and $n = n(m,\mu)$ we write

$$F_n f(z) = \int\limits_{D_n} \frac{a(\zeta)f(\zeta)g(\zeta)}{(1-z\overline{h(\zeta)})^2} dA(\zeta) \qquad \forall\, z \in \mathbb{D}.$$

For all $n \in \mathbb{N}$ we define $\mathcal{D}_n = \{D_\nu \,:\, \nu \in \mathbb{N},\ \overline{D}_\nu \cap \overline{D}_n \neq \emptyset\}$. There exist constants N, $M \in \mathbb{N}$ such that any set $\mathcal{D}_n$ contains at most N elements D_ν and on the other hand, any set D_ν belongs to at most M sets $\mathcal{D}_n$. By the choice of the sets D_n, for all given D_n and $w \in D_n$ the subdomain $\cup_{D\in\mathcal{D}_n} D$ always contains a Euclidean disc $B(w,R(n))$ such that $|B(w,R(n))| \geqslant C|D_n|$. Now, let $\tilde{f} : \mathbb{D} \to \mathbb{C}$ be an analytic analytic function and let g be as in the assumption. We claim that for each $n \in \mathbb{N}$, $j,k = 0,1$, and $w \in D_n$,

$$|\tilde{f}(w)\partial_r^j\partial_\theta^k g(w)| \leqslant \frac{C}{|D_n|} \sum_{D\in\mathcal{D}_n} \int\limits_D |\tilde{f}(\zeta)\partial_r^j\partial_\theta^k g(\zeta)| dA(\zeta). \tag{2.6}$$

To see this let $B(w,R(n)) \subset \cup_{D\in\mathcal{D}_n} D$ be as above. Then, $\tilde{f}$ has the subharmonicity property

$$|\tilde{f}(w)| \leqslant \frac{C}{|B(w,R(n))|} \int\limits_{B(w,R(n))} |\tilde{f}(\zeta)| dA(\zeta).$$

Moreover, by the choice of the family $\mathcal{D}_n$, there exists $S > 0$, which can be chosen independently of n, such that for every $w \in D_n$, $B(w,R(n))$ is contained in the hyperbolic disc $D_h(w,S)$. Then, by (2.1)

$$\begin{aligned}|\tilde{f}(w)\partial_r^j\partial_\theta^k g(w)| &\leqslant \frac{C}{|B(w,R)|} \int\limits_{D(w,R)} |\tilde{f}(\zeta)\partial_r^j\partial_\theta^k g(\zeta)| dA(\zeta) \\ &\leqslant \frac{C'}{|D_n|} \sum_{D\in\mathcal{D}_n} \int\limits_D |\tilde{f}(\zeta)\partial_r^j\partial_\theta^k g(\zeta)| dA(\zeta).\end{aligned}$$

For every n, a double integration by parts in polar coordinates yields

$$
\begin{aligned}
\int\limits_{D_n} \frac{a(\zeta)f(\zeta)g(\zeta)}{(1-z\bar\zeta)^2}dA(\zeta) &= \Big(\int\limits_{r_n}^{r_n'}\int\limits_{\theta_n}^{\theta_n'} a(\varrho e^{i\varphi})\varrho d\varphi d\varrho\Big)\frac{f(r_n'e^{i\theta_n'})g(r_n'e^{i\theta_n'})}{(1-z\overline{h(r_n'e^{i\theta_n'})})^2} \\
&\quad - \int\limits_{r_n}^{r_n'}\Big(\int\limits_{r_n}^{r}\int\limits_{\theta_n}^{\theta_n'} a(\varrho e^{i\varphi})\varrho d\varphi d\varrho\Big)\partial_r\frac{f(re^{i\theta_n'})g(re^{i\theta_n'})}{(1-z\overline{h(re^{i\theta_n'})})^2}dr \\
&\quad - \int\limits_{\theta_n}^{\theta_n'}\Big(\int\limits_{r_n}^{r_n'}\int\limits_{\theta_n}^{\theta} a(\varrho e^{i\varphi})\varrho d\varphi d\varrho\Big)\partial_\theta\frac{f(r_n'e^{i\theta})g(r_n'e^{i\theta})}{(1-z\overline{h(r_n'e^{i\theta})})^2}d\theta \\
&\quad + \int\limits_{r_n}^{r_n'}\int\limits_{\theta_n}^{\theta_n'}\Big(\int\limits_{r_n}^{r}\int\limits_{\theta_n}^{\theta} a(\varrho e^{i\varphi})\varrho d\varphi d\varrho\Big)\partial_r\partial_\theta\frac{f(re^{i\theta})g(re^{i\theta})}{(1-z\overline{h(re^{i\theta})})^2}d\theta dr \\
&=: F_{1,n}(z)+F_{2,n}(z)+F_{3,n}(z)+F_{4,n}(z) = F_n(z).
\end{aligned}
$$

We consider $F_{2,n}(z)$. By (2.3) and (4.8) of [11],

$$
|1-z\overline{h(re^{i\theta_n'})}| \geqslant C_h|1-zre^{-i\theta_n'}| \geqslant C_h'|1-z\bar\zeta| \tag{2.7}
$$

for all $z\in\mathbb{D}$, $\zeta\in D$, all $D\in\mathcal{D}_n$. Performing the differentiation, using (2.6) for f and its derivative in the place of $\tilde f$, and then using (2.1), (2.2), (2.7) we thus get

$$
\begin{aligned}
&\Big|\partial_r\frac{f(re^{i\theta_n'})g(re^{i\theta_n'})}{(1-z\overline{h(re^{i\theta_n'})})^2}\Big| \\
&\leqslant \frac{C}{|D_n|}\sum_{D\in\mathcal{D}_n}\int\limits_D\Big(\frac{|f(\zeta)||g(\zeta)|}{|1-z\overline{h(re^{i\theta_n'})}|^3}+\frac{|f(\zeta)||\partial_r g(\zeta)|}{|1-z\overline{h(re^{i\theta_n'})}|^2}+\frac{|f'(\zeta)||g(\zeta)|}{|1-z\overline{h(re^{i\theta_n'})}|^2}\Big)dA(\zeta) \\
&\leqslant \frac{C_h}{|D_n|}\sum_{D\in\mathcal{D}_n}\int\limits_D\Big(\frac{|f(\zeta)|\,\|g\|_\infty}{|1-z\bar\zeta|^3}+\frac{|f(\zeta)|\,W(\zeta)^{-1}\|W\partial_r g\|_\infty}{|1-z\bar\zeta|^2}+\frac{|f'(\zeta)|\,\|g\|_\infty}{|1-z\bar\zeta|^2}\Big)dA(\zeta) \\
&\leqslant \frac{C_{g,h}}{|D_n|}\sum_{D\in\mathcal{D}_n}\int\limits_D\Big(\frac{|f(\zeta)|}{|1-z\bar\zeta|^3}+\frac{|f(\zeta)|}{W(\zeta)|1-z\bar\zeta|^2}+\frac{|f'(\zeta)|}{|1-z\bar\zeta|^2}\Big)dA(\zeta)
\end{aligned}
$$

Thus, $F_{2,n}$ can be estimated by

$$
\begin{aligned}
|F_{2,n}(z)| &\leqslant \int_{r_n}^{r_n'} \Big| \int_{r_n}^{r} \int_{\theta_n}^{\theta_n'} a(\varrho e^{i\varphi}) \varrho d\varphi d\varrho \Big| \Big| \partial_r \frac{f(re^{i\theta_n'}) g(re^{i\theta_n'})}{(1 - z\overline{h(re^{i\theta_n'})})^2} \Big| dr \\
&\leqslant C_{g,h} \int_{r_n}^{r_n'} \Big| \int_{r_n}^{r} \int_{\theta_n}^{\theta_n'} a(\varrho e^{i\varphi}) \varrho d\varphi d\varrho \Big| \\
&\quad \times \frac{1}{|D_n|} \sum_{D \in \mathcal{D}_n} \int_D \Big(\frac{|f(\zeta)|}{|1 - z\bar{\zeta}|^3} + \frac{|f(\zeta)|}{W(\zeta)|1 - z\bar{\zeta}|^2} + \frac{|f'(\zeta)|}{|1 - z\bar{\zeta}|^2} \Big) dA(\zeta) dr \\
&\leqslant C_{g,h}' \sum_{D \in \mathcal{D}_n} \int_{r_n}^{r_n'} \int_D \Big(\frac{|f(\zeta)|}{|1 - z\bar{\zeta}|^3} + \frac{|f(\zeta)|}{W(\zeta)|1 - z\bar{\zeta}|^2} + \frac{|f'(\zeta)|}{|1 - z\bar{\zeta}|^2} \Big) dA(\zeta) dr \\
&\leqslant C_{g,h}'' \sum_{D \in \mathcal{D}_n} \int_D \Big(\frac{|f(\zeta)|}{|1 - z\bar{\zeta}|^2} + \frac{|f'(\zeta)| W(\zeta)}{|1 - z\bar{\zeta}|^2} \Big) dA(\zeta). \qquad (2.8)
\end{aligned}
$$

where the bound for the integral of a follows from (1.3) and we use $|r_n - r_n'| \leqslant CW(\zeta)$ to cancel the factors $|1-z\bar{\zeta}|^{-1}$ and $W(\zeta)^{-1}$. The terms $F_{1,n}$, $F_{3,n}$ and $F_{4,n}$ can be estimated with similar calculations, and we obtain for F_n the estimate (2.7) of [9]. From here on, the proof goes by a word-to-word repetition of the citation, except that the symbol g has a different meaning in [9]. ■

Corollary 2.3. *Let p, a and f be as in Lemma 2.1, and let $\lambda \in \mathbb{D}$ (considered as a fixed parameter). Then, the limit*

$$
\lim_{\rho \to 1} \int_{\varphi_\lambda(\mathbb{D}_\rho)} \frac{a \circ \varphi_\lambda(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta) \qquad (2.9)
$$

converges in A^p and there exists a constant $C = C(p, \lambda) > 0$ such that

$$
\Big| \lim_{\rho \to 1} \int_{\varphi_\lambda(\mathbb{D}_\rho)} \frac{a \circ \varphi_\lambda(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta) \Big| \leqslant C \int_{\mathbb{D}} \frac{|f(\zeta)| + |f'(\zeta)| W(\zeta) + |f''(\zeta)| W(\zeta)^2}{|1 - z\bar{\zeta}|^2} dA(\zeta), \qquad (2.10)
$$

Proof. We perform a change of variable ([11], Proposition 4.2) to obtain

$$
\int_{\varphi_\lambda(\mathbb{D}_\rho)} \frac{a \circ \varphi_\lambda(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta) = \int_{\mathbb{D}_\rho} \frac{a(\zeta) f \circ \varphi_\lambda(\zeta)}{(1 - z\overline{\varphi_\lambda(\zeta)})^2} |k_\lambda(\zeta)|^2 dA(\zeta).
$$

The result follows from the previous proposition by setting $g = |k_\lambda|^2$ and $h = \varphi_\lambda$: it is well-known or a routine matter to show that these functions satisfy the assumptions (2.1)–(2.3), since λ is fixed. Of course, we also have $f \circ \varphi_\lambda \in A^p$. ■

Corollary 2.3 implies the following observation.

Corollary 2.4. *If $a \in L^1$ satisfies* (1.3) *and $\lambda \in \mathbb{D}$, then $P(a \circ \varphi_\lambda) \in A^p$ for every $1 < p < \infty$.*

Indeed, if $f \equiv 1$ in Corollary 2.3, then f belongs to A^p for every $1 < p < \infty$, and the limit (2.9) converges in any A^p to $P(a \circ \varphi_\lambda)$.

Remark 2.5. We actually get the result of Corollary 2.4 for every $a \in L^1_{\mathrm{loc}}$ satisfying condition (1.3), if we generalize the expression $P(a \circ \varphi_\lambda)$ as

$$\mathbf{P}(a \circ \varphi_\lambda)(z) := \lim_{\rho \to 1} \int\limits_{\mathbb{D}_\rho} \frac{a \circ \varphi_\lambda(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta), \qquad z \in \mathbb{D}. \tag{2.11}$$

3. Characterization of compact Toeplitz operators.

In this section we generalize the compactness characterization result [7], Theorem 6, for unbounded symbols and A^p-spaces with arbitrary $p \in (1, \infty)$. In the following theorem, the operator $\mathbf{T}_a$ is the generalized Toeplitz-operator of (1.4), but as explained below Theorem 1.1 and in [9], it coincides with the usual definition, if $af \in L^1$ for every $f \in A^p$. Also, we use Corollary 2.4 to assure that the L^q-norm of $P(a \circ \varphi_\lambda)$ is finite for every $q \in (1, \infty)$. The theorem would hold true by merely assuming $a \in L^1_{\mathrm{loc}}$ instead of $a \in L^1$ and using Remark 2.5.

Theorem 3.1. *Assume that the symbol $a : \mathbb{D} \to \mathbb{C}$ belongs to L^1 and satisfies the condition* (1.3)*, and let $1 < p < \infty$. Then, the following conditions (i)–(iii) are equivalent:*

(i) $\mathbf{T}_a : A^p \to A^p$ *is compact,*
(ii) $\|P(a \circ \varphi_\lambda)\|_q \to 0$ *as* $\lambda \to \partial\mathbb{D}$ *for some* $q \in [1, \infty)$,
(iii) $\|P(a \circ \varphi_\lambda)\|_q \to 0$ *as* $\lambda \to \partial\mathbb{D}$ *for every* $q \in [1, \infty)$.

Before proceeding to the proof we need to generalize known facts to our setting. The next lemma would hold even in the case of locally integrable symbols, but the proof is less simple, see Remark 3.4.

Lemma 3.2. *Let $1 < p < \infty$ and let $a \in L^1$ satisfy* (1.3)*. Then, for every $\lambda \in \mathbb{D}$,*

$$\mathbf{T}_a K_\lambda = T_a K_\lambda = K_\lambda P(a \circ \varphi_\lambda) \circ \varphi_\lambda \tag{3.1}$$

Proof. First, notice that the function K_λ is bounded, hence the function aK_λ belongs to L^1, and indeed $\mathbf{T}_a K_\lambda$ coincides with the conventional definition.

The identity (3.1) is known to be true, if $a \in L^\infty$, see for example Proposition 1 of [7]. Hence, defining for every $R > 0$

$$a_{(R)}(z) = \begin{cases} a(z), & \text{if } |a(z)| \leqslant R, \\ a(z)|a(z)|^{-1}, & \text{if } |a(z)| > R, \end{cases}$$

we have

$$T_{a_{(R)}} K_\lambda = K_\lambda P(a_{(R)} \circ \varphi_\lambda) \circ \varphi_\lambda. \tag{3.2}$$

By the dominated convergence theorem, the left hand side converges pointwise to $T_a K_\lambda$, as $R \to \infty$. For the same reason we have on the right $P(a_{(R)} \circ \varphi_\lambda) \to P(a \circ \varphi_\lambda)$ pointwise. Thus also

$$K_\lambda P(a_{(R)} \circ \varphi_\lambda) \circ \varphi_\lambda \to K_\lambda P(a \circ \varphi_\lambda) \circ \varphi_\lambda$$

pointwise as $R \to \infty$, and the claim follows from (3.2). ■

Lemma 3.3. *Let $1 < p < \infty$ and let $a \in L^1$ satisfy* (1.3)*. Given $\varepsilon > 0$, we have for every $z \in \mathbb{D}$,*

$$\int_{\mathbb{D}} |P(a \circ \varphi_\lambda)(\varphi_\lambda(z))|\, |K_\lambda(z)| \frac{1}{W(\lambda)^\varepsilon} dA(\lambda) \leqslant \frac{C(a,\varepsilon)}{W(z)^\varepsilon} \tag{3.3}$$

Proof. By Lemmas 3.2 and 2.1 we have

$$\begin{aligned} |P(a \circ \varphi_\lambda)(\varphi_\lambda(z))|\, |K_\lambda(z)| &= \Big| \int_{\mathbb{D}} \frac{a(\zeta)}{(1 - z\bar{\zeta})^2} K_\lambda(\zeta) dA(\zeta) \Big| \\ &\leqslant C \int_{\mathbb{D}} \frac{|K_\lambda(\zeta)| + |K_\lambda'(\zeta)|\, W(\zeta) + |K_\lambda''(\zeta)|\, W(\zeta)^2}{|1 - z\bar{\zeta}|^2} dA(\zeta) \\ &\leqslant C' \int_{\mathbb{D}} \frac{1}{|1 - \lambda\bar{\zeta}|^2 |1 - z\bar{\zeta}|^2} dA(\zeta), \end{aligned}$$

where we also used at the end the evident estimates $|K_\lambda'(\zeta)|\, W(\zeta) \leqslant C|K_\lambda(\zeta)|$ and $|K_\lambda''(\zeta)|\, W^2(\zeta) \leqslant C|K_\lambda(\zeta)|$ for some constant $C > 0$, for all ζ and λ. Thus

$$\int_{\mathbb{D}} \frac{|P(a \circ \varphi_\lambda)(\varphi_\lambda(z))|}{|1 - z\bar{\lambda}|^2 (1 - |\lambda|^2)^\varepsilon} dA(\lambda) \leqslant C \int_{\mathbb{D}} \frac{1}{|1 - z\bar{\zeta}|^2} \int_{\mathbb{D}} \frac{1}{|1 - \lambda\bar{\zeta}|^2 (1 - |\lambda|^2)^\varepsilon} dA(\lambda) dA(\zeta).$$

The bound (3.3) follows by applying twice the Forelli-Rudin estimate, see [11], Lemma 3.10. ■

To prove that (iii) $\Rightarrow$ (i) in Theorem 3.1, we denote by p' the dual exponent of p, and for every $0<\rho<1$ we define the operator $S_\rho : A^{p'} \to L^{p'}$,

$$S_\rho g(\lambda) = \chi_\rho(\lambda) \int\limits_{\mathbb{D}} g(\zeta)\overline{P(a\circ\varphi_\lambda)(\varphi_\lambda(\zeta))K_\lambda(\zeta)}dA(\zeta) \tag{3.4}$$

where χ_ρ is the characteristic function of $\mathbb{D}_\rho$. We observe by Corollary 2.4 that $P(a\circ\varphi_\lambda)\in L^r$ for every $r\in(1,\infty)$ (in particular $r=p$) and every $\lambda\in\mathbb{D}$, and for some constants $C, C_r>0$,

$$\sup_{\lambda\in\mathbb{D}_\rho}\sup_{\zeta\in\mathbb{D}}|K_\lambda(\zeta)| \leqslant C \qquad \text{and} \qquad \sup_{\lambda\in\mathbb{D}_\rho}\|P(a\circ\varphi_\lambda)\|_r \leqslant C_r. \tag{3.5}$$

This implies that the integral in (3.4) converges, since $g\in A^{p'}$ is assumed. We show that the operator $S_\rho : A^{p'}\to L^{p'}$ is compact. To this end we fix $r<p'$ and estimate, for all $g\in A^r$,

$$\begin{aligned}\|S_\rho g\|_{p'} &\leqslant \|\chi_\rho\|_{p'} \sup_{\lambda\in\mathbb{D}_\rho}\Big|\int\limits_{\mathbb{D}} g(\zeta)\overline{P(a\circ\varphi_\lambda)(\varphi_\lambda(\zeta))K_\lambda(\zeta)}dA(\zeta)\Big| \\ &\leqslant \sup_{\lambda\in\mathbb{D}_\rho}\|g\|_r\Big(\int\limits_{\mathbb{D}_\rho}\Big|P(a\circ\varphi_\lambda)(\varphi_\lambda(\zeta))K_\lambda(\zeta)\Big|^{r'}dA(\zeta)\Big)^{1/r'} \end{aligned}\tag{3.6}$$

where $r'\in(1,\infty)$ is the dual exponent of r and the Hölder inequality was used. Using (3.5) we can bound (3.6) by

$$C\|g\|_r \sup_{\lambda\in\mathbb{D}_\rho}\|P(a\circ\varphi_\lambda)\|_{r'} \leqslant C'\|g\|_r$$

and together with (3.6) this shows that the operator S_ρ is bounded $A^r\to L^{p'}$. Since $r<p'$, the embedding $A^{p'}\hookrightarrow A^r$ is compact (see Chapter 4, Exercise 2 in [11]), and thus $S_\rho : A^{p'}\to L^{p'}$ is compact.

The rest of the proof goes as in [7], with straightforward changes. For the convenience of the reader we expose the details. By our assumptions on a, $T_a : A^p\to A^p$ is bounded and thus so is $T_a^* : A^{p'}\to A^{p'}$. The proof is completed by showing that $S_\rho\to T_a^*$ in the operator norm $A^{p'}\to A^{p'}$ as $\rho\to 1$, because then T_a^* and T_a are compact. The definition of an adjoint and a change of variables yield

$$\begin{aligned}T_a^*g(\lambda)-S_\rho g(\lambda) &= \langle T_a^*g, K_\lambda\rangle - S_\rho g(\lambda) \\ &= \int\limits_{\mathbb{D}} X_\rho(\lambda)g(\zeta)\overline{P(a\circ\varphi_\lambda)(\varphi_\lambda(\zeta))K_\lambda(\zeta)}dA(\zeta) \\ &=: \int\limits_{\mathbb{D}} R_\rho(\lambda,\zeta)g(\zeta)dA(\zeta),\end{aligned}$$

where $X_\rho = 1 - \chi_\rho$. We will find a function $h : \mathbb{D} \to \mathbb{R}^+$ such that

$$\begin{aligned} \int_{\mathbb{D}} R_\rho(\lambda,\zeta) h(\lambda)^p dA(\lambda) &\leqslant C_1(a) h(\zeta)^p, \\ \int_{\mathbb{D}} R_\rho(\lambda,\zeta) h(\lambda)^{p'} dA(\lambda) &\leqslant C_\rho h(\zeta)^{p'} \end{aligned} \tag{3.7}$$

where $C_1(a)$ is independent of ρ and $C_\rho \to 0$ as $\rho \to 1$. Then, the operator norm of $T_a^* - S_\rho : A^{p'} \to L^{p'}$ tends to 0, by the Schur test, [11], Theorem 3.6, and the proof is complete.

We define the test function

$$h(\lambda) = W(\lambda)^{-1/(1+p')}.$$

Lemma 3.3 with $\varepsilon = p/(1+p')$ yields for all $0 < \rho < 1$

$$\begin{aligned} \int_{\mathbb{D}} R_\rho(\lambda,\zeta) h(\lambda)^p dA(\lambda) &\leqslant \int_{\mathbb{D}} \big|P(a\circ\varphi_\lambda)(\varphi_\lambda(\zeta))\big|\,|K_\lambda(\zeta)| W(\lambda)^{-p/(1+p')} dA(\lambda) \\ &\leqslant C_1(a) W(\zeta)^{-p/(1+p')} = C_1(a) h(\zeta)^p, \end{aligned} \tag{3.8}$$

where obviously the constant $C_1(a)$ can be chosen independently of ρ.

Moreover, using the change of variables $\zeta = \varphi_\lambda(z)$ and the identities

$$\begin{aligned} W(\zeta) = W(\varphi_\lambda(z)) &= W(\lambda) W(z) |K_\lambda(z)| \\ |K_\lambda(\zeta)| = |K_\lambda(\varphi_\lambda(z)| &= \frac{1}{W(\lambda)^2\,|K_\lambda(z)|}\ , \end{aligned}$$

where the first one follows from Proposition 4.1 of [11] and the second one is an immediate consequence, and denoting $\delta = p'/(1+p') \in (\frac{1}{2}, 1)$, we can estimate

$$\begin{aligned} &\int_{\mathbb{D}} R_\rho(\lambda,\zeta) h(\zeta)^{p'} dA(\zeta) \\ &\quad\leqslant \int_{\mathbb{D}} X_\rho(\lambda) \big|P(a\circ\varphi_\lambda)(\varphi_\lambda(\zeta))\big|\,|K_\lambda(\zeta)|\, W(\zeta)^{-\delta} dA(\zeta) \\ &\quad= W(\lambda)^{-\delta} X_\rho(\lambda) \int_{\mathbb{D}} \big|P(a\circ\varphi_\lambda)(z)\big| \frac{|k_\lambda(z)|^2}{W(\lambda)^2\,|K_\lambda(z)|} |K_\lambda(z)|^{-\delta}\, W(z)^{-\delta} dA(z) \\ &\quad= W(\lambda)^{-\delta} X_\rho(\lambda) \int_{\mathbb{D}} \big|P(a\circ\varphi_\lambda)(z)\big|\,|K_\lambda(z)|^{1-\delta}\, W(z)^{-\delta} dA(z). \end{aligned} \tag{3.9}$$

We finally choose e.g. $r = 1 + (1-\delta)/10$ and recall $\delta > 1/2$ so that

$$r\delta = \delta + (1-\delta)\frac{\delta}{10} < 1, \qquad 2r(1-\delta) = 2 + \frac{2(1-\delta)}{10} - 2r\delta \leqslant 2 - \frac{1}{4} - r\delta,$$

and we thus can use the Forelli-Rudin-estimates [11], Lemma 3.10., to see that the integral

$$\int_{\mathbb{D}} |K_\lambda(z)|^{r(1-\delta)}\, W(z)^{-r\delta} dA(z) = \int_{\mathbb{D}} \frac{(1-|z|^2)^{-r\delta}}{|1-\bar{\lambda} z|^{2-\frac{1}{4}-r\delta}}\, dA(z)$$

converges and has a bound independent of $\lambda \in \mathbb{D}$. Denote the dual exponent of r by $\tilde{r}$. Then, (3.9) can be estimated using the Hölder inequality by

$$W(\lambda)^{-\delta}\Big(\int_{\mathbb{D}} |K_\lambda(z)|^{r(1-\delta)}\, W(z)^{-r\delta} dA(z)\Big)^{1/r} X_\rho(\lambda)\|P(a\circ\varphi_\lambda)\|_{\tilde{r}}$$
$$\leqslant Ch(\lambda)^{p'} X_\rho(\lambda)\|P(a\circ\varphi_\lambda)\|_{\tilde{r}}. \quad (3.10)$$

By the assumption (iii), $X_\rho(\lambda)\|P(a\circ\varphi_\lambda)\|_{\tilde{r}} \to 0$ as $\rho \to 1$. Thus, (3.8), (3.10) imply (3.7).

The proof for the implication (ii) $\Rightarrow$ (iii) is well-known (see [7], proof of Theorems 6 and 7): if (ii) is true for some $\tilde{q} \in [1,\infty)$, we trivially have (ii) for all $q \leqslant \tilde{q}$. For $q > \tilde{q}$ one uses the Hölder inequality,

$$\|P(a\circ\varphi_\lambda)\|_q^q \leqslant \|P(a\circ\varphi_\lambda)\|_{\tilde{q}}^{1/2}\, \|P(a\circ\varphi_\lambda)\|_s^{q-1/2}$$

with $s = \tilde{q}(2q-1)/(2\tilde{q}-1)$, and observes that the last factor is uniformly bounded with respect to λ, by the boundedness of the Bergman projection. Hence, (iii) holds true.

We finally consider the implication (i) $\Rightarrow$ (ii). We first assume that (i) holds and $1 < p \leqslant 2$ and denote $k_{\lambda,p} = W(\lambda)^{2-2/p} K_\lambda$ so that $\|k_{\lambda,p}\|_p \cong 1$. Thus,

$$\langle g, k_{\lambda,p}\rangle = W(\lambda)^{2-2/p}\langle g, K_\lambda\rangle = W(\lambda)^{2-2/p} g(\lambda)$$

for every $g \in H^\infty$, which implies that $k_{\lambda,p} \to 0$ weakly, due to the normalization of $k_{\lambda,p}$, since every $f \in A^p$ can be approximated by functions $g \in H^\infty$. Hence, $\|T_a k_{\lambda,p}\|_p^p \to 0$ as $\lambda \to 1$. On the other hand,

$$\begin{aligned}
\|T_a k_{\lambda,p}\|_p^p &= \int_{\mathbb{D}} \big|P(a\circ\varphi_\lambda)\circ\varphi_\lambda\big|^p |K_\lambda|^p W^{2p-2} dA \\
&= \int_{\mathbb{D}} \big|P(a\circ\varphi_\lambda)\circ\varphi_\lambda(\zeta)\big|^p \Big(\frac{1-|\lambda|^2}{|1-\bar{\lambda}\zeta|}\Big)^{2p} W(\zeta)^{-2} dA \\
&\geqslant C_p \int_{\mathbb{D}} \big|P(a\circ\varphi_\lambda)\circ\varphi_\lambda(\zeta)\big|^p \Big(\frac{1-|\lambda|^2}{|1-\bar{\lambda}\zeta|}\Big)^{4} W(\zeta)^{-2} dA \\
&\geqslant C_p \int_{\mathbb{D}} \big|P(a\circ\varphi_\lambda)\circ\varphi_\lambda\big|^p |k_\lambda|^2 dA = C_p\|P(a\circ\varphi_\lambda)\|_p^p
\end{aligned}$$

where we used $p \leqslant 2$, $(1-|\lambda|^2)/|1-\bar{\lambda}\zeta| \leqslant 2$ and (3.1). Hence, (ii) follows.

If $2 < p < \infty$, the operator $\mathbf{T}_a^* : A^{p'} \to A^{p'}$ is compact, by Schauder's theorem; here again $p' \in (1,2)$ is the dual exponent of p. By our Theorem 1.1, or Theorem 2.1 of [9], $\mathbf{T}_a^* = \mathbf{T}_{\bar{a}}$, and we get the condition (ii) for $\bar{a}$. This again implies the compactness of $\mathbf{T}_{\bar{a}} : A^2 \to A^2$, by what we have already proven. Then, $\mathbf{T}_a : A^2 \to A^2$ is compact by Schauder's theorem. By the above proof, we obtain (ii) for a. ■

Remark 3.4. If the symbol a is only in the space L^1_{loc}, the proof of Lemma 3.2 needs to be modified, since $T_a K_\lambda$ may not be defined directly by the integral formula and thus the use of the dominated convergence theorem cannot be justified. However, for every $0 < \rho < 1$, the symbol a_ρ belongs to L^1, the expression $T_{a_\rho} K_\lambda$ is defined by the conventional formula, and the identity (3.1),

$$T_{a_\rho} K_\lambda = K_\lambda P(a_\rho \circ \varphi_\lambda) \circ \varphi_\lambda. \tag{3.11}$$

holds by the existing proof of Lemma 3.2. By (1.4), the left hand side converges in L^p to $T_a K_\lambda$, as $\rho \to 1$. On the right we have

$$P(a_\rho \circ \varphi_\lambda)(z) = \int\limits_{\mathbb{D}_\rho} \frac{a \circ \varphi_\lambda(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta)$$

so that by Corollary 2.3, $P(a_\rho \circ \varphi_\lambda) \to P(a \circ \varphi_\lambda)$ in L^p. Thus also

$$K_\lambda P(a_\rho \circ \varphi_\lambda) \circ \varphi_\lambda \to K_\lambda P(a \circ \varphi_\lambda) \circ \varphi_\lambda$$

in L^p. The formula (3.1) follows from (3.11).

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