*Functiones et Approximatio*
*59.1 (2018), 129–139*
*doi: 10.7169/facm/1726*

# A NOTE ON COMPLEX SYMMETRIC COMPOSITION OPERATORS ON THE BERGMAN SPACE $A^2(\mathbb{D})$

Ted Eklund, Mikael Lindström, Paweł Mleczko

To the memory of our colleague
and friend Paweł Domański

**Abstract:** In this note complex symmetric composition operators $C_\varphi$ on the Bergman space $A^2(\mathbb{D})$ are studied. It is shown that if an operator $C_\varphi$ is complex symmetric on $A^2(\mathbb{D})$ then either $\varphi\colon \mathbb{D} \to \mathbb{D}$ has a Denjoy–Wolff point in $\mathbb{D}$ or is an elliptic automorphism of the disc. Moreover in the latter case $\varphi$ is either a rotation or has an order smaller than six.

**Keywords:** complex symmetric operator, composition operator, Denjoy-Wolff point, Bergman space.

## 1. Introduction

The space of analytic functions on the open unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ is denoted by $H(\mathbb{D})$. Every analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ induces a *composition operator* $C_\varphi f = f \circ \varphi$ on $H(\mathbb{D})$. Operators of this type have been considered on many spaces of analytic functions for several decades, starting from the papers on Hardy spaces $H^p(\mathbb{D})$ in the beginning of the 20th century. One of the main lines of research is to study the interplay between properties of the composition operator $C_\varphi$ and its generating function $\varphi$. We refer the reader to the monographs [4, 11] for more information on this topic.

A new class of Hilbert space operators, called *complex symmetric operators*, was recently introduced and studied in [7]. In [3] it was proved that if $\varphi$ is an automorphism of the disc which is not a rotation or elliptic of order three, then $C_\varphi\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is complex symmetric if and only if $\varphi = \varphi_\alpha$ for some $\alpha \in \mathbb{D} \setminus \{0\}$, where $\varphi_\alpha(z) = (\alpha - z)/(1 - \overline{\alpha} z)$. However, the question of which

The third author's research was supported by the National Science Centre, Poland (project no. 2015/17/B/ST1/00064). Part of the research was done while the third author visited Åbo Akademi University.
**2010 Mathematics Subject Classification:** primary: 47B33; secondary: 47B32, 47B38

composition operators are complex symmetric on the Hardy–Hilbert space $H^2(\mathbb{D})$ is still not fully answered. We refer the reader to the above papers and to article [6] and references therein.

In this note we study complex symmetric composition operators on the Bergman space $A^2(\mathbb{D})$. Our main results are contained in Theorem 2, where we show that a complex symmetric composition operator $C_\varphi$ on the Bergman space $A^2(\mathbb{D})$ needs either to be induced by an elliptic automorphism $\varphi\colon \mathbb{D} \to \mathbb{D}$ or has a Denjoy–Wolff point in $\mathbb{D}$, and Theorems 8 and 10, where we prove that if $\varphi$ is an elliptic automorphism (but not rotation) of order at least six (or infinite), then $C_\varphi$ is not complex symmetric. It should be mentioned that in general we follow the ideas used for the Hardy space $H^2(\mathbb{D})$ in [3], but still in the Bergman case some new facts are needed and the calculations are more involved.

## 2. Preliminaries

Let us recall that a bounded linear operator $T\colon \mathcal{H} \to \mathcal{H}$ on a separable Hilbert space $\mathcal{H}$ is called *complex symmetric* if $T = CT^*C$ for some conjugate-linear operator $C\colon \mathcal{H} \to \mathcal{H}$ satisfying $C^2 = I$ and $\langle Cf, Cg\rangle_{\mathcal{H}} = \langle f, g\rangle_{\mathcal{H}}$ for all $f, g \in \mathcal{H}$. An operator $C\colon \mathcal{H} \to \mathcal{H}$ with the above mentioned properties is called a *conjugation* (see [7]).

The *Bergman space* $A^2 = A^2(\mathbb{D})$ is the separable Hilbert space consisting of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^2} = \left(\int_{\mathbb{D}} |f(z)|^2 dA(z)\right)^2 < \infty,$$

where $dA(z)$ is the normalized area measure on $\mathbb{D}$ and the *Hardy space* $H^2 = H^2(\mathbb{D})$, consists of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^2} = \sup_{r\in[0,1)} \frac{1}{2\pi}\left(\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\right)^2 < \infty.$$

The inner product in $A^2$ is defined as (cf. [4])

$$\langle f, g\rangle_{A^2} = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z) = \sum_{n=0}^{\infty} \frac{\hat{f}(n)\overline{\hat{g}(n)}}{n+1}, \tag{1}$$

where $f, g \in A^2$ have series expansions

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n, \qquad z \in \mathbb{D}.$$

The *reproducing kernel* $K_\alpha$ on $A^2$ is given by

$$K_\alpha(z) = \frac{1}{(1-\overline{\alpha}z)^2}, \qquad \alpha, z \in \mathbb{D}, \tag{2}$$

and has the property that $\langle f, K_\alpha\rangle_{A^2} = f(\alpha)$ for every $f \in A^2$ (see [4, p. 17]). For more information on Bergman spaces we refer to the books [4, 5].

A function $f \in A^2$ is said to be *cyclic* in $A^2$ if the closed linear span of $f, zf, z^2f, \dots$ is all of $A^2$, and $f \in A^2$ is called $A^2$-*outer*, if every $g \in A^2$ such that $\|gp\|_{A^2} \leqslant \|fp\|_{A^2}$ holds for all polynomials $p$ has the property $|g(0)| \leqslant |f(0)|$. By Theorem 7.2 in [9] the cyclic elements in $A^2$ are known to be precisely the $A^2$-outer functions in $A^2$. Since the inequality $\|f\|_{A^2} \leqslant \|f\|_{H^2}$ holds for every $f \in H^2$, any cyclic element in $H^2$ is also cyclic in $A^2$.

## 3. Complex symmetric composition operators on the Bergman space

We will study complex symmetric composition operators on $A^2$. It is well known that $C_\varphi\colon A^2 \to A^2$ is a bounded operator for every analytic self-map $\varphi$ of the unit disc. The article [6] contains results on complex symmetric composition operators on the Bergman space. In fact from [6, Proposition 2.9] it follows that for any $\alpha \in \mathbb{C}$, $|\alpha| \leqslant 1$, if $\varphi(z) = \alpha z$, $z \in \mathbb{D}$, then the operator $C_\varphi$ is complex symmetric on $A^2$ and even a normal operator. See also the comment after Theorem 2 for a direct argument.

For $|\alpha| < 1$, let $\varphi_\alpha$ denote the automorphism of $\mathbb{D}$ given by

$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \qquad z \in \mathbb{D}. \tag{3}$$

A disc automorphism $\varphi$ is called *elliptic* if there exists $\lambda \in \partial\mathbb{D}$ such that

$$\varphi = \varphi_\alpha \circ (\lambda \varphi_\alpha), \qquad |\alpha| < 1. \tag{4}$$

Below we obtain that if $C_\varphi$ is a complex symmetric operator on $A^2$, then $\varphi$ is either an elliptic automorphism of the unit disc or has a Denjoy–Wolff point in $\mathbb{D}$. To do this we need the following lemma, where we use the relationship $\overline{\operatorname{Ran} T} = (\operatorname{Ker} T^*)^\perp$, which is valid for any operator $T\colon \mathcal{H} \to \mathcal{H}$. From this if follows that $T$ has dense range if and only if $\operatorname{Ker} T^* = \{\bar{0}\}$. For a $H^2$ version of the result see [2, Proposition 2.1].

**Lemma 1.** *Suppose that the analytic self-map $\varphi$ of $\mathbb{D}$ has a Denjoy–Wolff point in $\partial\mathbb{D}$. If $\lambda$ is an eigenvalue of $C_\varphi\colon A^2 \to A^2$ with an $A^2$-outer function as a corresponding eigenfunction, then $C_\varphi - \lambda I$ has dense range.*

**Proof.** By assumption $C_\varphi g = \lambda g$ for some nonzero $A^2$-outer function $g \in A^2$. By Theorem 7.2 in [9] the function $g$ is cyclic in $A^2$. The operator $C_\varphi - \lambda I$ has dense range if and only if $\operatorname{Ker}(C_\varphi^* - \bar{\lambda} I) = \{\bar{0}\}$. In order to reach a contradiction assume that $\bar{\lambda}$ is an eigenvalue of $C_\varphi^*$. Thus $C_\varphi^* h = \bar{\lambda} h$ for some nonzero $h \in A^2$. By assumption $\varphi$ has a Denjoy–Wolff point $\omega \in \partial\mathbb{D}$. For any integers $n, k \geqslant 0$,

$$\begin{aligned}
\lambda^k \big\langle z^n(\omega - z) g(z), h \big\rangle_{A^2} &= \big\langle z^n(\omega - z) g(z), \bar{\lambda}^k h \big\rangle_{A^2} \\
&= \big\langle z^n(\omega - z) g(z), \big(C_\varphi^*\big)^k h \big\rangle_{A^2} \\
&= \big\langle C_{\varphi_k}\big(z^n(\omega - z) g(z)\big), h \big\rangle_{A^2} \\
&= \big\langle \varphi_k^n(\omega - \varphi_k) g \circ \varphi_k, h \big\rangle_{A^2} \\
&= \lambda^k \big\langle \varphi_k^n(\omega - \varphi_k) g, h \big\rangle_{A^2},
\end{aligned}$$

where $\varphi_n := \varphi \circ \varphi_{n-1}$ for $n \geqslant 1$ and $\varphi_0 := \mathrm{Id}\colon \mathbb{D} \to \mathbb{D}$. This shows that

$$\big\langle z^n(\omega - z)g(z), h\big\rangle_{A^2} = \big\langle \varphi_k^n(\omega - \varphi_k)g, h\big\rangle_{A^2}. \tag{5}$$

Since $\big|\varphi_k^n(\omega-\varphi_k)g\bar{h}\big| \leqslant 2|gh| \in L^1(\mathbb{D})$ and the iterate sequence $\{\varphi_k\}_{k=0}^{\infty}$ converges pointwise to $\omega$ on $\mathbb{D}$ (even uniformly on compact subsets of the disc, see [4, Theorem 2.51]), we can use the Lebesgue Dominated Convergence Theorem and obtain from the equality (5) that

$$\big\langle z^n(\omega - z)g(z), h\big\rangle_{A^2} = \lim_{k\to\infty} \big\langle \varphi_k^n(\omega - \varphi_k)g, h\big\rangle_{A^2} = 0, \qquad n \geqslant 0. \tag{6}$$

The function $z \mapsto \omega - z$ belongs to $H^\infty$ and is outer in $H^2$, and consequently cyclic in $H^2$ by Beurling's theorem. Therefore the function $z \mapsto (\omega - z)g(z)$ is cyclic in $A^2$ by [5, Theorem 8.3.2], so the linear span $\mathcal{S}$ of the set of functions $\{z \mapsto z^n(\omega - z)g(z)\}_{n=0}^{\infty}$ is dense in $A^2$. This means that if $f \in A^2$ then there exists a sequence $\{f_k\}_{k=0}^{\infty} \subset \mathcal{S}$ converging to $f$ in the norm of $A^2$. This shows that $\langle f, h\rangle_{A^2} = \lim_{k\to\infty}\langle f_k, h\rangle_{A^2} = 0$ by (6), and we obtain the contradiction $h \equiv 0$. ■

The proof of the promised theorem heavily relies on Lemma 1 and mimics the steps of [3, Proposition 2.1], so no proof is given. Note that the function 1 is cyclic in $A^2$, since the set of polynomials is dense in $A^2$.

**Theorem 2.** *If the composition operator $C_\varphi\colon A^2 \to A^2$ is complex symmetric then $\varphi$ is either an elliptic automorphism of the unit disc or has a Denjoy–Wolff point in $\mathbb{D}$.*

In the rest of the paper we will only analyze elliptic automorphisms $\varphi_\alpha$ of $\mathbb{D}$ that induce complex symmetric operators $C_\varphi$ on the Bergman space $A^2$. The case when $\alpha = 0$, that is $\varphi_\alpha$ is a rotation, follows from [6, Proposition 2.9]. Indeed, if $\alpha = 0$ then it is easy to see that $C_\varphi^* = C_\psi$, where $\psi(z) = \overline{\lambda}z$. Thus $C_\varphi\colon A^2(\mathbb{D}) \to A^2(\mathbb{D})$ is a unitary operator, and hence complex symmetric since all normal operators have this property. To resolve the remaining cases $\alpha \in \mathbb{D} \setminus \{0\}$ we need some additional results.

Let $\varphi$ be an automorphism of the form

$$\varphi = \varphi_\alpha \circ (\lambda\varphi_\alpha), \qquad |\alpha| < 1.$$

If $N$ is the smallest positive integer such that $\lambda^N = 1$, then $\varphi$ is said to be of *finite order* $N$. If no such integer exists, then $\varphi$ is said to have *infinite order.*

**Lemma 3.** *Consider the multiplication operator $M_{\mathrm{Id}}\colon A^2 \to A^2$ with symbol* $\mathrm{Id}(z) := z$. *The adjoint operator acts on any function $f \in A^2$ with corresponding series expansion $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ as*

$$M_{\mathrm{Id}}^* f(z) = \sum_{n=0}^{\infty} \frac{n+1}{n+2}\hat{f}(n+1)z^n, \qquad z \in \mathbb{D}, \tag{7}$$

*and in particular for integers $m \geqslant 0$ we have*

$$M_{\mathrm{Id}}^* z^m = \begin{cases} 0, & m = 0 \\ \frac{m}{m+1} z^{m-1}, & m \geqslant 1. \end{cases} \tag{8}$$

**Proof.** By using the latter form of the Bergman inner product in (1) we obtain that

$$\begin{aligned} \frac{\hat{f}(n+1)}{n+2} &= \langle f, z^{n+1} \rangle_{A^2} = \langle f, M_{\mathrm{Id}} z^n \rangle_{A^2} \\ &= \langle M_{\mathrm{Id}}^* f, z^n \rangle_{A^2} = \frac{\widehat{M_{\mathrm{Id}}^* f}(n)}{n+1} \end{aligned}$$

and hence

$$\widehat{M_{\mathrm{Id}}^* f}(n) = \frac{n+1}{n+2} \hat{f}(n+1).$$

This proves the equality (7), from which the formula (8) follows. ∎

**Lemma 4.** *The sequence $\{e_n\}_{n=0}^\infty$ of functions $e_n := K_\alpha \varphi_\alpha^n$ is orthogonal in $A^2$, and $\|e_n\|_{A^2} = \frac{1}{(1-|\alpha|^2)\sqrt{n+1}}$.*

**Proof.** Choose arbitrary integers $n, m \geqslant 0$. The reproducing kernel $K_\alpha$ given in the equality (2) is related to the derivative of $\varphi_\alpha$ in the following manner

$$\varphi_\alpha'(z) = \frac{|\alpha|^2 - 1}{(1 - \overline{\alpha} z)^2} = (|\alpha|^2 - 1) K_\alpha(z).$$

After substituting $w = \varphi_\alpha(z)$ we obtain

$$\begin{aligned} \langle e_n, e_m \rangle_{A^2} &= \int_{\mathbb{D}} e_n(z) \overline{e_m(z)} dA(z) = \int_{\mathbb{D}} \varphi_\alpha(z)^n \overline{\varphi_\alpha(z)^m} |K_\alpha(z)|^2 dA(z) \\ &= \frac{1}{(|\alpha|^2 - 1)^2} \int_{\mathbb{D}} \varphi_\alpha(z)^n \overline{\varphi_\alpha(z)^m} |\varphi_\alpha'(z)|^2 dA(z) \\ &= \frac{1}{(|\alpha|^2 - 1)^2} \int_{\mathbb{D}} w^n \overline{w^m} dA(w) = \frac{1}{(|\alpha|^2 - 1)^2} \langle \mathrm{Id}^n, \mathrm{Id}^m \rangle_{A^2} \\ &= \frac{\delta_{n,m}}{(|\alpha|^2 - 1)^2 (n+1)}. \end{aligned}$$

The last equality with the Kronecker delta function $\delta_{n,m}$ holds in view of the latter form of the Bergman inner product in (1). This completes the proof. ∎

For the Hardy space version of the following result see [3, Lemma 2.2].

**Lemma 5.** *Let $\alpha \in \mathbb{D} \setminus \{0\}$, consider $C_{\varphi_\alpha} \colon A^2 \to A^2$ as an operator on the Bergman space $A^2$ and define $v_n := C_{\varphi_\alpha}^* z^n$ for integers $n \geqslant 0$. Then $v_n \perp v_m$ if and only if $|n - m| \geqslant 3$.*

**Proof.** According to [10, Theorem 2], the adjoint operator of $C_{\varphi_\alpha}$ takes the form

$$C^*_{\varphi_\alpha} = M_{K_\alpha} C_{\varphi_\alpha} M^*_{1/K_\alpha}. \tag{9}$$

Since

$$\frac{1}{K_\alpha(z)} = (1-\overline{\alpha}z)^2 = 1 - 2\overline{\alpha}z + \overline{\alpha}^2 z^2,$$

the following equations hold

$$M_{1/K_\alpha} = I - 2\overline{\alpha}M_{\mathrm{Id}} + \overline{\alpha}^2 M_{\mathrm{Id}^2} = I - 2\overline{\alpha}M_{\mathrm{Id}} + \overline{\alpha}^2 (M_{\mathrm{Id}})^2.$$

From the above and the equality (9) it follows that

$$C^*_{\varphi_\alpha} = M_{K_\alpha} C_{\varphi_\alpha}\big(I - 2\alpha M^*_{\mathrm{Id}} + \alpha^2 (M^*_{\mathrm{Id}})^2\big).$$

Applying this representation and the formula (8) on $v_n = C^*_{\varphi_\alpha} z^n$, we obtain $v_0 = K_\alpha$, $v_1 = K_\alpha(\varphi_\alpha - \alpha)$ and for integers $n \geqslant 2$:

$$\begin{aligned}
v_n &= M_{K_\alpha} C_{\varphi_\alpha}\big(z^n - 2\alpha M^*_{\mathrm{Id}} z^n + \alpha^2 (M^*_{\mathrm{Id}})^2 z^n\big)\\
&= M_{K_\alpha} C_{\varphi_\alpha}\big(z^n - 2\alpha \tfrac{n}{n+1} z^{n-1} + \alpha^2 \tfrac{n}{n+1} M^*_{\mathrm{Id}} z^{n-1}\big)\\
&= M_{K_\alpha} C_{\varphi_\alpha}\big(z^n - 2\alpha \tfrac{n}{n+1} z^{n-1} + \alpha^2 \tfrac{n}{n+1}\tfrac{n-1}{n} z^{n-2}\big)\\
&= K_\alpha \varphi_\alpha^n - 2\alpha \tfrac{n}{n+1} K_\alpha \varphi_\alpha^{n-1} + \alpha^2 \tfrac{n-1}{n+1} K_\alpha \varphi_\alpha^{n-2}.
\end{aligned}$$

The above results can be summarized in terms of the functions $e_n := K_\alpha \varphi_\alpha^n$ (consult Lemma 4, where it was shown that $\{e_n\}$ are orthogonal in $A^2$) as

$$\begin{cases} v_0 = e_0 \\ v_1 = e_1 - \alpha e_0 \\ v_n = e_n - 2\alpha\tfrac{n}{n+1} e_{n-1} + \alpha^2 \tfrac{n-1}{n+1} e_{n-2}, \quad n \geqslant 2. \end{cases} \tag{10}$$

Assume now that $n, m \geqslant 2$. By the last formula of (10), we have

$$\begin{aligned}
\langle v_n, v_m\rangle_{A^2} &= \Big\langle e_n - 2\alpha\tfrac{n}{n+1}e_{n-1} + \alpha^2\tfrac{n-1}{n+1}e_{n-2},\, e_m - 2\alpha\tfrac{m}{m+1}e_{m-1} + \alpha^2\tfrac{m-1}{m+1}e_{m-2}\Big\rangle_{A^2}\\
&= \langle e_n, e_m\rangle_{A^2} - 2\overline{\alpha}\tfrac{m}{m+1}\langle e_n, e_{m-1}\rangle_{A^2} + \overline{\alpha}^2\tfrac{m-1}{m+1}\langle e_n, e_{m-2}\rangle_{A^2}\\
&\quad - 2\alpha\tfrac{n}{n+1}\langle e_{n-1}, e_m\rangle_{A^2} + 4|\alpha|^2\tfrac{n}{n+1}\tfrac{m}{m+1}\langle e_{n-1}, e_{m-1}\rangle_{A^2}\\
&\quad - 2|\alpha|^2\overline{\alpha}\tfrac{n}{n+1}\tfrac{m-1}{m+1}\langle e_{n-1}, e_{m-2}\rangle_{A^2} + \alpha^2\tfrac{n-1}{n+1}\langle e_{n-2}, e_m\rangle_{A^2}\\
&\quad - 2|\alpha|^2\alpha\tfrac{n-1}{n+1}\tfrac{m}{m+1}\langle e_{n-2}, e_{m-1}\rangle_{A^2} + |\alpha|^4\tfrac{n-1}{n+1}\tfrac{m-1}{m+1}\langle e_{n-2}, e_{m-2}\rangle_{A^2}.
\end{aligned}$$

It follows immediately from the above expression that $v_n \perp v_m$ if $|n-m| \geqslant 3$, since the sequence $\{e_n\}_{n=0}^{\infty}$ is orthogonal. It remains to check the case when

$|n-m| < 3$, that is when $m = n-2$, $m = n-1$, $m = n$, $m = n+1$ and $m = n+2$. The corresponding inner products can be computed from the above general expression (10) by again using the orthogonality of $\{e_n\}_{n=0}^{\infty}$

$$\begin{aligned}
\langle v_n, v_{n-2}\rangle_{A^2} &= \alpha^2 \tfrac{n-1}{n+1}\|e_{n-2}\|_{A^2}^2 \\
\langle v_n, v_{n-1}\rangle_{A^2} &= -2\alpha\Big(\tfrac{n}{n+1}\|e_{n-1}\|_{A^2}^2 + |\alpha|^2 \tfrac{(n-1)^2}{(n+1)n}\|e_{n-2}\|_{A^2}^2\Big) \\
\langle v_n, v_n\rangle_{A^2} &= \|e_n\|_{A^2}^2 + 4|\alpha|^2\big(\tfrac{n}{n+1}\big)^2\|e_{n-1}\|_{A^2}^2 + |\alpha|^4\big(\tfrac{n-1}{n+1}\big)^2\|e_{n-2}\|_{A^2}^2 \\
\langle v_n, v_{n+1}\rangle_{A^2} &= -2\overline{\alpha}\Big(\tfrac{n+1}{n+2}\|e_n\|_{A^2}^2 + |\alpha|^2 \tfrac{n^2}{(n+1)(n+2)}\|e_{n-1}\|_{A^2}^2\Big) \\
\langle v_n, v_{n+2}\rangle_{A^2} &= \overline{\alpha}^2 \tfrac{n+1}{n+3}\|e_n\|_{A^2}^2.
\end{aligned}$$

None of these inner products can be zero since $\alpha \neq 0$, so for integers $n, m \geqslant 2$ it holds that $v_n \perp v_m$ if and only if $|n-m| \geqslant 3$. Again, from (10) it can also be seen that $v_0 \perp v_n$ if and only if $n \geqslant 3$ and $v_1 \perp v_n$ if and only if $n \geqslant 4$, so the proof is complete. ■

**Remark 6.** It follows from the above proof that $\|v_0\|_{A^2} = \|e_0\|_{A^2}$, $\|v_1\|_{A^2} = \big(\|e_1\|_{A^2}^2 + |\alpha|^2\|e_0\|_{A^2}^2\big)^{\frac{1}{2}}$ and

$$\|v_n\|_{A^2} = \Big(\|e_n\|_{A^2}^2 + 4|\alpha|^2\big(\tfrac{n}{n+1}\big)^2\|e_{n-1}\|_{A^2}^2 + |\alpha|^4\big(\tfrac{n-1}{n+1}\big)^2\|e_{n-2}\|_{A^2}^2\Big)^{\frac{1}{2}}$$

for $n \geqslant 2$. Hence $\|v_n\|_{A^2} \neq 0$ for every $n \in \mathbb{N}$ by Lemma 4.

The following fact was already used in [3, p. 108]. Therefore we leave out the proof.

**Lemma 7.** *If $T\colon \mathcal{H} \to \mathcal{H}$ is a complex symmetric operator with conjugation $C$ and the equation $C(T-\lambda I) = (T^* - \overline{\lambda} I)C$ holds for some $\lambda \in \mathbb{C}$, then*

$$f \in \operatorname{Ker}(T-\lambda I) \iff Cf \in \operatorname{Ker}(T^* - \overline{\lambda} I).$$

Recall, that an operator $T\colon \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is called *cyclic* if there exists a vector $x \in \mathcal{H}$ such that the *orbit*

$$\operatorname{Orb}(T, x) = \big\{T^n x : n \in \mathbb{N}\big\}$$

has dense linear span in $\mathcal{H}$.

The proof of the following result is based on the approach taken in [3, Proposition 3.1] for the case of $C_\varphi\colon H^2 \to H^2$.

**Theorem 8.** *Suppose $\varphi$ is an elliptic automorphism of infinite order and is not a rotation. Then $C_\varphi\colon A^2 \to A^2$ is not complex symmetric.*

**Proof.** The elliptic automorphism $\varphi$ is of the form (4), where $\alpha \in \mathbb{D} \setminus \{0\}$ and $\lambda$ is not a root of unity, so $C_\varphi = C_{\varphi_\alpha} C_{\lambda z} C_{\varphi_\alpha}$. We begin by showing that the adjoint operator

$$C_\varphi^* = C_{\varphi_\alpha}^* C_{\overline{\lambda} z} C_{\varphi_\alpha}^* \tag{11}$$

is cyclic. Since

$$\left(C_{\varphi_\alpha}^*\right)^2 = C_{\varphi_\alpha \circ \varphi_\alpha}^* = C_{\mathrm{Id}}^* = I, \tag{12}$$

we see from the equality (11) that

$$\left(C_\varphi^*\right)^n = C_{\varphi_\alpha}^* C_{\overline{\lambda} z}^n C_{\varphi_\alpha}^*, \quad n \in \mathbb{N}. \tag{13}$$

Choose some $\beta \in \mathbb{D} \setminus \{0\}$ and notice that

$$C_{\overline{\lambda} z}^n K_\beta(z) = K_\beta\left(\overline{\lambda}^n z\right) = K_{\lambda^n \beta}(z). \tag{14}$$

Now, using equations (13) and (14) we conclude that

$$\mathrm{Orb}(C_\varphi^*, C_{\varphi_\alpha}^* K_\beta) = \left\{C_{\varphi_\alpha}^* K_{\lambda^n \beta} : n \in \mathbb{N}\right\}$$

has dense linear span in $A^2$ since this is the case for the set

$$\left\{K_{\lambda^n \beta} : n \in \mathbb{N}\right\}.$$

This shows that $C_\varphi^*$ is cyclic. It is known that if an operator is cyclic, then its adjoint has simple eigenvalues (see [1, Proposition 2.7]). Thus $C_\varphi \colon A^2 \to A^2$ has simple eigenvalues.

In order to reach a contradiction suppose that $C_\varphi \colon A^2 \to A^2$ is complex symmetric with conjugation $C$. If we define $v_n := C_{\varphi_\alpha}^* z^n$ as in Lemma 5, then by formulas (11) and (12) we have that $v_n \in \mathrm{Ker}(C_\varphi^* - \overline{\lambda}^n I)$ and

$$\begin{aligned}\left(C_\varphi^* - \overline{\lambda}^n I\right) v_n &= C_\varphi^* C_{\varphi_\alpha}^* z^n - \overline{\lambda}^n C_{\varphi_\alpha}^* z^n \\ &= C_{\varphi_\alpha}^* C_{\overline{\lambda} z} (C_{\varphi_\alpha}^*)^2 z^n - \overline{\lambda}^n C_{\varphi_\alpha}^* z^n \\ &= \overline{\lambda}^n C_{\varphi_\alpha}^* z^n - \overline{\lambda}^n C_{\varphi_\alpha}^* z^n = 0.\end{aligned}$$

Furthermore by the complex symmetry

$$C\left(C_\varphi^* - \overline{\lambda}^n I\right) = C C_\varphi^* - \lambda^n C = C_\varphi C - \lambda^n C = (C_\varphi - \lambda^n I) C, \tag{15}$$

and it follows from Lemma 7 that $C v_n \in \mathrm{Ker}(C_\varphi - \lambda^n I)$ for every $n \in \mathbb{N}$, which means that $C v_n$ is an eigenfunction of $C_\varphi - \lambda^n I$. Indeed,

$$\|C v_n\|_{A^2}^2 = \langle C v_n, C v_n \rangle_{A^2} = \langle v_n, v_n \rangle_{A^2} = \|v_n\|_{A^2}^2 \neq 0,$$

as noted in Remark 6. But we also have that $\varphi_\alpha^n \in \mathrm{Ker}(C_\varphi - \lambda^n I)$:

$$\begin{aligned}\left(C_\varphi - \lambda^n I\right) \varphi_\alpha^n &= (\varphi_\alpha \circ \varphi)^n - \lambda^n \varphi_\alpha^n \\ &= (\varphi_\alpha \circ \varphi_\alpha \circ (\lambda \varphi_\alpha))^n - \lambda^n \varphi_\alpha^n \\ &= \lambda^n \varphi_\alpha^n - \lambda^n \varphi_\alpha^n = 0.\end{aligned}$$

Since $C_\varphi\colon A^2 \to A^2$ has simple eigenvalues, the function $Cv_n$ is a scalar multiple of $\varphi_\alpha^n$, say $Cv_n = \mu_n \varphi_\alpha^n$ for some nonzero constant $\mu_n$. Now using Lemma 5 we get that

$$\begin{aligned} 0 &= \langle v_0, v_3 \rangle_{A^2} = \langle Cv_0, Cv_3 \rangle_{A^2} \\ &= \mu_0 \overline{\mu_3} \langle 1, \varphi_\alpha^3 \rangle_{A^2} = \mu_0 \overline{\mu_3} \overline{\varphi_\alpha(0)}^3 \\ &= \mu_0 \overline{\mu_3} \overline{\alpha}^3, \end{aligned}$$

which implies that $\alpha = 0$, and so $\varphi$ is a rotation. This contradicts the assumption and the proof is complete. ■

**Lemma 9.** *Suppose $\varphi = \varphi_\alpha \circ (\lambda \varphi_\alpha)$ is an elliptic automorphism of finite order $N$ that is not a rotation, and define $V_n := \operatorname{Ker}(C_\varphi^* - \overline{\lambda}^n I)$ for $n \in \mathbb{N}$. Then $V_0 \perp V_3$ if and only if $N \geqslant 6$.*

**Proof.** Define $v_n := C_\varphi^* z^n$ as in Lemma 5 and recall that $v_n \in V_n$ for every $n \in \mathbb{N}$ as shown in the proof of Theorem 8. We first prove that $V_0 \not\perp V_3$ when $N < 6$. If $N = 1$ then $\varphi$ is a rotation, so this case needs not to be considered. If $N = 2$, then $\lambda^2 = 1$ and

$$V_2 = \operatorname{Ker}(C_\varphi^* - \overline{\lambda}^2 I) = \operatorname{Ker}(C_\varphi^* - I) = V_0.$$

Hence $v_2 \in V_2 = V_0$ and $v_3 \in V_3$. But $v_2 \not\perp v_3$ by Lemma 5 so $V_0 \not\perp V_3$ when $N = 2$. For the other cases we obtain similarly:

$$\begin{aligned} V_0 &= V_3, & N &= 3 \\ V_0 &= V_4, & N &= 4 \\ V_0 &= V_5, & N &= 5, \end{aligned}$$

and another usage of Lemma 5 shows that $V_0 \not\perp V_3$ for these cases.

Now suppose that $N \geqslant 6$. Since

$$\operatorname{Ker}(C_{\overline{\lambda} z} - \overline{\lambda}^n I) = \overline{\operatorname{span}}\{z^{kN+n}\}_{k \in \mathbb{N}}$$

we have that

$$\begin{aligned} f \in V_n &\Leftrightarrow (C_\varphi^* - \overline{\lambda}^n I) f = \overline{0} \Leftrightarrow C_{\varphi_\alpha}^* C_{\overline{\lambda} z} C_{\varphi_\alpha}^* f - \overline{\lambda}^n f = \overline{0} \\ &\Leftrightarrow \ C_{\overline{\lambda} z} C_{\varphi_\alpha}^* f - \overline{\lambda}^n C_{\varphi_\alpha}^* f = \overline{0} \\ &\Leftrightarrow C_{\varphi_\alpha}^* f \in \operatorname{Ker}(C_{\overline{\lambda} z} - \overline{\lambda}^n I) = \overline{\operatorname{span}}\{z^{kN+n}\}_{k \in \mathbb{N}} \\ &\Leftrightarrow f \in \overline{\operatorname{span}}\{C_{\varphi_\alpha}^* z^{kN+n}\}_{k \in \mathbb{N}} = \overline{\operatorname{span}}\{v_{kN+n}\}_{k \in \mathbb{N}}, \end{aligned}$$

and thus $V_n = \overline{\operatorname{span}}\{v_{kN+n}\}_{k \in \mathbb{N}}$. Now consider $v_{kN} \in V_0$ and $v_{jN+3} \in V_3$ for any $k, j \in \mathbb{N}$. Since $N \geqslant 6$ it holds that

$$\big| kN - (jN + 3) \big| = \big| (k - j)N - 3 \big| \geqslant 3,$$

so Lemma 5 gives that $v_{kN} \perp v_{jN+3}$, and hence $V_0 \perp V_3$. ■

Below we show that, as in the case of the Hardy space $H^2$ (see [3, Proposition 3.3]), the class of disc self-maps which induce complex symmetric composition operators on the Bergman space $A^2$ is quite sparse.

**Theorem 10.** *Suppose $\varphi$ is an elliptic automorphism of finite order $N \geqslant 6$ and is not a rotation. Then $C_\varphi \colon A^2 \to A^2$ is not complex symmetric.*

**Proof.** In order to reach a contradiction assume that $C_\varphi \colon A^2 \to A^2$ is complex symmetric, with a conjugation $C$. By the formula (15) and Lemma 7 it follows that

$$f \in V_n := \operatorname{Ker}(C_\varphi^* - \overline{\lambda}^n I) \Longleftrightarrow Cf \in \operatorname{Ker}(C_\varphi - \lambda^n I).$$

Now, using the property $C^2 = I$ we see that $C$ maps $V_n$ onto $\operatorname{Ker}(C_\varphi - \lambda^n I)$ for every $n \in \mathbb{N}$. Thus if $f \in \operatorname{Ker}(C_\varphi - I)$ and $g \in \operatorname{Ker}(C_\varphi - \lambda^3 I)$ then there exist functions $u \in V_0$ and $w \in V_3$ such that $f = Cu$ and $g = Cw$. Hence since $N \geqslant 6$ from Lemma 9 it follows

$$\langle f, g\rangle_{A^2} = \langle Cu, Cw\rangle_{A^2} = \langle u, w\rangle_{A^2} = 0.$$

This shows that

$$\operatorname{Ker}(C_\varphi - I) \perp \operatorname{Ker}(C_\varphi - \lambda^3 I),$$

and in particular $1 \perp \varphi_\alpha^3$ because $\varphi_\alpha^n \in \operatorname{Ker}(C_\varphi - \lambda^n I)$ for every $n \in \mathbb{N}$ (cf. the proof of Theorem 8). This gives the contradiction $\alpha = 0$ and the proof is complete. ∎

After summarizing what has been proven we see that if the composition operator $C_\varphi \colon A^2 \to A^2$ is complex symmetric then $\varphi$ has a Denjoy–Wolff point in the disc $\mathbb{D}$, is a rotation (and in this case $C_\varphi \colon A^2 \to A^2$ is a normal operator) or is an elliptic automorphism of finite order $N = 2, 3, 4$ or $5$. Using the following result from [8] we can solve the case $N = 2$.

**Theorem 11 ([8, Theorem 2]).** *If an operator $T \colon \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ satisfies $p(T) = 0$ for some polynomial of degree $2$ or less, then $T$ is complex symmetric.*

**Theorem 12.** *Suppose $\varphi = \varphi_\alpha \circ (\lambda\varphi_\alpha)$ is an elliptic automorphism of order two. Then $C_\varphi \colon A^2 \to A^2$ is complex symmetric.*

**Proof.** The $n$-th iterate of $C_\varphi$ can be written as in the formula (13)

$$C_\varphi^n = C_{\varphi_\alpha} C_{\lambda z}^n C_{\varphi_\alpha} = C_{\varphi_\alpha} C_{\lambda^n z} C_{\varphi_\alpha}, \qquad n \in \mathbb{N}.$$

Using this with $n = 2$ and recalling that $\lambda^2 = 1$, we see that $C_\varphi$ satisfies a polynomial equation of order two. Indeed,

$$C_\varphi^2 = C_{\varphi_\alpha}^2 = I,$$

so $C_\varphi \colon A^2 \to A^2$ is complex symmetric by Theorem 11. ∎

## References

[1] P. S. Bourdon and J. H. Shapiro, *Cyclic phenomena for composition operators*, Mem. Amer. Math. Soc. **125** (1997).

[2] P. S. Bourdon, *Spectra of some composition operators and associated weighted composition operators*, J. Operator Theory **67**(2) (2012), 537–560.

[3] P. S. Bourdon and S. Waleed Noor, *Complex symmetry of invertible composition operators*, J. Math. Anal. Appl. **429** (2015), 105–110.

[4] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.

[5] P. Duren and A. Schuster, *Bergman Spaces*, American Mathematical Society, Providence, RI 2004.

[6] S. R. Garcia and C. Hammond, *Which weighted composition operators are complex symmetric?*, Concrete operators, spectral theory, operators in harmonic analysis and approximation, Oper. Theory Adv. Appl. **236**, Birkhäuser/Springer, Basel 2014, 171–179.

[7] S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), 1285–1315.

[8] S. R. Garcia and W. R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362** (2010), no. 11, 6065–6077.

[9] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces*, Graduate Texts in Mathematics 199, Springer-Verlag, New York, 2000.

[10] P. R. Hurst, *Relating composition operators on different weighted Hardy spaces*, Arch. Math. **68** (1997) 503–513.

[11] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, 1993.

**Addresses:** Ted Eklund and Mikael Lindström: Department of Mathematics, Åbo Akademi University, FI-20500 Åbo, Finland;
Paweł Mleczko: Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, Umultowska 87, 61-614 Poznań, Poland.

**E-mail:** ted.eklund@abo.fi, mikael.lindstrom@abo.fi, pml@amu.edu.pl

**Received:** 16 December 2017; **revised:** 23 January 2018