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CONICAL MEASURES AND CLOSED VECTOR MEASURES

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To the memory of Paweł Domański

Abstract: Let X be a locally convex Hausdorff space with topological dual X^* and m be a (σ-additive) X-valued vector measure defined on a σ-algebra. The completeness of the associated L^1-space of m is determined by the *closedness* of m, a concept introduced by I. Kluvánek in the early 1970's. He characterized the closedness of m via the existence of a certain kind of localizable, $[0, \infty]$- valued measure ι such that every scalar measure $\langle m, x^*\rangle : E \mapsto \langle m(E), x^*\rangle$, for $x^* \in X^*$, satisfies $\langle m, x^*\rangle << \iota$. The construction of ι relies on the theory of conical measures. Unfortunately, in this generality the characterization is invalid; a counterexample is exhibited. However, by restricting ι to the class of *Maharam measures* and strengthening the requirement of absolute continuity to the condition that every $\langle m, x^*\rangle$, for $x^* \in X^*$, is *truly continuous* with respect to ι (a notion investigated by D. Fremlin in connection with the Radon Nikodým Theorem), it is shown that an adequate characterization of the closedness of m is indeed available.

Keywords: Boolean algebra, conical measure, closed vector measure, truly continuous, localizable measure.

1. Introduction and main results

The theory of vector measures has a well established place in modern analysis. Recall, if X is a locally convex Hausdorff space (briefly, lcHs) and (Ω, Σ) is a measurable space, then a σ-additive set function $m : \Sigma \to X$ is called an X-valued vector measure; see, for example, [5], [12, Chapter 1], [16], [23, Chapter 3], [25]. One of the fundamental notions associated with a vector measure is that of its *closedness* (see Section 2 for the definition), introduced by I. Kluvánek in [13] and further developed in [14], [15], [16]. Important from the viewpoint of analysis is that m always generates an associated lcHs $L^1(m)$ consisting of all the m-integrable functions together with a continuous, linear, X-valued integration operator $f \mapsto \int_\Omega f\,dm$ for $f \in L^1(m)$. Under mild assumptions on X it turns out

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that the completeness properties of $L^1(m)$ are determined by whether or not m is a closed vector measure. Many sufficient criteria are known which imply the closedness of a vector measure; see Section 2. Some of these criteria involve properties of X (e.g., if X is metrizable , then every X-valued vector measure is closed), whereas others involve more intrinsic properties of m (e.g., the measure algebra of m and its order properties). Characterizations of closedness, without any a priori conditions on X are not so common.

One approach, due to I. Kluvánek, is to consider the family of scalar measures $\langle m, x^* \rangle : E \mapsto \langle m(E), x^* \rangle$, for $E \in \Sigma$, as x^* varies through the topological dual space X^* of X, with the idea being that some appropriate sort of "global control" over this family should provide a characterization of the closedness of m. This led to the following statement, [16, Theorem IV.7.3].

Assertion K-1. *Let $m : \Sigma \to X$ be a lcHs-valued vector measure. If there exists a localizable measure $\iota : \Sigma \to [0, \infty]$ such that $\langle m, x^* \rangle$ is absolutely continuous with respect to ι for each $x^* \in X^*$, then m is closed.*

The terminology "localizable measure" is perhaps not so well known and is not uniquely fixed in the literature. So, let us formulate it more precisely. For any σ-additive scalar measure $\iota : \Sigma \to [0, \infty]$ let $J_\iota : L^\infty(\iota) \to (L^1(\iota))^*$ denote the canonical linear map which sends $\varphi \in L^\infty(\iota)$ to the continuous linear functional on $L^1(\iota)$ given by $f \mapsto \int_\Omega \varphi f \, d\iota$. Consider the following conditions:

(a) J_ι is surjective;
(b) J_ι is injective;
(c) J_ι is bijective.

In Assertion K-1 the measure ι being localizable means precisely that (a) is satisfied, [16, p. 9]. Condition (b) is equivalent to ι being semifinite, [9, Theorem 243G]. Finally, (c) is equivalent to ι being both semifinite and its measure algebra being a complete Boolean algebra, [9, Theorem 243G]; precise definitions of these notions are given in Section 2. In the setting of (c) the measure ι is also called localizable (or *Maharam*); see, for example, the extensive works of D. Fremlin, [7, 8, 9, 10], and the references therein. The localizable measures in the sense of [16] form a *more extensive* class than those of Fremlin; this is illustrated by examples in Appendix B of Section 4. Unfortunately, in the generality formulated above it turns out that Assertion K-1 is incorrect; see examples in Appendix B of Section 4. One of our main aims is to present a modified version of Assertion K-1. Henceforth, a localizable measure ι *always* means that it satisfies condition (c) above. This restriction on ι is still insufficient to rectify Assertion K-1. It is also important to have available an adequate form of the Radon-Nikodým Theorem for localizable measures ι which may fail to be σ-finite. The central notion here is that of a $\mathbb{C}$-valued measure ξ on Σ being *truly continuous with respect to* ι (see Section 2 for the definition). This is a *genuinely stronger* requirement than absolute continuity of ξ with respect to ι (cf. Appendix B). Lemma 2.1 below shows that a good Radon-Nikodým Theorem is available for true continuity. One of our aims is to establish the following (correct) analogue of Assertion K-1.

Theorem 1. *Let $m : \Sigma \to X$ be a lcHs-valued vector measure. If there exists a localizable measure $\iota : \Sigma \to [0, \infty]$ such that $\langle m, x^* \rangle$ is truly continuous with respect to ι for each $x^* \in X^*$, then m is closed.*

Of course, this result is not a characterization of closedness. In two well known papers Kluvánek revealed some remarkable connections between vector measures and conical measures, [14], [15]. Combining Corollary 13 of [15] with the fact that, for *any* measure $\iota : \Sigma \to [0, \infty]$, every $\mathbb{C}$-valued measure of the form $E \mapsto \int_E f \, d\iota$, for $E \in \Sigma$, with f any ι-integrable function is necessarily truly continuous with respect to ι, [9, Proposition 232D], leads to the following

Assertion K-2. *Let $m : \Sigma \to X$ be a lcHs-valued vector measure. Then there exists (in the sense of [16, p. 9]) a localizable measure $\iota : \Sigma \to [0, \infty]$ such that $\langle m, x^* \rangle$ is truly continuous with respect to ι for each $x^* \in X^*$.*

The difficulty with Assertion K-2 is that it is based on Corollary 13 of [15], which in turn is an apparent consequence of earlier results on conical measures in that paper, some of which are known to have incomplete proofs. For instance, some effort by various authors was invested to provide a detailed proof of Theorem 1 in [15]; see the discussion immediately after the statement of Proposition 2.5 below. Moreover, the proof of the last claim in the statement of Theorem 1 of [15] is also rather sketchy with some apparent gaps. We present a detailed argument of this claim (see Lemmas 3.2 and 3.4 below) but, *only* for the *order ideal* H_m generated by $\{|\langle m, x^* \rangle| : x^* \in X^*\}$ in the Riesz space $ca(\Sigma)$ of all $\mathbb{R}$-valued, σ-additive measures on Σ. We are unable to verify it for a general vector sublattice of $ca(\Sigma)$ in place of H_m, as is claimed to be the case in [15]. Fortunately, our result for H_m suffices to establish our second main result (Theorem 2 below), which also incorporates an adequate analogue of Assertion K-2 above. We recall that localizability in Theorem 2 is meant in the sense of condition (c) above being satisfied.

Given a lcHs X, let $(X^*)^a$ denote the algebraic dual of X^*, in which case there is a dual pairing given by

$$\langle x^*, \xi \rangle := \xi(x^*), \qquad x^* \in X^*, \ \xi \in (X^*)^a.$$

Then $(X^*)^a$ is a weakly complete lcHs for the topology $\sigma((X^*)^a, X^*)$. The following result *characterizes* the closedness of an X-valued vector measure in terms of the family of scalar measures $\langle m, x^* \rangle$, for $x^* \in X^*$.

Theorem 2. *Let X be a lcHs and m be an X-valued vector measure defined on a measurable space (Ω, Σ). The following assertions are equivalent.*

(i) *The vector measure m is closed.*
(ii) *There exists a localizable measure $\iota : \Sigma \to [0, \infty]$ such that $\langle m, x^* \rangle$ is truly continuous with respect to ι for each $x^* \in X^*$.*
(iii) *There exists a localizable measure $\iota : \Sigma \to [0, \infty]$ and a function $F : \Omega \to (X^*)^a$ such that each scalar-valued function $\langle F, x^* \rangle : \omega \mapsto \langle F(\omega), x^* \rangle$, for*

$\omega \in \Omega$, is ι-integrable for every $x^* \in X^*$ and satisfies

$$\langle m(E), x^* \rangle = \int_E \langle F(\omega), x^* \rangle \, d\iota(\omega), \qquad E \in \Sigma. \tag{1.1}$$

In particular $\langle m, x^* \rangle$ is truly continuous with respect to ι for every $x^* \in X^*$.

It is possible to choose such a localizable measure ι in (ii) and (iii) which has the same null sets as m.

We have already alluded to the point that (part of) Theorem 2 is essentially presented in Corollary 13 in [15]; the latter is in a slightly different format but is equivalent to the relevant part of Theorem 2. The proof of Theorem 2 that we present in Section 3 is based directly on Proposition 2.4 and Lemma 3.4 below. On the other hand, the proof of Corollary 13 given in [15] appears to be a consequence of Theorem 12 and its proof (as given in [15]). However, there is an inherent difficulty in this process, as pointed out to us by Prof. R. Becker. It arises due to the fact that the vector lattice taken in [15] (see the proof of Theorem 12 there) is the one generated by the family of measures $\{|\langle m, x^* \rangle| : x^* \in X^*\}$ *together with* all the Dirac measures δ_ω, for $\omega \in \Omega$. It is precisely the presence of the Dirac measures which cause the difficulty; this is explained in Remark 3.5. Our proof only uses the *order ideal* H_m; fortunately, this suffices.

Finally, let $m : \Sigma \to X$ be a lcHs-valued vector measure and $X_{\sigma(X,X^*)}$ denote X equipped with its weak topology $\sigma(X, X^*)$. Then m, when considered as taking its values in the lcHs $X_{\sigma(X,X^*)}$ is also σ-additive; denote this vector measure by m_σ. A consequence of Assertions K-1 and K-2 was to show, in Theorem 2 of [26], that m is a closed vector measure if and only if m_σ is a closed vector measure. In view of the above discussions it is clear that the proof of this fact presented in [26] cannot be correct. Fortunately, its statement is still correct (see Proposition 2.4 below); an alternate proof, based on completely different arguments as those used in [26], is provided in Appendix C of Section 4. This (correctly proved) result can then (and will) be used in the proof of Theorem 2 above; see Section 3.

The structure of this paper is as follows. Section 2 presents various definitions and preliminary results (with detailed proofs) that are needed in the sequel, both for conical measures and for vector measures. Section 3 is mainly devoted to the proof of Theorem 2. Crucial for its proof is the availability of both Theorem 1 (proved in Appendix A of Section 4) and Proposition 2.4 (proved in Appendix C of Section 4). Relevant examples and counterexamples which illustrate the difficulties associated with Assertions K-1 and K-2 are formulated in Appendix B of Section 4.

2. Preliminaries

Throughout this section, let (Ω, Σ) denote a measurable space, that is, Σ is a σ-algebra of subsets of a non-empty set Ω. In particular, Σ is a σ-complete Boolean algebra (briefly B.a.). Indeed, it is clear that Σ is a lattice, with $\emptyset$ as zero and Ω as unit, in the order defined by set inclusion. Moreover, Σ is both distributive and complemented. Here the complement E' in the B.a. sense of a set $E \in \Sigma$

is the set-theoretic complement $E^c = \Omega \setminus E$. In other words, E is a B.a. of sets such that $E \wedge F = E \cap F$ and $E \vee F = E \cup F$ for $E, F \in \Sigma$ (see [10, Ch. 31], for example). The σ-completeness of Σ is obvious.

Let $\iota : \Sigma \to [0, \infty]$ be a scalar measure; namely, it is a σ-additive set function. The subfamily of Σ consisting of all ι-null sets is denoted by $\mathscr{N}_0(\iota)$, that is,

$$\mathscr{N}_0(\iota) := \{E \in \Sigma : \iota(E) = 0\}.$$

Then, $\mathscr{N}_0(\iota)$ is a σ-ideal of the B.a. Σ. Define an equivalence relation by $E \sim F$ for $E, F \in \Sigma$ if the symmetric difference $E \triangle F \in \mathscr{N}_0(\iota)$, where $E \triangle F := (E \cup F) \setminus (E \cap F)$. Let $\pi_\iota(E) := \{F \in \Sigma : E \sim F\}$ for each $E \in \Sigma$. The quotient

$$\Sigma / \mathscr{N}_0(\iota) := \{\pi_\iota(E) : E \in \Sigma\}$$

is a B.a. with the operations induced by Σ as follows:

$$\pi_\iota(E) \wedge \pi_\iota(F) := \pi_\iota(E \cap F), \;\; \pi_\iota(E) \vee \pi_\iota(F) := \pi_\iota(E \cup F), \;\; (\pi_\iota(E))' := \pi_\iota(\Omega \setminus E)$$

for $E, F \in \Sigma$. Since $\mathscr{N}_0(\iota)$ is a σ-ideal of Σ, the quotient B.a. $\Sigma / \mathscr{N}_0(\iota)$ is σ-complete and the so defined quotient map $\pi_\iota : \Sigma \to \Sigma / \mathscr{N}_0(\iota)$ is a B.a. σ-homomorphism.

The measure ι factors through $\Sigma / \mathscr{N}_0(\iota)$. In fact, observe that whenever $E, F \in \Sigma$ satisfy $\pi_\iota(E) = \pi_\iota(F)$, we have $E \triangle F \in \mathscr{N}_0(\iota)$, which implies that $\iota(E) = \iota(F)$. This enables us to define a function

$$\bar{\iota} : \Sigma / \mathscr{N}_0(\iota) \to [0, \infty]$$

by $\bar{\iota}(\pi_\iota(E)) := \iota(E)$ for $E \in \Sigma$, so that $\bar{\iota} \circ \pi_\iota = \iota$ on Σ. The function $\bar{\iota}$ has the following two properties:

$$(\bar{\iota})^{-1}(\{0\}) = \{0\}, \qquad \text{and} \qquad \bar{\iota}(\vee_{n=1}^{\infty} \eth_n) = \sum_{n=1}^{\infty} \bar{\iota}(\eth_n), \tag{2.1}$$

whenever $\{\eth_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence in $\Sigma / \mathscr{N}_0(\iota)$. The pair $(\Sigma / \mathscr{N}_0(\iota), \bar{\iota})$ is called the *measure algebra* of the measure space (Ω, Σ, ι). For the terminology and further details see [7, 61D], [8, 2.4], [10, 321H].

Given $E \in \Sigma$, we write $\Sigma \cap E := \{F \cap E : F \in \Sigma\} \subseteq \Sigma$. We say that ι is *localizable* if the quotient B.a. $\Sigma / \mathscr{N}_0(\iota)$ is complete and ι is semifinite. Here, by ι being semifinite, we mean that, given $E \in \Sigma$ with $\iota(E) = \infty$, there is $F \in \Sigma \cap E$ such that $0 < \iota(F) < \infty$, [8, 1.2(b)(v)], [9, Def. 211F]. Note that ι is localizable in our sense if and only if the measure algebra $(\Sigma / \mathscr{N}_0(\iota), \bar{\iota})$ is localizable in the sense of [7, Definitions 53A and 64A], because the latter requires $\bar{\iota}$ being semifinite, [7, 61F(b)], and because ι is semifinite if and only if $\bar{\iota}$ is semifinite, [10, Theorem 322B(d)].

Finite or more generally σ-finite measures are localizable. A wider class of localizable measures consists of the *decomposable measures*. This can be found in [7, 64H(b)], [8, Theorem 2.11]. We say that the measure space (Ω, Σ, ι) is decomposable, or simply ι is *decomposable*, if there exists a family $\{(\Omega_\kappa, \Sigma_\kappa, \iota_\kappa)\}_{\kappa \in K}$ of finite measure spaces such that $\{\Omega_\kappa : \kappa \in K\}$ is a family of pairwise disjoint subsets of Ω whose union equals Ω, a set $A \subseteq \Omega$ belongs to Σ if and only if $A \cap \Omega_\kappa \in \Sigma_\kappa$ for all $\kappa \in K$, and $\iota(A) = \sum_{\kappa \in K} \iota_\kappa(A \cap \Omega_\kappa)$ for $A \in \Sigma$; see [7, 64G(a)], [8, 1.2(b)(iv)], [9, Definition 211E], [11, Definition 19.25], for example.

Let us return to the general $[0, \infty]$-valued measure ι on Σ. Take a complex measure ξ on Σ. Its total variation measure $|\xi|$ is a positive, finite measure on Σ, [29, §6.1]. A set $E \in \Sigma$ is called *ξ-null*, if $|\xi|(E) = 0$. By $\mathscr{N}_0(\xi)$ we denote the subfamily of Σ consisting of all ξ-null sets, so that $\mathscr{N}_0(\xi) = \mathscr{N}_0(|\xi|)$. We say that ξ is *absolutely continuous with respect to ι*, denoted by $\xi << \iota$, if, given $\varepsilon > 0$, there is a $\delta > 0$ such that $|\xi(E)| < \varepsilon$ whenever a set $E \in \Sigma$ satisfies $0 \leqslant \iota(E) < \delta$. It turns out, [29, Theorem 6.11], that ξ is absolutely continuous with respect to ι if and only if every ι-null set is ξ-null. In other words

$$\xi << \iota \quad \text{if and only if} \quad \mathscr{N}_0(\iota) \subseteq \mathscr{N}_0(\xi). \tag{2.2}$$

A measure $\mu : \Sigma \to \mathbb{R}$ is called *truly continuous with respect to* a $[0, \infty]$-valued measure ι on Σ if $\mu << \iota$ and if, whenever $E \in \Sigma$ satisfies $\mu(E) \neq 0$, then there exists $F \in \Sigma$ such that $\iota(F) < \infty$ and $\mu(E \cap F) \neq 0$, [9, Definition 232A and Proposition 232B(b)]. If ι is σ-finite, then μ is truly continuous with respect to ι if and only if $\mu << \iota$, [9, Proposition 232B(c)]. For examples of ι and μ such that $\mu << \iota$ but μ is not truly continuous with respect to ι see Appendix B. We say that a $\mathbb{C}$-valued measure ξ is truly continuous with respect to ι if both its real part $\mathrm{Re}(\xi)$ and its imaginary part $\mathrm{Im}(\xi)$ are truly continuous with respect to ι, where $\mathrm{Re}(\xi) : E \mapsto \mathrm{Re}(\xi(E))$ and $\mathrm{Im}(\xi) : E \mapsto \mathrm{Im}(\xi(E))$, for $E \in \Sigma$.

Lemma 2.1. *Let $\iota : \Sigma \to [0, \infty]$ be a scalar measure.*

(i) *The following assertions are equivalent for a measure $\xi : \Sigma \to \mathbb{C}$.*
  - (a) *The measure ξ is truly continuous with respect to ι*
  - (b) *There exists a ι-integrable function ϕ_ξ such that $\xi(E) = \int_E \phi_\xi \, d\iota$, for $E \in \Sigma$, that is, ξ admits a Radon-Nikodým derivative with respect to ι.*
  - (c) *There exists a sequence $\{E_n\}_{n=1}^\infty$ in Σ satisfying $\iota(E_n) < \infty$ for all $n \in \mathbb{N}$ such that $|\xi|(\Omega \setminus \bigcup_{n=1}^\infty E_n) = 0$.*

(ii) *A measure $\xi : \Sigma \to \mathbb{C}$ is truly continuous with respect to ι if and only if so is its total variation measure $|\xi| : \Sigma \to [0, \infty)$*

Proof.

(i) For (a)$\Leftrightarrow$(b) see [9, Proposition 232D and Theorem 232E] and for (a)$\Leftrightarrow$(c) see [9, 232X(a)].

(ii) Apply (a)$\Leftrightarrow$(c) in part (i) to $\mathrm{Re}(\xi)$ and $\mathrm{Im}(\xi)$ and use the fact that $F \in \Sigma$ is ξ-null if and only if it is $|\xi|$-null if and only if it is null for both $\mathrm{Re}(\xi)$ and $\mathrm{Im}(\xi)$. ■

An example is given in Appendix B.1(ii) of measures ι and ξ such that $\xi << \iota$ but (i)(b) of Lemma 2.1 fails. This shows that, in general, absolute continuity does not suffice to ensure the existence of a Radon-Nikodým derivative.

All vector spaces to be considered will be over $\mathbb{R}$ or $\mathbb{C}$; the corresponding scalar field will be indicated clearly. Let X be a lcHs over $\mathbb{C}$. The duality between between X and X^* is denoted by $\langle x, x^*\rangle := x^*(x)$ for $x \in X$ and $x^* \in X^*$. By $\mathscr{P}(X)$ we denote the set of all continuous seminorms on X.

Consider a vector measure $m : \Sigma \to X$. Recall, for $x^* \in X^*$, that the complex measure $E \mapsto \langle m(E), x^*\rangle$ on Σ is denoted by $\langle m, x^*\rangle$; its range is a bounded subset of $\mathbb{C}$. So, the range $\mathscr{R}(m)$ of m in X is weakly bounded and hence, bounded in the initial topology. Given $p \in \mathscr{P}(X)$, the *p-semivariation* $p(m)$ is defined by

$$p(m)(E) := \sup_{x^*} |\langle m, x^*\rangle|(E), \qquad E \in \Sigma,$$

where the supremum is formed over those $x^* \in X^*$ satisfying $|\langle x, x^*\rangle| \leqslant p(x)$ for all $x \in X$. Equivalently, the supremum is taken over all $x^* \in (p^{-1}([0,1]))^\circ$ for the polar set

$$U_p^\circ = (p^{-1}([0,1]))^\circ := \{u^* \in X^* : |\langle x, u^*\rangle| \leqslant 1 \text{ for all } x \in X \text{ satisfying } p(x) \leqslant 1\}.$$

Then we have

$$\sup_{F \in \Sigma \cap E} p(m(F)) \leqslant p(m)(E) \leqslant 4 \sup_{F \in \Sigma \cap E} p(m(F)), \qquad E \in \Sigma, \tag{2.3}$$

[18, p.158]. Consequently, $p(m)(E) < \infty$ for all $E \in \Sigma$ because the boundedness of $\mathscr{R}(m)$ ensures that the right-side of (2.3) is finite. We say that a set $E \in \Sigma$ is *m-null* if $p(m)(E) = 0$ for all $p \in \mathscr{P}(X)$. It follows from (2.3) that a set $E \in \Sigma$ is m-null if and only if $m(\Sigma \cap E) = \{0\}$. The subfamily $\mathscr{N}_0(m) \subseteq \Sigma$ of all m-null sets satisfies

$$\mathscr{N}_0(m) = \bigcap_{x^* \in X^*} \mathscr{N}_0(|\langle m, x^*\rangle|) = \bigcap_{x^* \in X^*} \mathscr{N}_0(\langle m, x^*\rangle). \tag{2.4}$$

It is clear that $\mathscr{N}_0(m)$ is a σ-ideal of the B.a. Σ, so that we can consider the quotient B.a. $\Sigma/\mathscr{N}_0(m)$, analogous to the case of $\Sigma/\mathscr{N}_0(\iota)$. Let

$$q_m : \Sigma \to \Sigma/\mathscr{N}_0(m)$$

denote the corresponding quotient map, which is a B.a. σ-homomorphism. Given $p \in \mathscr{P}(X)$, the p-semivariation $p(m)$ defines a natural pseudometric on Σ via

$$(E, F) \mapsto p(m)(E \triangle F), \qquad (E, F) \in \Sigma \times \Sigma. \tag{2.5}$$

These pseudometrics with p varying through $\mathscr{P}(X)$ generate a uniformity $\tau(m)$ on Σ. We then equip Σ with the topology induced by $\tau(m)$. The uniformity $\tau(m)$ may not be separated, or equivalently its induced topology on Σ may not be Hausdorff. The associated Hausdorff space turns out to be the quotient space $\Sigma/\mathscr{N}_0(m)$. To be precise, given $p \in \mathscr{P}(X)$, define a function $\hat{p}(m) : \Sigma/\mathscr{N}_0(m) \to [0, \infty)$ by

$$\hat{p}(m)(q_m(E)) := p(m)(E), \qquad E \in \Sigma;$$

it is well defined because $\mathscr{N}_0(m) \subseteq p(m)^{-1}(\{0\})$. The pseudometric on $\Sigma/\mathscr{N}_0(m)$ induced by $\hat{p}(m)$ can be shown to equal the function

$$(q_m(E), q_m(F)) \mapsto p(m)(E \triangle F), \qquad E, F \in \Sigma, \tag{2.6}$$

because q_m is a B.a. homomorphism. Let $\hat{\tau}(m)$ denote the uniformity on $\Sigma/\mathscr{N}_0(m)$ generated by those pseudometrics $\hat{p}(m)$ with p varying through $\mathscr{P}(X)$. The topology on $\Sigma/\mathscr{N}_0(m)$ induced by $\hat{\tau}(m)$ is Hausdorff. In other words, $\Sigma/\mathscr{N}_0(m)$ is a Hausdorff uniform space.

A vector measure m is said to be *closed* if $\Sigma/\mathscr{N}_0(m)$ is $\hat{\tau}(m)$-complete, [13, p. 49], [16, p. 71]. A characterization of closed vector measures is given by the following lemma. It has originally been presented in [6, Proposition 1.1] with extra assumptions on X. The current general form is in [20, Lemma 1.4].

Lemma 2.2. *A lcHs-valued vector measure $m : \Sigma \to X$ is closed if and only if the B.a. $\Sigma/\mathscr{N}_0(m)$ is complete and has the property that, whenever $\{E_\kappa\}_\kappa$ is a net with $\{q_m(E_\kappa)\}_\kappa$ filtering to 0 in the order of $\Sigma/\mathscr{N}_0(m)$, the net $\{m(E_\kappa)\}_\kappa$ converges to 0 in X.*

The following sample result is from [16, Theorem IV.7.1] and [26, Proposition 1].

Lemma 2.3. *Let $m : \Sigma \to X$ be a lcHs-valued vector measure. If X is metrizable, in particular, if X is normable, then m is closed. Actually, it suffices that the range $\mathscr{R}(m)$ of m is metrizable for the relative topology from X.*

Further sufficient criteria for closedness of vector measures occur in [20], [21, §1], [22], [26]; see also the references therein..

Let us return to the general lcHs-valued vector measure $m : \Sigma \to X$. Given $x^* \in X^*$, the seminorm $p_{x^*} : x \mapsto |\langle x, x^* \rangle|$ on X is $\sigma(X, X^*)$-continuous and hence, continuous in the initial topology. According to [22, (3.16) and (3.17), p.26] we have

$$p_{x^*}(m)(E) = |\langle m, x^* \rangle|(E), \qquad E \in \Sigma. \tag{2.7}$$

Recall that $m_\sigma : \Sigma \to X_{\sigma(X,X^*)}$ denotes the vector measure m when considered as taking its values in $X_{\sigma(X,X^*)}$. Then m_σ is σ-additive as the natural identity map i_σ from X onto $X_{\sigma(X,X^*)}$ is continuous and linear. In view of (2.4) and (2.7), we have the identity $\mathscr{N}_0(m_\sigma) = \mathscr{N}_0(m)$, so that

$$q_{m_\sigma} = q_m \ \text{ and } \ \Sigma/\mathscr{N}_0(m_\sigma) = \Sigma/\mathscr{N}_0(m). \tag{2.8}$$

By (2.7), we can deduce that the uniformity $\tau(m_\sigma)$ on Σ is generated by the pseudometrics

$$(E, F) \mapsto |\langle m, x^* \rangle|(E \triangle F), \qquad E, F \in \Sigma, \tag{2.9}$$

with x^* varying through X^*. The corresponding uniformity $\hat{\tau}(m_\sigma)$ on $\Sigma/\mathscr{N}_0(m)$ is generated by the pseudometrics

$$(q_m(E), q_m(F)) \mapsto |\langle m, x^* \rangle|(E \triangle F), \qquad E, F \in \Sigma, \tag{2.10}$$

with x^* varying through X^*; see both (2.6) with p_{x^*} in place of p and (2.7) as well as (2.8).

As indicated in Section 1, the following result is needed in the sequel.

Proposition 2.4. *A lcHs-valued vector measure $m : \Sigma \to X$ is closed if and only if $m_\sigma : \Sigma \to X_{\sigma(X,X^*)}$ is closed.*

For the special case when m happens to be a spectral measure, Proposition 2.4 has already been verified (independently), [22, Proposition 3.8].

Now let us turn our attention to conical measures over weakly complete, real lcHs'. For conical measures over general lcHs' over $\mathbb{R}$, we refer to the monographs [3], [4, Sections 30 and 38-40]. Let Y be a weakly complete, real lcHs. By $h(Y)$ we denote the vector lattice of $\mathbb{R}$-valued functions on Y which is generated, with respect to the pointwise order, by all linear functionals in the continuous dual space Y^* of Y. Every function $f \in h(Y)$ is of the form

$$f(y) = \sup\{\langle y, y_j^*\rangle : j = 1, \dots, k\} - \sup\{\langle y, y_j^*\rangle : j = (k+1), \dots, l\}, \qquad y \in Y, \tag{2.11}$$

that is,

$$f = \bigvee_{j=1}^{k} y_j^* - \bigvee_{j=k+1}^{l} y_j^* \tag{2.12}$$

as elements of $h(Y)$ for some $y_1^*, \dots y_l^* \in Y^*$ and $l \in \mathbb{N}$ with $l \geqslant 2$, where $\bigvee$ (resp. $\bigwedge$) denotes the least upper (resp. greatest lower) bound in a lattice. We adopt the usual notation $h(Y)^+$ for the positive cone of $h(Y)$, i.e., $f \in h(Y)^+$ if and only if $f(y) \geqslant 0$ for all $y \in Y$. The restriction of each $f \in h(Y)$ to a subset $U \subseteq Y$ is denoted by $f|_U$, except possibly when we can clearly see that f is considered on such a set U. Write

$$h(Y)|_U \coloneqq \{f|_U : f \in h(Y)\}.$$

Each positive linear functional u on $h(Y)$ is called a *conical measure over Y*. By 'positive' we mean the value $u(f) \geqslant 0$ for all $f \in h(Y)^+$. The set $M^+(Y)$ of all conical measures over Y is a lattice in the order given by $u \geqslant v$ with $u, v \in M^+(Y)$ if and only if $u(f) \geqslant v(f)$ for all $f \in h(Y)^+$.

We shall adopt the setting of the proof of Theorem 2.8 in [27], with some alterations. As Y is weakly complete, we may assume that $Y = \mathbb{R}^{\mathbb{A}}$ with $\mathbb{A}$ equal to a closed ordinal interval $[0, \Gamma]$. Both symbols Y and $\mathbb{R}^{\mathbb{A}}$ will be used interchangeably. Given $\alpha \in \mathbb{A}$, let $e_\alpha^* : Y = \mathbb{R}^{\mathbb{A}} \to \mathbb{R}$ denote the corresponding coordinate functional, i.e., $e_\alpha(y) = y_\alpha$ for $y = (y_\beta)_{\beta \in \mathbb{A}}$. Then Y^* equals the linear hull $\mathrm{span}\{e_\alpha^* : \alpha \in \mathbb{A}\}$ and $\{e_\alpha^* : \alpha \in \mathbb{A}\}$ is a Hamel basis for Y^*. Define pairwise disjoint subsets $T(\alpha) \subseteq Y$, for $\alpha \in \mathbb{A}$, as follows: if $\alpha = 0$, then

$$T(0) \coloneqq \{y \in Y : |\langle y, e_0^*\rangle| = 1\}$$

and for $\alpha > 0$,

$$T(\alpha) \coloneqq \{y \in Y : |\langle y, e_\beta^*\rangle| = 0 \text{ for all } \beta \in [0, \alpha) \text{ and } |\langle y, e_\alpha^*\rangle| = 1\}.$$

Then, for every $\alpha \in \mathbb{A}$, the restriction of $|e^*_\alpha|$ to $T(\alpha)$ equals the constant function $\mathbb{1}$ on $T(\alpha)$, that is, $\mathbb{1} \in h(Y)|_{T(\alpha)}$. Given $\alpha \in \mathbb{A}$, let $\mathscr{S}_\alpha$ denote the σ-algebra of subsets of $T(\alpha)$ generated by $h(Y)|_{T(\alpha)}$; namely, it is the smallest σ-algebra which makes all functions in $h(Y)|_{T(\alpha)}$ measurable. Next define a σ-algebra of subsets of the disjoint union

$$T := \bigcup_{\alpha\in\mathbb{A}} T(\alpha) \subseteq Y$$

by

$$\mathscr{S} := \{A \subseteq T : A \cap T(\alpha) \in \mathscr{S}_\alpha \text{ for all } \alpha \in \mathbb{A}\}.$$

Fix $u \in M^+(Y)$ for the moment. The proof of Theorem 2.8 in [27] constructs pairs $(u_\alpha, \lambda_\alpha)$ for each $\alpha \in \mathbb{A}$ of a conical measure u_α over Y and a positive finite measure $\lambda_\alpha : \mathscr{S}_\alpha \to [0, \infty)$ such that

$$u(f) = \sum_{\alpha\in\mathbb{A}} u_\alpha(f), \qquad f \in h(Y),$$

with the right-side absolutely summable in $\mathbb{R}$ and

$$u_\alpha(f) = \int_{T(\alpha)} f \, d\lambda_\alpha, \qquad f \in h(Y),\ \alpha \in \mathbb{A}. \tag{2.13}$$

In (2.13) the right-side should have been written as $\int_{T(\alpha)} f|_{T(\alpha)} \, d\lambda_\alpha$. However, we have written f instead of $f|_{T(\alpha)}$ as the set $T(\alpha)$ over which f is integrated is clearly indicated. Now define a decomposable measure $\lambda : \mathscr{S} \to [0, \infty]$ by

$$\lambda(A) := \sum_{\alpha\in\mathbb{A}} \lambda_\alpha(A \cap T(\alpha)), \qquad A \in \mathscr{S}.$$

Our arguments in Section 3 will depend on the following result; the notation is as above.

Proposition 2.5. *Let Y be a weakly complete, real lcHs and $u : h(Y) \to \mathbb{R}$ be a conical measure.*

(i) *There exists a decomposable measure λ on a σ-algebra $\mathscr{S}$ of subsets of a non-empty set $T \subseteq Y$ such that every $f \in h(Y)$ (more precisely its restriction $f|_T$) is λ-integrable and*

$$u(f) = \int_T f \, d\lambda. \tag{2.14}$$

(ii) *The subset $h(Y)|_T$ is dense in the real Banach space $L^1(\lambda)$ of all $\mathbb{R}$-valued, λ-integrable functions, equipped with the usual L^1-norm.*

Some comments are in order. The above result has been presented originally in [15, Theorem 1]. However, according to [3, p.131], [27, p.29], there are problems with the proof given in [15] of part (i) of Proposition 2.5 above. A correct proof of

Proposition 2.5(i) above is given in [27]; see Theorem 2.8 there. It proceeds via the construction outlined prior to Proposition 2.5. We point out that an earlier proof than the one in [27] occurs in [2, Theorem 21] but, with an additional assumption on the cardinality of $\mathbb{A}$. The general case, without such cardinality assumptions on $\mathbb{A}$, has been presented later in [3, Theorem VI.1.11].

Regarding part (ii) of Proposition 2.5 above, the arguments presented in [15, p.90] do not seem to provide a full proof. So, we now present a more detailed proof of this fact.

Concerning some terminology, the scalar measure λ in part (i) above is said to represent u, [15, p.93]. According to the terminology of [27, p.27], u is localized in the measurable space $(T, \mathscr{S})$, which generalizes the concept of localizing conical measures on compact sets, [4, Definition 30.4, Vol. II], [14, p.328].

Proof of Proposition 2.5(ii). Fix $A \in \mathscr{S}$ with $0 < \lambda(A) < \infty$. We shall show that its characteristic function χ_A can be approximated by functions from $h(Y)|_T$ in the norm of $L^1(\lambda)$.

Step 1: For every $\alpha \in \mathbb{A}$, the space $h(Y)|_{T(\alpha)}$ is dense in the real Banach space $L^1(\lambda_\alpha)$.

To verify this consider the linear functional I_α, on the vector lattice $h(Y)|_{T(\alpha)}$, given by

$$I_\alpha(f|_{T(\alpha)}) := \int_{T(\alpha)} f \, d\lambda_\alpha = \int_{T(\alpha)} f|_T \, d\lambda_\alpha, \qquad f \in h(Y).$$

The Monotone Convergence Theorem for λ_α implies that I_α is a Daniell integral (the definition of Daniell integrals can be found in [28, Section 1, Ch. 13], [31, Ch. 6], for example). Recall from above that $h(Y)|_{T(\alpha)}$ contains the constant function $\mathbb{1}$ on $T(\alpha)$ and that $\mathscr{S}_\alpha$ is the σ-algebra generated by $h(Y)|_{T(\alpha)}$. So, λ_α is the unique (finite) measure on $\mathscr{S}_\alpha$ with the property that a function on $T(\alpha)$ is I_α-Daniell integrable if and only if it is λ_α-integrable and such that, for every $g \in L^1(\lambda_\alpha)$, the Daniell integral of $|g|$ equals $\int_{T(\alpha)} |g| \, d\lambda_\alpha$; see [28, Theorem 13.20 and Proposition 13.21], for example.

Next, $h(Y)|_{T(\alpha)}$ is known to be dense in the space of all I_α-Daniell integrable functions with respect to the norm which assigns to each g the Daniell integral of $|g|$, [31, Theorem 6-4 VI].

Step 2: For every $\alpha \in \mathbb{A}$ and $\varepsilon > 0$, there exists $f_\alpha \in h(Y)$ such that

$$\int_T |\chi_{A \cap T(\alpha)} - f_\alpha| \, d\lambda < \varepsilon. \tag{2.15}$$

To verify Step 2, fix $\alpha \in \mathbb{A}$ and $\varepsilon > 0$. First select $\tilde{f} \in h(Y)$ such that $\int_{T(\alpha)} |\chi_{A \cap T(\alpha)} - \tilde{f}| \, d\lambda_\alpha < \frac{\varepsilon}{3}$; this is possible via Step 1 as $\chi_{A \cap T(\alpha)} \in L^1(\lambda_\alpha)$. Set $f_0 := |\tilde{f}| \wedge |e^*_\alpha| \in h(Y)$. Then

$$\int_{T(\alpha)} |\chi_{A \cap T(\alpha)} - f_0| \, d\lambda_\alpha < \frac{\varepsilon}{3} \tag{2.16}$$

because $|e_\alpha^*| = \mathbb{1}$ on $T(\alpha)$ and hence,

$$|\chi_{A\cap T(\alpha)} - f_0| = |(\chi_{A\cap T(\alpha)} \wedge |e_\alpha^*|) - (|\tilde f| \wedge |e_\alpha^*|)| \\ \leqslant |\chi_{A\cap T(\alpha)} - |\tilde f|\,| \leqslant |\chi_{A\cap T(\alpha)} - \tilde f|$$

pointwise on $T(\alpha)$, where we have used the Birkoff inequality: $|(a\wedge c) - (b\wedge c)| \leqslant |a-b|$, [1, Theorem 1.1(12)]. By the definition of e_α^*, we can see that e_α^* vanishes on $\bigcup_{\beta\in(\alpha,\Gamma]} T(\beta)$; recall that $\mathbb{A} = [0,\Gamma]$. Hence, also f_0 vanishes on $\bigcup_{\beta\in(\alpha,\Gamma]} T(\beta)$.

Suppose first that $\alpha = 0$. Since f_0 vanishes on $\bigcup_{\beta\in(0,\Gamma]} T(\beta)$, we have, by (2.16) with $\alpha = 0$, that

$$\int_T |\chi_{A\cap T(0)} - f_0|\, d\lambda = \int_{T(0)} |\chi_{A\cap T(0)} - f_0|\, d\lambda_0 < \frac{\varepsilon}{3}.$$

So, (2.15) holds when $\alpha = 0$.

Now assume that $\alpha > 0$. Since $f_0 \in h(Y)$, part (i) implies that $f_0|_T \in (L^1(\lambda))^+$. Consequently, as the disjoint union $\bigcup_{\beta\in[0,\alpha)} T(\beta) \subseteq T$ we have

$$\sum_{\beta\in[0,\alpha)} \int_{T(\beta)} f_0\, d\lambda \leqslant \int_T f_0\, d\lambda < \infty$$

and hence, the set $\{\beta \in [0,\alpha) : \int_{T(\beta)} f_0\, d\lambda > 0\}$ is at most countable. So, there exists a finite, non-empty subset $\mathbb{A}_0 \subseteq [0,\alpha)$ such that

$$\sum_{\beta\in[0,\alpha)\setminus\mathbb{A}_0} \int_{T(\beta)} f_0\, d\lambda_\beta < \frac{\varepsilon}{3}. \tag{2.17}$$

With n denoting the number of elements in $\mathbb{A}_0$, let $\{\beta(j) : j = 1,\dots,n\}$ be an enumeration of $\mathbb{A}_0$. For each $j = 1,\dots,n$, since $|e_{\beta(j)}^*(t)| = 1$ for every $t \in T(\beta(j))$, we have $f_0 \wedge |ke_{\beta(j)}^*| \uparrow f_0$ pointwise on $T(\beta(j))$ as $k \to \infty$. According to the Monotone Convergence Theorem, there exists $k_j \in \mathbb{N}$ such that

$$\int_{T(\beta(j))} (f_0 - (f_0 \wedge |k_j e_{\beta(j)}^*|))\, d\lambda_{\beta(j)} < \frac{\varepsilon}{3n}. \tag{2.18}$$

Now let

$$f_\alpha := \bigwedge_{j=1}^{n} (f_0 - (f_0 \wedge |k_j e_{\beta(j)}^*|)).$$

Then $f_\alpha \in h(Y)$ and we have that $f_\alpha = f_0$ pointwise on $T(\alpha)$, because $e_{\beta(j)}^*$ vanishes on $T(\alpha)$ as $\beta(j) < \alpha$ for all $j = 1,\dots,n$, and also that $0 \leqslant f_\alpha \leqslant f_0$ on $h(Y)$. So, (2.16) and (2.17) give

$$\int_{T(\alpha)} |\chi_{A\cap T(\alpha)} - f_\alpha|\, d\lambda_\alpha = \int_{T(\alpha)} |\chi_{A\cap T(\alpha)} - f_0|\, d\lambda_\alpha < \frac{\varepsilon}{3} \tag{2.19}$$

and

$$\sum_{\beta\in[0,\alpha)\setminus\mathbb{A}_0}\int_{T(\beta)} f_\alpha\,d\lambda_\beta \leqslant \sum_{\beta\in[0,\alpha)\setminus\mathbb{A}_0}\int_{T(\beta)} f_0\,d\lambda_\beta < \frac{\varepsilon}{3}, \tag{2.20}$$

respectively. Next, by (2.18) and the definition of f_α we have

$$\sum_{\beta\in\mathbb{A}_0}\int_{T(\beta)} f_\alpha\,d\lambda_\beta \leqslant \sum_{j=1}^{n}\int_{T(\beta(j))}(f_0-(f_0\wedge|k_je^*_{\beta(j)}|))\,d\lambda_{\beta(j)} < \sum_{j=1}^{n}\frac{\varepsilon}{3n}=\frac{\varepsilon}{3}. \tag{2.21}$$

Combining (2.19), (2.20) and (2.21) yields

$$\begin{aligned}\int_T |\chi_{A\cap T(\alpha)}-f_\alpha|\,d\lambda &= \sum_{\beta\in\mathbb{A}_0}\int_{T(\beta)} f_\alpha\,d\lambda_\beta + \sum_{\beta\in[0,\alpha)\setminus\mathbb{A}_0}\int_{T(\beta)} f_\alpha\,d\lambda_\beta\\ &\quad+\int_{T(\alpha)}|\chi_{A\cap T(\alpha)}-f_\alpha|\,d\lambda_\alpha < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

We have thereby established Step 2.

Step 3: For every $\varepsilon>0$, there exists $f\in h(Y)$ such that $\int_T|\chi_A-f|\,d\lambda<\varepsilon$.

To establish Step 3, observe first that $\sum_{\alpha\in\mathbb{A}}\lambda(A\cap T(\alpha))=\lambda(A)<\infty$, so that $\lambda(A\cap T(\alpha))=0$ except for at most countably many α's. So, there exists a finite, non-empty subset $\mathbb{A}_1\subseteq\mathbb{A}$ such that

$$\int_T|\chi_A-\sum_{\alpha\in\mathbb{A}_1}\chi_{A\cap T(\alpha)}|\,d\lambda=\sum_{\alpha\in\mathbb{A}\setminus\mathbb{A}_1}\lambda(A\cap T(\alpha))<\frac{\varepsilon}{2}. \tag{2.22}$$

With m denoting the number of elements in $\mathbb{A}_1$, let $\{\alpha(1),\dots,\alpha(m)\}$ be an enumeration of $\mathbb{A}_1$. For each $j=1,\dots,m$, choose $f_{\alpha(j)}\in h(Y)$ such that

$$\int_T|\chi_{A\cap T(\alpha(j))}-f_{\alpha(j)}|\,d\lambda<\frac{\varepsilon}{2m}; \tag{2.23}$$

see Step 2 with $\alpha(j)$ in place of α and $\frac{\varepsilon}{2m}$ in place of ε.

Let $f:=\sum_{j=1}^{m}f_{\alpha(j)}\in h(Y)$. Then it follows from (2.22) and (2.23) that

$$\begin{aligned}\int_T|\chi_A-f|\,d\lambda &= \int_T|(\chi_A-\sum_{\alpha\in\mathbb{A}_1}\chi_{A\cap T(\alpha)})+(\sum_{\alpha\in\mathbb{A}_1}\chi_{A\cap T(\alpha)}-f)|\,d\lambda\\ &\leqslant \int_T|(\chi_A-\sum_{\alpha\in\mathbb{A}_1}\chi_{A\cap T(\alpha)})|\,d\lambda\\ &\quad+\sum_{j=1}^{m}\int_{T(\alpha(j))}|\chi_{A\cap T(\alpha(j))}-f_{\alpha(j)}|\,d\lambda\\ &<\frac{\varepsilon}{2}+\sum_{j=1}^{m}\frac{\varepsilon}{2m}=\varepsilon,\end{aligned}$$

so that Step 3 is verified.

Finally, since span$\{\chi_A : A \in \Sigma$ with $0 < \lambda(A) < \infty\}$, i.e., the λ-simple functions in the terminology of [9], is dense in $L^1(\lambda)$, [9, Proposition 242M], Step 3 ensures that $h(Y)|_T$ is also dense in $L^1(\lambda)$, which completes the proof of Proposition 2.5(ii). ■

Now we present a consequence of Proposition 2.5. To this end, let the notation be as in Proposition 2.5. First, observe that $L^1(\lambda)$ is a vector lattice with respect to the λ-a.e. pointwise order. Denote its positive cone by $(L^1(\lambda))^+$. According to Proposition 2.5(i), $h(Y)|_T$ is a vector sublattice of $L^1(\lambda)$.

By $L^\infty(\lambda)$ we denote the real vector space of all (equivalence classes of) $\mathbb{R}$-valued, λ-essentially bounded, $\mathscr{S}$-measurable functions on T. Recall that an $\mathscr{S}$-measurable function g is called λ-*essentially bounded* if there is $a \in (0, \infty)$ such that $\{t \in T : |g(t)| > a\}$ is λ-null, [9, Def. 243A]. Since the *decomposable* measure λ in Proposition 2.5 has the property that

$$\lambda(A) = 0 \Longleftrightarrow \lambda(A \cap B) = 0 \text{ for all B} \in \mathscr{S} \text{ with } \lambda(B) < \infty,$$

[9, 213J, 214J], the above definition of λ-essential boundedness is equivalent to that in [11, Definition 20.11]. Functions in $L^\infty(\lambda)$ which coincide λ-a.e. on T are identified, except when we need to distinguish between *individual* functions and their corresponding equivalence classes. We shall use the well known identification $(L^1(\lambda))^* = L^\infty(\lambda)$, [7, 64B, 64G, 64H], [9, 243G],[11, Theorems 20.16 and 20.19].

Corollary 2.6. *With $u : h(Y) \to \mathbb{R}$ and $(T, \mathscr{S}, \lambda)$ as in Proposition 2.5, the following statements hold.*

(i) *Let $\psi \in L^\infty(\lambda)$ be a function such that $\int_T f\psi\, d\lambda = 0$ for all $f \in h(Y)$. Then $\psi = 0$ (λ-a.e.).*

(ii) *If $v : h(Y) \to \mathbb{R}$ is another conical measure such that $v \leqslant u$ on $h(Y)$, then there is a unique positive function $\varphi \in L^\infty(\lambda)$ such that*

$$v(f) = \int_T f\varphi\, d\lambda, \qquad f \in h(Y). \tag{2.24}$$

Proof.

(i) This is a consequence of Proposition 2.5(ii) because the assumption says that the continuous linear functional $g \mapsto \int_T g\psi\, d\lambda$ on $L^1(\lambda)$ vanishes on the dense linear subspace $h(Y)|_T$ of $L^1(\lambda)$.

(ii) We define a linear functional $\eta : h(Y)|_T \to \mathbb{R}$ by

$$\eta(f|_T) := v(f), \qquad f \in h(Y).$$

To see that η is well-defined, take $f_1, f_2 \in h(Y)$ such that $f_1|_T = f_2|_T$ (λ-a.e.) on T. Since $(f_1 - f_2)|_T = 0$ (λ-a.e.), it follows that

$$0 \leqslant v((f_1 - f_2) \vee 0) \leqslant u((f_1 - f_2) \vee 0) = \int_T ((f_1 - f_2) \vee 0)\, d\lambda = 0$$

and hence, that $v((f_1-f_2)\vee 0)=0$. Similarly, we have $v((f_2-f_1)\vee 0)=0$. Thus

$$v(f_1-f_2)=v((f_1-f_2)\vee 0-((f_2-f_1)\vee 0))=0,$$

so that $v(f_1)=v(f_2)$. This ensures that η is well defined. Moreover, it is clear that η is linear.
Take $f\in h(Y)|_T$ such that $f|_T\geqslant 0$ (λ-a.e.). Then $v((-f)\vee 0)=0$ because $((-f)\vee 0)=0$ (λ-a.e. on T) and so

$$0\leqslant v((-f)\vee 0)\leqslant u((-f)\vee 0)=\int_T((-f)\vee 0)\,d\lambda=0.$$

Hence,

$$\eta(f|_T)=v(f)=v((f\vee 0)-((-f)\vee 0))=v(f\vee 0)\geqslant 0.$$

This implies that η is a positive linear functional.
Next, since $v:h(Y)\to\mathbb{R}$ is a positive linear functional, it follows that

$$|\eta(f|_T)|=|v(f)|\leqslant v(|f|)\leqslant u(|f|)=\int_T|f|\,d\lambda=\int_T|f|_T\,d\lambda$$

for every $f\in h(Y)$ and hence, that η is continuous on $h(Y)|_T$ for the induced norm from $L^1(\lambda)$. So, η admits a unique continuous linear extension $\tilde{\eta}:L^1(\lambda)\to\mathbb{R}$ as $h(Y)|_T$ is dense in $L^1(\lambda)$ by Proposition 2.5(ii). According to the discussion prior to the Corollary we have $(L^1(\lambda))^*=L^\infty(\lambda)$ and so there is $\varphi\in L^\infty(\lambda)$ such that $\langle g,\tilde{\eta}\rangle=\int_T g\varphi\,d\lambda$ for $g\in L^1(\lambda)$.
To see that $\varphi\geqslant 0$ (λ-a.e.), fix $g\in(L^1(\lambda))^+$. Select functions $f_n\in h(Y)$ with $n\in\mathbb{N}$ such that $\lim_{n\to\infty}\int_T|f_n-g|\,d\lambda=0$; see Proposition 2.5(ii). Since $|(f_n\vee 0)-g|\leqslant|f_n-g|$ pointwise on T, it then follows that $\lim_{n\to\infty}\int_T|(f_n\vee 0)-g|\,d\lambda=0$, in other words, $\lim_{n\to\infty}(f_n\vee 0)|_T=g$ in the norm of $L^1(\lambda)$. So we have, as η is positive, that

$$\int_T g\varphi\,d\lambda=\langle g,\tilde{\eta}\rangle=\langle\lim_{n\to\infty}(f_n\vee 0)|_T,\tilde{\eta}\rangle=\lim_{n\to\infty}\eta((f_n\vee 0)|_T)\geqslant 0.$$

Since $g\in L^1(\lambda)$ is an arbitrary positive function, it then follows that $\varphi\geqslant 0$ (λ-a.e.). Moreover, (2.24) clearly holds as

$$\int_T f\varphi\,d\lambda=\langle f|_T,\tilde{\eta}\rangle=\eta(f|_T)=v(f),\qquad f\in h(Y).$$

Finally, take another positive function $\varphi_1\in L^\infty(\lambda)$ satisfying $v(f)=\int_T f\varphi_1\,d\lambda$ for all $f\in h(Y)$, so that $\int_T f(\varphi-\varphi_1)\,d\lambda=0$ for all $f\in h(Y)$. Applying part (i) with $\psi:=(\varphi-\varphi_1)$ yields $\varphi=\varphi_1$ (λ-a.e.), which completes the proof. ■

3. Kluvánek's characterization of closed vector measures

The aim of this section is to prove Theorem 2 of Section 1, which is a correct version of Kluvánek's characterization of closed vector measures (cf. Assertions K-1 and K-2 in Section 1). Throughout this section, let X be a complex lcHs and m be an X-valued vector measure defined on a measurable space (Ω, Σ), unless stated otherwise.

The real vector space $ca(\Sigma)$ of all $\mathbb{R}$-valued, σ-additive measures on Σ is a vector lattice (or Riesz space, [1]) for the setwise order, so that $\mu_1 \geqslant \mu_2$ if and only if $\mu_1(A) \geqslant \mu_2(A)$ for all $A \in \Sigma$. Its positive cone $ca^+(\Sigma)$ consists of all positive measures and the modulus of each $\mu \in ca(\Sigma)$ in the lattice sense coincides with its total variation measure $|\mu| : \Sigma \to [0, \infty)$. Let H_m denote the *order ideal* in $ca(\Sigma)$ generated by $\{|\langle m, x^*\rangle| : x^* \in X^*\} \subseteq ca^+(\Sigma)$. Since $\alpha|\langle m, x^*\rangle| = |\langle m, \alpha x^*\rangle|$ for all $\alpha \geqslant 0$, it follows that a measure $\mu \in ca(\Sigma)$ belongs to H_m if and only if

$$|\mu| \leqslant \sum_{j=1}^{n} |\langle m, x_j^*\rangle|, \qquad \text{on } \Sigma, \tag{3.1}$$

for some $x_1^*, \ldots, x_n^* \in X^*$ and $n \in \mathbb{N}$, [1, p.4].

Let Y denote the algebraic dual of the real vector space H_m (i.e., Y is the space of *all* linear functionals $y : H_m \to \mathbb{R}$). We equip Y with the pointwise convergence topology $\sigma(Y, H_m)$ on H_m, that is, the lcHs-topology generated by the seminorms

$$p_\mu(y) := |\langle \mu, y\rangle| = |y(\mu)|, \qquad y \in Y,$$

as μ varies through H_m. Then Y is a weakly complete, real lcHs, [4, II Theorem 22.16], and there is a natural vector space isomorphism from H_m onto Y^*. It is the assignment sending each $\mu \in H_m \subseteq ca(\Sigma)$ to the continuous linear functional $\tilde{\mu} \in Y^* \subseteq h(Y)$ given by

$$\langle y, \tilde{\mu}\rangle := \langle \mu, y\rangle = y(\mu), \qquad y \in Y.$$

Then a function $f \in h(Y)$, expressed as (2.12) in Section 2, now has the form

$$f = \bigvee_{j=1}^{k} \tilde{\mu}_j - \bigvee_{j=k+1}^{l} \tilde{\mu}_j \tag{3.2}$$

for some $\mu_1, \ldots, \mu_l \in H_m$ and $l \in \mathbb{N}$ with $l \geqslant 2$. In particular, each $\mu_j, 1 \leqslant j \leqslant l$, satisfies a condition of the form (3.1).

The following result is a special case of [15, Lemma 7]. The spaces H_m and Y are as defined above.

Lemma 3.1. *There exists a unique vector-lattice homomorphism $\Phi : h(Y) \to H_m$ satisfying $\Phi(\tilde{\mu}) = \mu$ for every $\mu \in H_m$. Consequently, for each $f \in h(Y)$ of the form (3.2), the element $\Phi(f) \in H_m$ is expressed as*

$$\Phi(f) = \bigvee_{j=1}^{k} \mu_j - \bigvee_{j=k+1}^{l} \mu_j.$$

According to [1, Theorem 1.17(vi)], the vector-lattice homomorphism Φ obtained in Lemma 3.1 above satisfies

$$|\Phi(f)| = \Phi(|f|), \qquad f \in h(Y). \tag{3.3}$$

Moreover, Φ is positive, [1, p.9], so that the linear functional $u : h(Y) \to \mathbb{R}$ defined by

$$u(f) := \Phi(f)(\Omega), \qquad f \in h(Y), \tag{3.4}$$

is a conical measure. Now, let us take the decomposable (scalar) measure λ defined on the measure space $(T, \mathscr{S})$ with $T \subseteq Y$ and representing u via Proposition 2.5, so that (2.14) holds for $f \in h(Y)$. In view of (3.4), we have

$$\Phi(f)(\Omega) = \int_T f \, d\lambda, \qquad f \in h(Y). \tag{3.5}$$

Next, given any set $E \in \Sigma$, consider the linear functional $v_E : h(Y) \to \mathbb{R}$ specified by

$$v_E(f) := \Phi(f)(E), \qquad f \in h(Y). \tag{3.6}$$

Then v_E is positive and hence, is also a conical measure. Moreover, given $f \in h(Y)^+$, since $\Phi(f) \in ca^+(\Sigma)$, it follows that

$$v_E(f) = \Phi(f)(E) \leqslant \Phi(f)(\Omega) = u(f),$$

which implies, in the order of $M^+(Y)$, that $0 \leqslant v_E \leqslant u$.

Lemmas 3.2 and 3.4 below are taken from Theorem 8 and its proof in [15]. The setting in [15] is more general. However, the arguments there are rather sketchy and seem to have gaps. We shall provide detailed proofs of these two lemmas which are needed to prove Theorem 2 in Section 1. The order ideal $H_m \subseteq ca(\Sigma)$, the weakly complete lcHs Y, the homomorphism $\Phi : h(Y) \to H_m$, the decomposable measure λ on $(T, \mathscr{S})$ with $T \subseteq Y$, and the conical measure v_E, for $E \in \Sigma$, are as specified above.

Lemma 3.2. *The following statements hold.*

(i) *Given $E \in \Sigma$, there is a unique positive function $\varphi_E \in L^\infty(\lambda)$ such that*

$$\Phi(f)(E) = v_E(f) = \int_T f \varphi_E \, d\lambda, \qquad f \in h(Y). \tag{3.7}$$

(ii) *The following results hold for each set $E \in \Sigma$.*

(a) *$v_E \wedge v_{\Omega \setminus E} = 0$ in the lattice $M^+(Y)$.*

(b) *$\varphi_E \wedge \varphi_{\Omega \setminus E} = 0$ (λ-a.e.) on T.*

(c) *$\varphi_E + \varphi_{\Omega \setminus E} = \mathbb{1}$ (λ-a.e.) on T.*

(d) *φ_E is $\{0, 1\}$-valued (λ-a.e.) on T.*

(iii) *The identity $\varphi_E \wedge \varphi_F = \varphi_{E \cap F}$ holds (λ-a.e.) on T for all $E, F \in \Sigma$.*

(iv) *Let* $g \in (L^1(\lambda))^+$ *and let* λ_g *denote its corresponding indefinite integral:* $E \mapsto \int_E g\,d\lambda$ *on* Σ*. Then, the linear space* $\operatorname{span}\{\varphi_E : E \in \Sigma\}$ *is dense in the real Banach space* $L^1(\lambda_g)$.

Proof.

(i) The first equality in (3.7) is exactly (3.6) whereas the second equality is a special case of Corollary 2.6(ii), after recalling (3.4) and that $0 \leqslant v_E \leqslant u$.

(ii) (a) Fix $E \in \Sigma$ and let $w := v_E \wedge v_{\Omega\setminus E} \in M^+(Y)$. Given any $\mu \in H_m$, we first show that

$$w(|\tilde{\mu}|) = 0, \tag{3.8}$$

where $|\tilde{\mu}|$ is the modulus in $h(Y)$ of $\tilde{\mu} \in Y^* \subseteq h(Y)$. To see this, define the restriction measures μ_E and $\mu_{\Omega\setminus E}$ on Σ by

$$\mu_E(F) := \mu(E \cap F) \quad \text{and} \quad \mu_{\Omega\setminus E}(F) := \mu((\Omega \setminus E) \cap F), \qquad F \in \Sigma,$$

respectively. Clearly, both μ_E and $\mu_{\Omega\setminus E}$ belong to the order ideal H_m; see (3.1). For ease of notation, write $\tilde{\mu}_E := (\mu_E)^\sim$ and $\tilde{\mu}_{\Omega\setminus E} := (\mu_{\Omega\setminus E})^\sim$. Since $\tilde{\mu}_E \in h(Y)$ and $w \leqslant v_{\Omega\setminus E}$ in $M^+(Y)$, it follows from (3.6) that the vector-lattice homomorphism Φ satisfies

$$\begin{aligned} 0 \leqslant w(\tilde{\mu}_E \vee 0) &\leqslant v_{\Omega\setminus E}(\tilde{\mu}_E \vee 0) = \Phi(\tilde{\mu}_E \vee 0)(\Omega \setminus E) \\ &= (\Phi(\tilde{\mu}_E) \vee \Phi(0))(\Omega \setminus E) = (\mu_E \vee 0)(\Omega \setminus E) \\ &= \sup_{F \in \Sigma \cap (\Omega\setminus E)} \mu_E(F) = 0, \end{aligned}$$

which gives $w(\tilde{\mu}_E \vee 0) = 0$. Similarly, we have $w((-\tilde{\mu}_E) \vee 0) = 0$ and so

$$w(|\tilde{\mu}_E|) = w((\tilde{\mu}_E \vee 0) + ((-\tilde{\mu}_E) \vee 0)) = 0.$$

Interchanging the roles of E and $\Omega \setminus E$ gives $w(|\tilde{\mu}_{\Omega\setminus E}|) = 0$. Accordingly, (3.8) holds because

$$|\tilde{\mu}| = |(\mu_E + \mu_{\Omega\setminus E})^\sim| = |\tilde{\mu}_E + \tilde{\mu}_{\Omega\setminus E}| \leqslant |\tilde{\mu}_E| + |\tilde{\mu}_{\Omega\setminus E}|$$

implies that

$$0 \leqslant w(|\tilde{\mu}|) \leqslant w(|\tilde{\mu}_E|) + w(|\tilde{\mu}_{\Omega\setminus E}|) = 0.$$

Next, let $f \in h(Y)$ be of the form (3.2). Then, $w(f) = 0$ as

$$\begin{aligned} 0 \leqslant |w(f)| \leqslant w(|f|) &\leqslant w\left(\bigvee_{j=1}^{k} |\tilde{\mu}_j|\right) + w\left(\bigvee_{j=k+1}^{l} |\tilde{\mu}_j|\right) \\ &= \sum_{j=1}^{l} w(|\tilde{\mu}_j|) = 0, \end{aligned}$$

where we have applied (3.8) with μ_j in place of μ for each $j = 1, \ldots, n$. Thus, $w = 0$, that is, (a) holds.

(b) Fix $E \in \Sigma$ and let $\varphi_E, \varphi_{\Omega\setminus E}$ be as in part (i). Since $\varphi_E \wedge \varphi_{\Omega\setminus E} \in (L^\infty(\lambda))^+$ and $h(Y)|_T \subseteq L^1(\lambda)$ (see Proposition 2.5(i)), we can define a conical measure z by

$$z(f) := \int_T f(\varphi_E \wedge \varphi_{\Omega\setminus E})\, d\lambda, \qquad f \in h(Y).$$

It is clear from (3.7) that $0 \leqslant z \leqslant v_E$ and $0 \leqslant z \leqslant v_{\Omega\setminus E}$ in $M^+(Y)$. An appeal to (a) ensures that $z = 0$. By Corollary 2.6(i) with $(\varphi_E \wedge \varphi_{\Omega\setminus E})$ in place of φ, we have (b).

(c) Again let $E \in \Sigma$ and $\varphi_E, \varphi_{\Omega\setminus E}$ be as in part (i). It follows from (3.5), (3.7) and part (i) above that, for every $f \in h(Y)$, we have

$$\int_T f(\varphi_E + \varphi_{\Omega\setminus E} - \mathbb{1})\, d\lambda = \Phi(f)(E) + \Phi(f)(\Omega \setminus E) - \Phi(f)(\Omega) = 0,$$

because $\Phi(f) \in ca(\Sigma)$. By Corollary 2.6(i) with $(\varphi_E + \varphi_{\Omega\setminus E} - \mathbb{1})$ in place of φ, we can conclude that $\varphi_E + \varphi_{\Omega\setminus E} - \mathbb{1} = 0$ (λ-a.e.), that is, (c) holds.

(d) This is immediate from (b) and (c).

(iii) First, given disjoint sets $G(1), G(2) \in \Sigma$, we claim that

$$\varphi_{G(1)} + \varphi_{G(2)} = \varphi_{G(1)\cup G(2)} \ (\lambda\text{-a.e.}) \quad \text{and} \quad \varphi_{G(1)} \wedge \varphi_{G(2)} = 0\, (\lambda\text{-a.e.}). \tag{3.9}$$

Indeed, the first identity can be proved as for (ii)(c) above.
Next, since $\Omega \setminus G(2)$ is the disjoint union of $G(1)$ and $\Omega \setminus (G(1) \cup G(2))$, we can apply the first identity with $\Omega \setminus (G(1) \cup G(2))$ in place of $G(2)$ to obtain

$$\varphi_{\Omega\setminus G(2)} = \varphi_{G(1)} + \varphi_{\Omega\setminus(G(1)\cup G(2))} \ (\lambda\text{-a.e.}).$$

In particular, $0 \leqslant \varphi_{G(1)} \leqslant \varphi_{\Omega\setminus G(2)}$, which gives $\varphi_{G(1)} \wedge \varphi_{G(2)} = 0$ because

$$0 \leqslant \varphi_{G(1)} \wedge \varphi_{G(2)} \leqslant \varphi_{\Omega\setminus G(2)} \wedge \varphi_{G(2)} = 0 \ (\lambda\text{-a.e.})$$

by (ii)(b) with $E := G(2)$. Hence, (3.9) is verified.
By (3.9) we have $\varphi_E = \varphi_{E\cap F} + \varphi_{E\setminus F}$ and $\varphi_F = \varphi_{F\cap E} + \varphi_{F\setminus E}$ as well as $\varphi_{E\setminus F} \wedge \varphi_{F\setminus E} = 0$. Now apply [1, Theorem 1.1(6)] to obtain (iii) as follows:

$$\begin{aligned}\varphi_E \wedge \varphi_F &= (\varphi_{E\cap F} + \varphi_{E\setminus F}) \wedge (\varphi_{E\cap F} + \varphi_{F\setminus E}) \\ &= \varphi_{E\cap F} + (\varphi_{E\setminus F} \wedge \varphi_{F\setminus E}) = \varphi_{E\cap F} \ (\lambda\text{-a.e.}).\end{aligned}$$

(iv) Note first that the λ-essentially bounded functions φ_E with $E \in \Sigma$ (see part (i) above) belong to $L^1(\lambda_g)$ because λ_g is a positive, finite measure on $\mathscr{S}$. Let $\xi : L^1(\lambda_g) \to \mathbb{R}$ be a continuous linear functional such that $\langle \varphi_E, \xi \rangle = 0$ for all $E \in \Sigma$. Our aim is to deduce that $\xi = 0$. Since λ_g being a positive, finite measure guarantees that $L^\infty(\lambda_g) = (L^1(\lambda_g))^*$ is valid, we can select an individual bounded function ψ_0 representing the linear functional ξ. That is,

$$\langle \phi, \xi \rangle = \int_T \phi \psi_0 g \, d\lambda, \qquad \phi \in L^1(\lambda_g),$$

and there exist a λ_g-null set $A \in \mathscr{S}$ and a positive number M such that $|\psi_0(t)| \leqslant M$ for all $t \in T \backslash A$. The function $\psi_1 := \psi_0 \chi_{T \backslash A}$ (defined pointwise on T) is clearly bounded and $\mathscr{S}$-measurable. Moreover, ψ_1 also represents ξ because $\psi_1 = \psi_0$ (λ_g-a.e.). In particular, since $\psi_1 g \in L^1(\lambda)$, we have

$$\int_T \varphi_E \psi_1 g \, d\lambda = \int_T \varphi_E \psi_0 g \, d\lambda = \langle \varphi_E, \xi \rangle = 0, \qquad E \in \Sigma. \tag{3.10}$$

By Proposition 2.5(ii), the function $\psi_1 g$ can be approximated in $L^1(\lambda)$ by elements of $h(Y)|_T$. So, fix any $\varepsilon > 0$ and select $f \in h(Y)$ such that

$$\int_T |\psi_1 g - f| \, d\lambda < \frac{\varepsilon}{2}. \tag{3.11}$$

Now, (3.3) gives $\Phi(|f|)(\Omega) = |\Phi(f)|(\Omega)$, so that

$$\int_T |f| \, d\lambda = |\Phi(f)|(\Omega) \tag{3.12}$$

by (3.5) with $|f| \in h(Y)$ in place of f. The Hahn Decomposition Theorem, [29, 6.14], applied to $\Phi(f) \in ca(\Sigma)$ provides a set $F \in \Sigma$ satisfying

$$|\Phi(f)|(\Omega) = |\Phi(f)(F)| + |\Phi(f)(\Omega \setminus F)|. \tag{3.13}$$

Observe that $\Phi(f)(F) = \int_T f \varphi_F \, d\lambda$ by (3.7) and that $\Phi(f)(\Omega \setminus F) = \int_T f \varphi_{\Omega \setminus F} \; d\lambda$ by (3.7) with $\Omega \setminus F$ in place of E. Therefore, from (ii)(c) above, as well as (3.10), (3.11), (3.12) and (3.13), it follows that

$$\begin{aligned} \int_T |f| \, d\lambda = |\Phi(f)(F)| + |\Phi(f)(\Omega \setminus F)| &= \left| \int_T f \varphi_F \, d\lambda \right| + \left| \int_T f \varphi_{\Omega \setminus F} \, d\lambda \right| \\ &= \left| \int_T \varphi_F (f - \psi_1 g) \, d\lambda \right| + \left| \int_T \varphi_{\Omega \setminus F} (f - \psi_1 g) \, d\lambda \right| \\ &\leqslant \int_T (\varphi_F + \varphi_{\Omega \setminus F}) |f - \psi_1 g| \, d\lambda = \int_T |f - \psi_1 g| \, d\lambda < \frac{\varepsilon}{2}. \end{aligned}$$

This and (3.11) imply that

$$\int_T |\psi_1 g| \, d\lambda \leqslant \int_T |f - \psi_1 g| \, d\lambda + \int_T |f| \, d\lambda < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\int_T |\psi_1 g| \, d\lambda = \int_T |\psi_1| g \, d\lambda = 0$, that is, $\psi_1 = 0$ (λ_g-a.e.). As ψ_1 represents $\xi \in (L^1(\lambda_g))^*$, we then have

$\langle \phi, \xi \rangle = \int_T \phi \psi_1 g \, d\lambda = 0$, for $\phi \in L^1(\lambda_g)$, that is, $\xi = 0$. So, we have proved that every continuous linear functional on $L^1(\lambda_g)$ vanishing on the subset $\{\varphi_E : E \in \Sigma\} \subseteq L^1(\lambda_g)$ is necessarily the zero functional. So, part (iv) follows from the Hahn-Banach Theorem. ■

Remark 3.3. We have obtained part (i) of Lemma 3.2 as a special case of an elementary fact, namely Corollary 2.6(ii). On the other hand, its proof as given in [15, Proof of Theorem 8], uses the Radon-Nikodým derivative (depending on E) of the scalar measure λ_E (representing the conical measure v_E) with respect to the decomposable measure λ.

Regarding (ii)(a) of Lemma 3.2, on which subsequent arguments are dependent, our proof uses the assumption that H_m is an *order ideal* of $ca(\Sigma)$. It seems to be open whether or not this assumption can be weakened to the requirement that H_m is merely a vector sublattice of $ca(\Sigma)$ as stated in [15, pp. 91-92 & proof of Theorem 8].

We now turn our attention to the measure algebra $(\mathscr{S}/\mathscr{N}_0(\lambda), \bar{\lambda})$ of the decomposable measure space $(T, \mathscr{S}, \lambda)$ and the corresponding quotient map

$$\pi_\lambda : \mathscr{S} \to \mathscr{S}/\mathscr{N}_0(\lambda);$$

see Section 2 for the notation and relevant definitions. Given $g \in L^1(\lambda)$, define a pseudometric d_g on $\mathscr{S}$ by $d_g(A,B) := \int_{A \triangle B} |g| \, d\lambda$ for $A, B \in \mathscr{S}$. Let $\rho(\lambda)$ be the uniformity generated by all the pseudometrics d_g with g varying through $L^1(\lambda)$. The associated *Hausdorff* uniform space is $\mathscr{S}/\mathscr{N}_0(\lambda)$ and its corresponding uniformity $\hat{\rho}(\lambda)$ is generated by the pseudometrics

$$\hat{d}_g : (\pi_\lambda(A), \pi_\lambda(B)) \mapsto d_g(A,B), \qquad A, B \in \mathscr{S}, \tag{3.14}$$

with g varying through $L^1(\lambda)$. This is a consequence of the fact that sets $A, B \in \mathscr{S}$ satisfy $\lambda(A \triangle B) = 0$ if and only if $d_g(A,B) = 0$ for all $g \in L^1(\lambda)$, which follows from the fact that $\chi_E \in L^1(\lambda)$ whenever $\lambda(E) < \infty$ and λ is decomposable; see the proof of Proposition 213J in [9].

It is worth noting that there exists a lcHs-valued vector measure ν on $\mathscr{S}$ such that the uniformity $\tau(\nu)$ on $\mathscr{S}$ induced by ν (see Section 2) satisfies $\tau(\nu) = \rho(\lambda)$ and $\hat{\tau}(\nu) = \hat{\rho}(\lambda)$. For example, this is the case for the vector measure $\nu : \mathscr{S} \to \mathbb{C}^{L^1(\lambda)}$ defined by

$$\nu(A) := \left(\int_A g \, d\lambda \right)_{g \in L^1(\lambda)}, \qquad A \in \mathscr{S}.$$

Recall that the weakly complete lcHs Y (hence, also the vector lattice $h(Y)$) is specified via a vector measure $m : \Sigma \to X$ defined in the measurable space (Ω, Σ). We proceed to define a map $\gamma : \Sigma \to \mathscr{S}/\mathscr{N}_0(\lambda)$. Given $E \in \Sigma$, take any function $\varphi_E \in L^\infty(\lambda)$ satisfying (3.7). Via Lemma 3.2(ii)(d), select $A \in \mathscr{S}$ such that $\varphi_E = \chi_A$ (λ-a.e.). Define $\gamma(E) := \pi_\lambda(A) \in \mathscr{S}/\mathscr{N}_0(\lambda)$. This definition does not depend on the choice of such a set $A \in \mathscr{S}$ because, for any other set $B \in \mathscr{S}$ satisfying $\varphi_E = \chi_B$ (λ-a.e.), we have $\chi_A = \chi_B$ (λ-a.e.) and hence, $\pi_\lambda(A) = \pi_\lambda(B)$.

Recall the identities $\mathscr{N}_0(m_\sigma) = \mathscr{N}_0(m)$, $q_{m_\sigma} = q_m$ and $\Sigma/\mathscr{N}_0(m_\sigma) = \Sigma/\mathscr{N}_0(m)$; see (2.8).

Lemma 3.4. *The following statements hold for the map* $\gamma : \Sigma \to \mathscr{S}/\mathscr{N}_0(\lambda)$.

(i) *The map* γ *is a* σ*-homomorphism between B.a.'s such that*

$$\gamma^{-1}(\{0\}) = \mathscr{N}_0(m_\sigma) = \mathscr{N}_0(m). \tag{3.15}$$

(ii) *Equip each of* Σ, $\Sigma/\mathscr{N}_0(m)$ *and* $\mathscr{S}/\mathscr{N}_0(\lambda)$ *with the uniformities* $\tau(m_\sigma)$, $\hat{\tau}(m_\sigma)$ *and* $\hat{\rho}(\lambda)$, *and their associated topologies, respectively.*

(a) *The map* γ *is uniformly continuous and has dense range in* $\mathscr{S}/\mathscr{N}_0(\lambda)$.

(b) *In view of (3.15) define a map* $\hat{\gamma} : \Sigma/\mathscr{N}_0(m) \to \mathscr{S}/\mathscr{N}_0(\lambda)$ *by*

$$\hat{\gamma}(q_m(E)) := \gamma(E), \qquad E \in \Sigma.$$

Then, $\hat{\gamma}$ *is a uniform isomorphism onto its range, that is,* $\hat{\gamma}$ *is bi-uniformly continuous when its range is equipped with the uniformity induced by* $\hat{\rho}(\lambda)$.

(iii) *The set function* $\iota_m := \bar{\lambda} \circ \gamma$ *on* Σ *is a* $[0, \infty]$*-valued measure such that*

$$\mathscr{N}_0(\iota_m) = \mathscr{N}_0(m). \tag{3.16}$$

Proof.

(i) To prove that γ is a B.a. homomorphism, let $E, F \in \Sigma$. Then $\varphi_E, \varphi_F \in L^\infty(\lambda)$. Select sets $A, B \in \mathscr{S}$ satisfying $\varphi_E = \chi_A$ (λ-a.e.) and $\varphi_F = \chi_B$ (λ-a.e.), so that $\gamma(E) = \pi_\lambda(A)$ and $\gamma(F) = \pi_\lambda(B)$; recall the discussion prior to this lemma. Now, it follows from Lemma 3.2(iii) that

$$\varphi_{E\cap F} = \varphi_E \wedge \varphi_F = \chi_A \wedge \chi_B = \chi_{A\cap B} \qquad (\lambda\text{-a.e.})$$

and hence, that

$$\gamma(E \cap F) = \pi_\lambda(A \cap B) = \pi_\lambda(A) \wedge \pi_\lambda(B) = \gamma(E) \wedge \gamma(F) \tag{3.17}$$

in $\mathscr{S}/\mathscr{N}_0(\lambda)$ because π_λ is a B.a. homomorphism. Moreover, as $\varphi_E = \chi_A$ (λ-a.e.), we have

$$\varphi_{\Omega\setminus E} = \mathbb{1} - \varphi_E = \mathbb{1} - \chi_A = \chi_{T\setminus A} \qquad (\lambda\text{-a.e.})$$

via Lemma 3.2(ii)(c). So, $\gamma(\Omega \setminus E) = \pi_\lambda(T \setminus A)$, of which the right-side equals the complement of $\pi_\lambda(A)$ in $\mathscr{S}/\mathscr{N}_0(\lambda)$. This together with (3.17) imply that γ is a B.a. homomorphism. An immediate consequence is that its range $\mathscr{R}(\gamma)$ is a Boolean subalgebra of $\mathscr{S}/\mathscr{N}_0(\lambda)$.

In order to verify that γ is a B.a. σ-homomorphism, take an increasing sequence $\{E(n)\}_{n=1}^\infty$ in Σ and let $E := \bigcup_{n=1}^\infty E(n)$. Select sets $A(n) \in \mathscr{S}$

for $n \in \mathbb{N}$ with $\varphi_{E(n)} = \chi_{A(n)}$ (λ-a.e.), so that $\gamma(E(n)) = \pi_\lambda(A(n))$ in $\mathscr{S}/\mathscr{N}_0(\lambda)$. Set $A := \bigcup_{n=1}^\infty A(n) \in \mathscr{S}$. The claim is that

$$\int_T f\varphi_E \, d\lambda = \int_T f\chi_A \, d\lambda, \qquad f \in h(Y). \tag{3.18}$$

Indeed, first suppose that $f \in h(Y)^+$. Since $\Phi(f) \in ca(\Sigma)$, we have $\lim_{n\to\infty} \Phi(f)(E(n)) = \Phi(f)(E)$. Moreover, $\chi_{A(n)}(t) \uparrow \chi_A(t)$ for λ-a.e. $t \in T$ because it follows from (3.9), with $G(1) := E(n+1) \setminus E(n)$ and $G(2) := E(n)$, that

$$\chi_{A(n+1)} = \varphi_{E(n+1)} = \varphi_{E(n+1)\setminus E(n)} + \varphi_{E(n)} \geqslant \varphi_{E(n)} = \chi_{A(n)}$$

holds λ-a.e. on T for every $n \in \mathbb{N}$. So the Monotone Convergence Theorem for λ ensures the validity of (3.18) as

$$\begin{aligned}\int_T f\varphi_E \, d\lambda = \Phi(f)(E) = \lim_{n\to\infty} \Phi(f)(E(n)) = \lim_{n\to\infty} \int_T f\varphi_{E(n)} \, d\lambda \\ = \lim_{n\to\infty} \int_T f\chi_{A(n)} \, d\lambda = \int_T f\chi_A \, d\lambda,\end{aligned}$$

in which we have used Lemma 3.2(i). So, (3.18) is valid for $f \in h(Y)^+$. Since each $f \in h(Y)$ is given by $f = f^+ - f^-$ with $f^+, f^- \in h(Y)^+$, [1, Theorem 1.1(2)], the identity (3.18) actually holds for all $f \in h(Y)$. Consequently, we can apply Corollary 2.6(i) with $(\varphi_E - \chi_A)$ in place of ψ there to deduce that $\varphi_E = \chi_A$ (λ-a.e.). Therefore

$$\gamma(E) := \pi_\lambda(A) = \pi_\lambda\Big(\bigcup_{n=1}^\infty A(n)\Big) = \bigvee_{n=1}^\infty \pi_\lambda(A(n)) = \bigvee_{n=1}^\infty \gamma(E(n))$$

in $\mathscr{S}/\mathscr{N}_0(\lambda)$ as π_λ is a B.a. σ-homomorphism. So, γ is also a B.a. σ-homomorphism.

To obtain (3.15), let us first verify that the following three conditions for any set $E \in \Sigma$ are equivalent:

(i-a) $\gamma(E) = 0$;

(i-b) $\Phi(f)(E) = 0$ for all $f \in h(Y)$; and

(i-c) $|\langle m, x^*\rangle|(E) = 0$ for all $x^* \in X^*$.

(i-a)$\Leftrightarrow$(i-b). Select $A \in \mathscr{S}$ satisfying $\varphi_E = \chi_A$ (λ-a.e.) so that $\gamma(E) = \pi_\lambda(A)$. Then, we have

$$\text{(i-a)} \Leftrightarrow A \in \mathscr{N}_0(\lambda) \Leftrightarrow \varphi_E = 0 \ (\lambda\text{-a.e.}).$$

Via Corollary 2.6(i), with $\psi := \varphi_E$, the identity $\varphi_E = 0$ (λ-a.e.) is equivalent to the condition that $\int_T f\varphi_E \, d\lambda = 0$ for all $f \in h(Y)$. The latter is equivalent to (i-b) via (3.7).

(i-b)⇒(i-c). Choose any $x^* \in X^*$. Then $|\langle m, x^*\rangle| \in H_m$ and, with $f := |\langle m, x^*\rangle|^\sim \in Y^* \subseteq h(Y)$, we have via Lemma 3.1 that $\Phi(f) = |\langle m, x^*\rangle|$. Accordingly $\Phi(f)(E) = |\langle m, x^*\rangle|(E)$. So (i-b) implies (i-c).
(i-c)⇒(i-b). Fix any $f \in h(Y)$. As $\Phi(f) \in H_m$, there exists $n \in \mathbb{N}$ and $x_1^*, \ldots, x_n^* \in X^*$ such that

$$|\Phi(f)| \leqslant \sum_{j=1}^{n} |\langle m, x_j^*\rangle| \quad \text{on } \Sigma; \tag{3.19}$$

see (3.1) with $\mu := \Phi(f)$. Then (i-c) implies (i-b) because

$$|\Phi(f)(E)| \leqslant |\Phi(f)|(E) \leqslant \sum_{j=1}^{n} |\langle m, x_j^*\rangle|(E).$$

So we have established all the stated equivalences.
Now, (3.15) holds because (2.4) implies that (i-c) holds if and only if $E \in \mathscr{N}_0(m)$ and because of the equivalence (i-a)⇔(i-c). Part (i) is thereby established.

(ii) To verify (a), fix $g \in L^1(\lambda)$. Let $\varepsilon > 0$. By Proposition 2.5(ii) select $f \in h(Y)$ such that $\int_T |g - f|\, d\lambda < \varepsilon$. Given any $E, F \in \Sigma$, choose $A, B \in \mathscr{S}$ satisfying $\varphi_E = \chi_A$ (λ-a.e.) and $\varphi_F = \chi_B$ (λ-a.e.). Since $\gamma(E) = \pi_\lambda(A)$ and $\gamma(F) = \pi_\lambda(B)$, it follows from (3.14) that

$$\begin{aligned} \hat{d}_g(\gamma(E), \gamma(F)) &= \int_{A\triangle B} |g|\, d\lambda \\ &\leqslant \int_{A\triangle B} |g - f|\, d\lambda + \int_{A\triangle B} |f|\, d\lambda \\ &< \varepsilon + \int_{A\triangle B} |f|\, d\lambda. \end{aligned} \tag{3.20}$$

We claim that

$$\int_{A\triangle B} |f|\, d\lambda = |\Phi(f)|(E\triangle F). \tag{3.21}$$

To verify this note that (3.9), with $G(1) := E \setminus F$ and $G(2) := E \cap F$, and Lemma 3.2(iii) imply that

$$\varphi_{E\setminus F} = \varphi_E - \varphi_{E\cap F} = \varphi_E - (\varphi_E \wedge \varphi_F) = \chi_A - (\chi_A \wedge \chi_B) = \chi_{A\setminus B}$$

holds λ-a.e. on T. Similarly, $\varphi_{F\setminus E} = \chi_{B\setminus A}$ (λ-a.e.). Clearly $\chi_{A\setminus B} + \chi_{B\setminus A} = \chi_{A\triangle B}$. Again via (3.9), with $G(1) := E \setminus F$ and $G(2) := F \setminus E$, we also have $\varphi_{E\triangle F} = \varphi_{E\setminus F} + \varphi_{F\setminus E}$. It follows that $\varphi_{E\triangle F} = \chi_{A\triangle B}$ (λ-a.e.). Hence, (3.21) holds, via (3.3) and (3.7) with $E\triangle F$ in place of E, because

$$|\Phi(f)|(E\triangle F) = \Phi(|f|)(E\triangle F) = \int_T |f|\varphi_{E\triangle F}\, d\lambda = \int_T |f|\chi_{A\triangle B}\, d\lambda.$$

Now, take $x_1^*, \dots, x_n^* \in X^*$ satisfying (3.19). Then, (3.20) and (3.21) yield

$$\hat{d}_g(\gamma(E), \gamma(F)) \leqslant \varepsilon + \sum_{j=1}^{n} |\langle m, x_j^* \rangle|(E \triangle F).$$

Since the function $g \in L^1(\lambda)$, the sets $E, F \in \Sigma$, and $\varepsilon > 0$ are arbitrary, this inequality verifies the uniform continuity of γ (via the definition of $\tau(m_\sigma)$).

To prove that $\mathscr{R}(\gamma)$ is dense in $\mathscr{S}/\mathscr{N}_0(\lambda)$ fix $g \in (L^1(\lambda))^+$ and $\varepsilon > 0$. Let $A \in \mathscr{S}$. Then there exists an individual function $\psi \in \operatorname{span}\{\varphi_E : E \in \Sigma\}$ such that

$$\int_T |\chi_A - \psi| g \, d\lambda < \frac{\varepsilon}{2};$$

see Lemma 3.2(iv). Let $B := \{t \in T : |1 - \psi(t)| \leqslant \frac{1}{2}\} \in \mathscr{S}$. It is routine to verify that $|\chi_A(t) - \chi_B(t)| \leqslant 2|\chi_A(t) - \psi(t)|$ for each $t \in T$ and hence,

$$\int_T |\chi_A - \chi_B| g \, d\lambda \leqslant 2 \int_T |\chi_A - \psi| g \, d\lambda < \varepsilon. \tag{3.22}$$

To see that $\pi_\lambda(B) \in \mathscr{R}(\gamma)$, let us write $\psi = \sum_{j=1}^{n} a_j \varphi_{E(j)}$ for some $a_1, \dots, a_n \in \mathbb{R}$ and $E(1), \dots, E(n) \in \Sigma$ with $n \in \mathbb{N}$. Select $A(1), \dots, A(n)$ from $\mathscr{S}$ satisfying $\chi_{A(j)} = \varphi_{E(j)}$ (λ-a.e.) and hence, $\pi_\lambda(A(j)) = \gamma(E(j))$ for $j = 1, \dots, n$. Since Σ and $\mathscr{S}$ are σ-algebras, π_λ and γ are B.a. homomorphisms, and $\mathscr{R}(\gamma)$ is a Boolean subalgebra of $\mathscr{S}/\mathscr{N}_0(\lambda)$, it follows that there exist distinct real numbers $b_1, \dots, b_l$ and pairwise disjoint sets $B(1), \dots, B(l) \in \mathscr{S}$ with $T = \bigcup_{k=1}^{l} B(k)$ such that $\sum_{j=1}^{n} a_j \chi_{A(j)} = \sum_{k=1}^{l} b_k \chi_{B(k)}$ pointwise on T and such that $\pi_\lambda(B(k) \in \mathscr{R}(\gamma)$ for $k = 1, \dots, l$. So, we may assume that $\psi = \sum_{k=1}^{l} b_k \chi_{B(k)}$ pointwise on T. Let $K \subseteq \{1, \dots, l\}$ denote the subset of all those k's satisfying $|1 - b_k| \leqslant \frac{1}{2}$. Then the set B equals $\bigcup_{k \in K} B(k)$, from which it follows that

$$\pi_\lambda(B) = \pi_\lambda\Big(\bigcup_{k \in K} B(k)\Big) = \bigvee_{k \in K} \pi_\lambda(B(k)) \in \mathscr{R}(\gamma) \tag{3.23}$$

as both π_λ and γ are B.a. σ-homomorphisms and because $\mathscr{R}(\gamma)$ is a Boolean subalgebra of $\mathscr{S}/\mathscr{N}_0(\lambda)$. An appeal to (3.14) and (3.22) gives

$$\hat{d}_g(\pi_\lambda(A), \pi_\lambda(B)) = d_g(A, B) = \int_T |\chi_A - \chi_B| g \, d\lambda < \varepsilon.$$

This together with (3.23) imply that $\mathscr{R}(\gamma)$ is dense in $\mathscr{S}/\mathscr{N}_0(\lambda)$ because $A \in \mathscr{S}$ is arbitrary.

Next let us verify (b). It is clear from (3.15) that $\hat{\gamma}$ is injective. The uniform continuity of $\hat{\gamma}$ follows from that of γ (cf. part (a)) and the definition of $\hat{\tau}(m_\sigma)$; see the discussion prior to Proposition 2.4. To prove that $\hat{\gamma}$ admits a uniformly continuous inverse, let $x^* \in X^*$ and set $\mu := |\langle m, x^*\rangle| \in H_m$. Consider the function $f := \tilde{\mu} \in Y^* \subseteq h(Y)$, which satisfies $\Phi(f) = \Phi(\tilde{\mu}) = \mu$. Then, given $E, F \in \Sigma$, we have

$$|\Phi(f)|(E \triangle F) = |\mu|(E \triangle F) = |\langle m, x^*\rangle|(E \triangle F). \tag{3.24}$$

For ease of notation, let us write $d_f(\cdot,\cdot) := d_{f|_T}(\cdot,\cdot)$, after noting that $f|_T \in h(Y)|_T \subseteq L^1(\lambda)$. Take $A, B \in \mathscr{S}$ satisfying $\pi_\lambda(A) = \gamma(E)$ and $\pi_\lambda(B) = \gamma(F)$. From (3.14), (3.21) and (3.24) we have

$$\begin{aligned} \hat{d_f}(\gamma(E), \gamma(F)) = d_f(A, B) = \int_{A \triangle B} |f|\, d\lambda &= |\Phi(f)|(E \triangle F) \\ = |\langle m, x^*\rangle|(E \triangle F). \end{aligned} \tag{3.25}$$

Recall that $\hat{\tau}(m_\sigma)$ is generated by the pseudometrics (2.10), with x^* varying through X^*. So, (3.25) implies that $\hat{\gamma}$ admits a uniformly continuous inverse because

$$\hat{d_f}(\gamma(E), \gamma(F)) = \hat{d_f}(\hat{\gamma}(q_m(E)), \hat{\gamma}(q_m(F))), \qquad E, F \in \Sigma.$$

(iii) Let $\{E_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint sets in Σ. Since γ is a B.a. σ-homomorphism (see part (i) above), it follows that the sequence $\{\gamma(E_n)\}_{n=1}^\infty$ is also pairwise disjoint and satisfies $\gamma(\bigcup_{n=1}^\infty E_n) = \bigvee_{n=1}^\infty \gamma(E_n)$ in $\mathscr{S}/\mathscr{N}_0(\lambda)$. So, we have

$$\begin{aligned} \iota_m\Big(\bigcup_{n=1}^\infty E_n\Big) &= \bar{\lambda}\Big(\gamma_m\Big(\bigcup_{n=1}^\infty E_n\Big)\Big) = \bar{\lambda}\Big(\bigvee_{n=1}^\infty \gamma(E_n)\Big) \\ &= \sum_{n=1}^\infty \bar{\lambda}(\gamma(E_n)) = \sum_{n=1}^\infty \iota_m(E_n) \end{aligned}$$

from (2.1) with $\bar{\iota} := \bar{\lambda}$ and $\eth_n := E_n$ for $n \in \mathbb{N}$. Thus, $\iota_m : \Sigma \to [0,\infty]$ is a measure.

Next, by (2.1) with $\bar{\iota} := \bar{\lambda}$ and (3.15), we have

$$\mathscr{N}_0(\iota_m) = (\iota_m)^{-1}(\{0\}) = \gamma^{-1}\left(\left(\bar{\lambda}\right)^{-1}(\{0\})\right) = \gamma^{-1}(\{0\}) = \mathscr{N}_0(m).$$

This completes the proof of Lemma 3.4. ■

We now come to the proof of our main result, namely Theorem 2 (see Section 1).

Proof of Theorem 2. (i)⇒(ii). Let the notation be as in Lemma 3.4 and define $\iota : \Sigma \to [0,\infty]$ to be the scalar measure $\iota_m := \bar{\lambda}\circ\gamma : \Sigma \to [0,\infty]$; see Lemma 3.4(iii). We first show that γ is surjective. By Proposition 2.4 the vector measure $m_\sigma : \Sigma \to X_{\sigma(X,X^*)}$ is also closed and hence, $\Sigma/\mathscr{N}_0(m) = \Sigma/\mathscr{N}_0(m_\sigma)$ is $\hat{\tau}(m_\sigma)$-complete. Recall the uniform isomorphism $\hat{\gamma} : \Sigma/\mathscr{N}_0(m) \to \mathscr{S}/\mathscr{N}_0(\lambda)$, considered as mapping onto its range; see Lemma 3.4(ii)(b). Since the domain $\Sigma/\mathscr{N}_0(m) = \Sigma/\mathscr{N}_0(m_\sigma)$ of $\hat{\gamma}$ is $\tau(m_\sigma)$-complete, its range $\mathscr{R}(\hat{\gamma})$ is then also $\hat{\rho}(\lambda)$-complete and hence, is a closed set in the Hausdorff uniform space $\mathscr{S}/\mathscr{N}_0(\lambda)$. So, $\mathscr{R}(\gamma)$ which equals $\mathscr{R}(\hat{\gamma})$ is closed in $\mathscr{S}/\mathscr{N}_0(\lambda)$. This implies that $\mathscr{R}(\gamma) = \mathscr{S}/\mathscr{N}_0(\lambda)$ as we already know that $\mathscr{R}(\gamma)$ is dense in $\mathscr{S}/\mathscr{N}_0(\lambda)$; see Lemma 3.4(ii)(a). We have thereby established the surjectivity of γ.

Fixing $x^* \in X^*$, let us show that $|\langle m, x^*\rangle|$ is truly continuous with respect to ι. To this end, we may assume that $|\langle m, x^*\rangle|$ is not the zero measure. First, we have $|\langle m, x^*\rangle| << \iota$. This is a consequence of (2.2) (with $\xi := |\langle m, x^*\rangle|$), (2.4) and (3.16). Define $f := |\langle m, x^*\rangle|^\sim \in Y^* \subseteq h(Y)$ and recall that $|\langle m, x^*\rangle| \in H_m$. Then, $\int_\Omega |f|\, d\lambda < \infty$ by Proposition 2.5(i) and $\Phi(f) = |\langle m, x^*\rangle|$ on Σ by Lemma 3.1.

Let us observe the general fact that, given any $E \in \Sigma$,

$$\int_A |f| d\lambda = |\langle m, x^*\rangle|(E), \qquad \forall A \in \mathscr{S} \text{ satisfying } \pi_\lambda(A) = \gamma(E). \tag{3.26}$$

This is a consequence of the definition of $\gamma(E)$ together with (3.3) and (3.7) as follows:

$$\int_A |f|\, d\lambda = \int_T |f|\varphi_E\, d\lambda = \Phi(|f|)(E) = |\Phi(f)|(E) = |\langle m, x^*\rangle|(E).$$

Now select $E \in \Sigma$ with $|\langle m, x^*\rangle|(E) > 0$ and choose $A \in \mathscr{S}$ satisfying $\pi_\lambda(A) = \gamma(E)$. As λ is *decomposable* and $\int_A |f|\, d\lambda > 0$ by (3.26), there exists $\alpha \in \mathbb{A}$ such that $\int_{A\cap T(\alpha)} |f|\, d\lambda > 0$. Since γ is surjective, there exists $F \in \Sigma$ with $\gamma(F) = \pi_\lambda(A \cap T(\alpha))$. As γ and π_λ are both B.a. homomorphisms and $\gamma(E) = \pi_\lambda(A)$, we have

$$\begin{aligned}\gamma(E \cap F) &= \gamma(E) \wedge \gamma(F) = \gamma(E) \wedge \pi_\lambda(A \cap T(\alpha)) \\ &= \gamma(E) \wedge \pi_\lambda(A) \wedge \pi_\lambda(T(\alpha)) = \pi_\lambda(A \cap T(\alpha)).\end{aligned}$$

So, from (3.26) with $E \cap F$ in place of E and $A \cap T(\alpha)$ in place of A, it follows that

$$|\langle m, x^*\rangle|(E \cap F) = \int_{A\cap T(\alpha)} |f|\, d\lambda > 0. \tag{3.27}$$

On the other hand, we have

$$0 < \iota(E \cap F) \leqslant \iota(F) < \infty. \tag{3.28}$$

To see this, note that $\iota(E \cap F) > 0$ via (3.27) because $|\langle m, x^*\rangle| << \iota$. Moreover, we also have

$$\iota(F) = \bar{\lambda} \circ \gamma(F) = \bar{\lambda} \circ \pi_\lambda(A \cap T(\alpha)) = \lambda(A \cap T(\alpha)) \leqslant \lambda(T(\alpha)) < \infty.$$

So, (3.28) holds. This, together with (3.27), ensures that $|\langle m, x^*\rangle|$ is truly continuous with respect to ι as we already know that $|\langle m, x^*\rangle| << \iota$. Now, from Lemma 2.1(ii) with $\xi := \langle m, x^*\rangle$, we conclude that $\langle m, x^*\rangle$ is also truly continuous with respect to ι.

Finally, let us show that ι is localizable. First the B.a. $\Sigma/\mathscr{N}_0(\iota)$ is complete because $\Sigma/\mathscr{N}_0(\iota) = \Sigma/\mathscr{N}_0(m)$ as B.a.'s (via (3.16) with ι in place of ι_m) and because the B.a. $\Sigma/\mathscr{N}_0(m)$ is complete for the closed vector measure m (see Lemma 2.2). To see that ι is semifinite, take any $E \in \Sigma$ with $\iota(E) = \infty$. Then $E \notin \mathscr{N}_0(m)$ and hence, there exists $x^* \in X^*$ such that $E \notin \mathscr{N}_0(\langle m, x^*\rangle)$ (see (2.4)), that is, $|\langle m, x^*\rangle|(E) > 0$. By the first part of this proof there exists $F \in \Sigma$ satisfying (3.28), which implies that ι is semifinite as $(E \cap F) \subseteq E$. Thus, ι is localizable. So, we have deduced (ii) from (i).

(ii)⇒(iii). Let $x^* \in X^*$. Since $\langle m, x^*\rangle$ is truly continuous with respect to ι, there is a ι-integrable function $\phi_{x^*} : \Omega \to \mathbb{C}$ such that $\langle m, x^*\rangle(E) = \int_E \phi_{x^*}\, d\iota$ for each $E \in \Sigma$; see Lemma 2.1(i). Define a function $F : \Omega \to (X^*)^a$ by $\langle F(\omega), x^*\rangle := \phi_{x^*}(\omega)$ for each $\omega \in \Omega$ and $x^* \in X^*$. Then, $\langle F, x^*\rangle$ is ι-integrable for each $x^* \in X^*$ and satisfies (1.1).

(iii)⇒(ii). For each $x^* \in X^*$, set $\phi_{x^*} := \langle F, x^*\rangle$. Then the function $\phi_{x^*} : \Omega \to \mathbb{C}$ is ι-integrable and $\langle m, x^*\rangle(E) = \int_E \phi_{x^*}\, d\iota$ for $E \in \Sigma$. Again, by Lemma 2.1(i), the measure $\langle m, x^*\rangle$ is truly continuous with respect to ι. So, the implication (iii)⇒(ii) is established.

(ii)⇒(i). This is precisely Theorem 1 of Section 1.

Finally, in the proof of (i)⇒(ii), we have chosen $\iota := \iota_m$, for which we have $\mathscr{N}_0(\iota) = \mathscr{N}_0(\iota_m) = \mathscr{N}_0(m)$; see (3.16). The proof of Theorem 2 is thereby complete. ∎

Remark 3.5. Let us return to the discussion immediately after Theorem 2 in Section 1. In the notion from there, let H_1 denote the order ideal in $ca(\Sigma)$ generated by

$$\{|\langle m, x^*\rangle| : x^* \in X^*\} \cup \{\delta_\omega : \omega \in \Omega\}.$$

Denote the algebraic dual of H_1 by Y_1 (which is a weakly complete lcHs for $\sigma(Y_1, H_1)$) and let $h(Y_1)$ be the vector lattice generated by $(Y_1)^*$ in $\mathbb{R}^Y$. As in Lemma 3.1 we can define $\Phi_1 : h(Y_1) \to H_1$ which then induces the conical measure u_1 via

$$u_1(f) := \Phi_1(f)(\Omega), \qquad f \in h(Y_1).$$

Apply Proposition 2.5 to select a decomposable measure $(T_1, \mathscr{S}_1, \lambda_1)$ representing u_1 and then a B.a. σ-homomorphism $\gamma_1 : \Sigma \to \mathscr{S}_1/\mathscr{N}_0(\lambda_1)$ as in Lemma 3.4. However, since now $\{\delta_\omega : \omega \in \Omega\} \subseteq H_1$, it turns out that $\gamma_1^{-1}(\{0\}) = \{\emptyset\}$, which prevents γ_1 from factoring through the quotient B.a. $\Sigma/\mathscr{N}_0(m)$. This is what causes the difficulty mentioned in Section 1. It is in contrast with our B.a. σ-homomorphism $\gamma : \Sigma \to \mathscr{S}/\mathscr{N}_0(\lambda)$ which *does* factor through $\Sigma/\mathscr{N}_0(m)$ as $\gamma = \hat{\gamma} \circ q_m$; see Lemma 3.4(ii)(b). Since the proof of Corollary 13 given in [15] requires a B.a. isomorphism defined on $\Sigma/\mathscr{N}_0(m)$, it appears to be the case that Theorem 12 of [15] (and its proof) are not applicable to establish Corollary 13.

We also point out that the proof of Corollary 13 in [15] relies on the fact that m is a closed vector measure if and only if so is m_σ, without any explanation. This fact is exactly our Proposition 2.4; it first appeared in [26], albeit with an incorrect proof, and does not seem to have appeared before 1984.

4. Appendices

A. Proof of Theorem 1

The standing assumption throughout Section 4 is that m is a vector measure, defined on a measurable space (Ω, Σ), with values in a (complex) lcHs X.

The *sequential completion* $\tilde{X}$ of X is defined as the smallest sequentially closed linear subspace of the quasi-completion of X, [17, pp.296-297], [24, p.14]. So, the initial topology on X is the induced topology by $\tilde{X}$. Let $J_X : X \to \tilde{X}$ denote the natural embedding. Since X is dense in $\tilde{X}$, the dual space $(\tilde{X})^*$ of $\tilde{X}$ is identified with the dual space X^* of X. In precise terms, the linear map $\xi^* \mapsto \xi^* \circ J_X$ for $\xi^* \in (\tilde{X})^*$ is a linear isomorphism from $(\tilde{X})^*$ onto X^*.

Every $p \in \mathscr{P}(X)$ admits a unique extension $\tilde{p} \in \mathscr{P}(\tilde{X})$ and conversely, every continuous seminorm on $\tilde{X}$ is realized as such an extension. In other words,

$$\mathscr{P}(\tilde{X}) = \{\tilde{p} : p \in \mathscr{P}(X)\}.$$

We will require the identity

$$U_p^\circ = \{\xi^* \circ J_X : \xi^* \in U_{\tilde{p}}^\circ\}, \qquad p \in \mathscr{P}(X). \tag{A.1}$$

To establish (A.1) let $\xi^* \in (\tilde{X})^*$. Then the following conditions are equivalent.

(a) $\xi^* \circ J_X \in U_p^\circ$.
(b) $|\langle x, \xi^* \circ J_X\rangle| \leqslant p(x), \qquad x \in X$.
(c) $|\langle J_X(x), \xi^*\rangle| \leqslant \tilde{p}(J_X(x)), \qquad x \in X$.
(d) $|\langle \xi, \xi^*\rangle| \leqslant \tilde{p}(\xi), \qquad \xi \in \tilde{X}$.

Indeed, (a)⇔(b) is clear by the definition of U_p°. Further, the equivalence (b)⇔(c) can be obtained via the definition of $\tilde{p}$. To verify the implication (c)⇒(d), let $\xi \in \tilde{X}$. Then there exists a net $\{x_\kappa\}$ in X convergent to ξ in $\tilde{X}$, i.e., $\lim_\kappa J_X(x_\kappa) = \xi$ in $\tilde{X}$, because X is dense in $\tilde{X}$. Then (c) implies (with x_κ in place of x) that

$$|\langle \xi, \xi^*\rangle| = |\langle \lim_\kappa J_X(x_\kappa), \xi^*\rangle| = \lim_\kappa |\langle J_X(x_\kappa), \xi^*\rangle| \leqslant \lim_\kappa \tilde{p}(J_X(x_\kappa)) = \tilde{p}(\xi).$$

We have thereby established (d). The reverse implication (d)⇒(c) is clear because $J(x) \in \tilde{X}$ for all $x \in X$. The four equivalences (a)–(d) easily imply (A.1).

The composition

$$J_X \circ m : \Sigma \to \tilde{X}$$

is a vector measure because J_X is continuous and linear.

Lemma A.1. *The following statements hold.*

(i) *For every* $p \in \mathscr{P}(X)$,

$$p(m)(E) = \tilde{p}(J_X \circ m)(E), \qquad E \in \Sigma. \tag{A.2}$$

(ii) *The identity* $\mathscr{N}_0(m) = \mathscr{N}_0(J_X \circ m)$ *holds, so that we have* $\Sigma/\mathscr{N}_0(m) = \Sigma/\mathscr{N}_0(J_X \circ m)$ *as B.a.'s.*
(iii) *The uniformities* $\hat{\tau}(m)$ *and* $\hat{\tau}(J_X \circ m)$ *coincide on* $\Sigma/\mathscr{N}_0(m)$.
(iv) *The vector measure* m *is closed if and only if so is the vector measure* $J_X \circ m : \Sigma \to \tilde{X}$.

Proof.

(i) We acquire (A.2) from (A.1) as follows:

$$\begin{aligned} p(m)(E) &= \sup\{|\langle m, x^*\rangle|(E) : x^* \in U_p^\circ\} \\ &= \sup\{|\langle m, \xi^* \circ J_X\rangle|(E) : \xi^* \in U_{\tilde{p}}^\circ\} \\ &= \sup\{|\langle J_X \circ m, \xi^*\rangle|(E) : \xi^* \in U_{\tilde{p}}^\circ\} = \tilde{p}(J_X \circ m)(E). \end{aligned}$$

(ii) This follows from (i) once we recall the definitions of $\mathscr{N}_0(m)$ and $\mathscr{N}_0(J_X \circ m)$ from Section 2; see also (2.4).
(iii) By (i) we have $p(m) = \tilde{p}(J_X \circ m)$ on Σ and hence, via part (ii), that $\hat{p}(m) = (\tilde{p})\hat{}\,(J_X \circ m)$ on $\Sigma/\mathscr{N}_0(m)$ whenever $p \in \mathscr{P}(X)$. So, the uniformities $\hat{\tau}(m)$ and $\hat{\tau}(J_X \circ m)$ coincide; see also [24, Lemma 2.5(iii)].
(iv) This follows from (iii) and the definition of a closed vector measure. ■

Lemma A.2. *Given* $p \in \mathscr{P}(X)$, *there exists* $x_p^* \in X^*$ *such that* $\mu := |\langle m, x_p^*\rangle|$ *has the property*

$$\lim_{\mu(E)\to 0} p(m)(E) = 0. \tag{A.3}$$

Proof. Let X_p denote the Banach space completion of the quotient normed space $X/p^{-1}(\{0\})$ with respect to the norm induced by p. By $\Lambda_p : X \to X_p$ we denote the corresponding quotient map. Then, we have

$$U_p^\circ = \{\xi^* \circ \Lambda_p : \xi^* \in \mathbb{B}[X_p^*]\}. \tag{A.4}$$

Here, $\mathbb{B}[X_p^*]$ denotes the closed unit ball of the dual Banach space X_p^* of X_p.

The composition $\Lambda_p \circ m : \Sigma \to X_p$ is also a vector measure because Λ_p is continuous and linear. Let $\|\Lambda_p \circ m\| : \Sigma \to [0, \infty)$ denote the *semivariation* of $\Lambda_p \circ m$ with respect to the norm on X_p, [5, Definition I.1.4], that is,

$$\|\Lambda_p \circ m\|(E) := \sup\{|\langle \Lambda_p \circ m, \xi^*\rangle|(E) : \xi^* \in \mathbb{B}[X_p^*]\}, \qquad E \in \Sigma.$$

It follows from (A.4) that

$$p(m)(E) = \|\Lambda_p \circ m\|(E), \qquad E \in \Sigma; \tag{A.5}$$

see also the formula (2.11) with $f := \chi_E$ in [24, p.11]. Via [5, Theorem I.2.1] and Rybakov's Theorem, [5, Theorem IX.2.2], there exists $\xi^* \in X_p^*$ such that

$\lim_{\nu(E)\to 0}\|\Lambda_p\circ m\|(E)=0$ with $\nu:=|\langle\Lambda_p\circ m,\xi^*\rangle|$. Then $x_p^*:=\xi^*\circ\Lambda_p$ belongs to X^* and $\mu:=|\langle m,x_p^*\rangle|$ is precisely ν. It follows from (A.5) that

$$\lim_{\mu(E)\to 0} p(m)(E)=\lim_{\nu(E)\to 0}\|\Lambda_p\circ m\|(E)=0,$$

which is precisely (A.3). ■

A Σ-measurable function $f:\Omega\to\mathbb{C}$ is called *m-integrable* if the following two conditions are satisfied: f is $\langle m,x^*\rangle$-integrable for every $x^*\in X^*$ and, given $E\in\Sigma$, there exists a unique vector $\int_E f\,dm\in X$ such that

$$\langle\int_E f\,dm,x^*\rangle=\int_E f\,d\langle m,x^*\rangle,\qquad x^*\in X^*.$$

In this case, the indefinite integral of f with respect to m is the X-valued set function

$$m_f:E\mapsto\int_E f\,dm,\qquad E\in\Sigma.$$

It follows from the Orlicz-Pettis Theorem, [19, Theorem 1], that m_f is σ-additive.

By $\mathscr{L}^1(m)$ we denote the complex vector space of all m-integrable functions on Ω. Every $\mathbb{C}$-valued, Σ-simple function on Ω is m-integrable. Indeed, for every $E\in\Sigma$, its characteristic function χ_E is m-integrable with $\int_F\chi_E\,dm=m(E\cap F)$ for $F\in\Sigma$. Furthermore, if $f\in\mathscr{L}^1(m)$ and $E\in\Sigma$, then $f\chi_E\in\mathscr{L}^1(m)$ and $\int_F f\chi_E\,dm:=\int_{E\cap F}f\,dm$ for $F\in\Sigma$.

Given $p\in\mathscr{P}(X)$, define a function $p(m)_1:\mathscr{L}^1(m)\to[0,\infty)$ by $p(m)_1(f):=p(m_f)(\Omega)<\infty$ for $f\in\mathscr{L}^1(m)$. Then,

$$p(m)_1(f)=\sup\left\{\int_\Omega|f|\,d|\langle m,x^*\rangle|:x^*\in U_p^\circ\right\},\qquad f\in\mathscr{L}^1(m),\tag{A.6}$$

[16, Lemma II.2.2(ii)], [24, p.11], by which it is clear that $p(m)_1$ is a seminorm. Equip $\mathscr{L}^1(m)$ with the locally convex topology (called the mean convergence topology) generated by the seminorms $p(m)_1$ with p varying through $\mathscr{P}(X)$. Its associated lcHs is the quotient space

$$L^1(m):=\mathscr{L}^1(m)/\mathscr{N}(m)$$

with respect to the closed linear subspace

$$\mathscr{N}(m):=\bigcap_{p\in\mathscr{P}(X)}p(m)_1^{-1}(\{0\}).$$

It is clear that a function $f\in\mathscr{L}^1(m)$ belongs to $\mathscr{N}(m)$ if and only if m_f is the zero vector measure, that is, $\int_E f\,dm=0$ for all $E\in\Sigma$. Moreover, a Σ-measurable function $f:\Omega\to\mathbb{C}$ is equal to 0 outside of some m-null set if and only if f is both m-integrable and m-null. Given $p\in\mathscr{P}(X)$, define $\bar{p}(m)_1:L^1(m)\to[0,\infty)$ by

$$\bar{p}(m)_1(f+\mathscr{N}(m)):=p(m)_1(f),\qquad f\in\mathscr{L}^1(m).$$

The topology in the lcHs $L^1(m)$ is, of course, generated by the seminorms $\bar{p}(m)_1$ with p varying through $\mathscr{P}(X)$.

Define

$$\Sigma(m) := \{\chi_E + \mathscr{N}(m) : E \in \Sigma\} \subseteq L^1(m).$$

In [16, p.25 and p.71], the identification of $\Sigma(m)$ with $\Sigma/\mathscr{N}_0(m)$ is adopted and the vector measure m is defined to be *closed* when $\Sigma(m)$ is a complete subset of the lcHs $L^1(m)$. Recalling our definition of a closed vector measure from Section 2, we shall formally verify in Lemma A.3(i) below that the two definitions are equivalent.

Lemma A.3. *The following statements hold for the vector measure $m : \Sigma \to X$.*

(i) *The vector measure m is closed if and only if $\Sigma(m)$ is a complete subset of the lcHs $L^1(m)$.*

(ii) *If $\Sigma(m)$ is relatively weakly compact in $L^1(m)$, then m is a closed vector measure. The converse holds if, in addition, X is sequentially complete.*

(iii) *The following assertions are equivalent.*

  (a) *The vector measure $m : \Sigma \to X$ is closed.*

  (b) *The vector measure $J_X \circ m : \Sigma \to \tilde{X}$ is closed.*

  (c) *The subset $\Sigma(J_X \circ m) := \{\chi_E + \mathscr{N}(J_X \circ m) : E \in \Sigma\}$ is relatively weakly compact in the lcHs $L^1(J_X \circ m)$.*

Proof.

(i) Via (A.6) with $f := \chi_E$ and the definition of $p(m)$ as given Section 2, we have

$$p(m)(E) = p(m)_1(\chi_E), \qquad E \in \Sigma, \tag{A.7}$$

for each $p \in \mathscr{P}(X)$. So, the quotient map $q_m : \Sigma \to \Sigma/\mathscr{N}_0(m)$ (see Section 2) induces the canonical map

$$q_m(E) := E + \mathscr{N}_0(m) \mapsto (\chi_E + \mathscr{N}(m)), \qquad E \in \Sigma, \tag{A.8}$$

from $\Sigma/\mathscr{N}_0(m)$ onto $\Sigma(m)$, which is well defined and is a bijection. Moreover, we have, via (A.7) for each $p \in \mathscr{P}(X)$, that

$$\begin{aligned}\hat{p}(m)(q_m(E\triangle F)) &:= p(m)(E\triangle F) = p(m)_1(\chi_{E\triangle F}) = p(m)_1(\chi_E - \chi_F) \\ &= \bar{p}(m)_1((\chi_E + \mathscr{N}(m)) - (\chi_F + \mathscr{N}(m)))\end{aligned}$$

whenever $E, F \in \Sigma$. This implies that the canonical map (A.8) is a uniform isomorphism with respect to $\hat{\tau}(m)$ on $\Sigma/\mathscr{N}_0(m)$ and the uniformity on $\Sigma(m)$ induced by $L^1(m)$. Thus, $\Sigma/\mathscr{N}_0(m)$ is $\hat{\tau}(m)$-complete if and only if $\Sigma/\mathscr{N}_0(m)$ is a complete subset of $L^1(m)$. This establishes (i).

(ii) See [22, Proposition 2.4 and Remark 2.6(vi)].

(iii) For (a)⇔(b), see Lemma A.1(iv). The equivalence (b)⇔(c) follows from part (ii) above with $\tilde{X}$ in place of X and $J_X \circ m$ instead of m. ■

From now on, let us identify $\mathscr{L}^1(m)$ with $L^1(m)$ except when precise arguments require a distinction between those two spaces. So, we treat an equivalence class $f + \mathscr{N}(m)$ in $L^1(m)$ as the function f, and functions which are equal outside of an m-null set will be identified.

To consider the weak topology on $L^1(m)$ later, we will require the following result.

Lemma A.4. *Let $\eta \in (L^1(m))^*$.*

(i) *The set function $\nu_\eta : \Sigma \to \mathbb{C}$ defined by $\nu_\eta(E) := \langle \chi_E, \eta \rangle$, for $E \in \Sigma$, is σ-additive.*
(ii) *Every m-integrable function f is also ν_η-integrable and $\langle f, \eta \rangle = \int_\Omega f \, d\nu_\eta$.*
(iii) *Suppose that $\iota : \Sigma \to [0, \infty]$ is a scalar measure such that $\langle m, x^* \rangle$ is truly continuous with respect to ι for all $x^* \in X^*$. Then, ν_η is also truly continuous with respect to ι. Consequently, ν_η admits a Radon-Nikodým derivative $\psi_\eta \in L^1(\iota)$ with respect to ι, that is,*

$$\nu_\eta(E) = \int_E \psi_\eta \, d\iota, \qquad E \in \Sigma.$$

Proof.

(i) The finite additivity of ν_η follows from the linearity of η and the identity $\chi_{E \cup F} = \chi_E + \chi_F$ whenever $E, F \in \Sigma$ are disjoint. To prove the σ-additivity of ν_η, select $p \in \mathscr{P}(X)$ such that

$$|\langle f, \eta \rangle| \leqslant p(m)_1(f), \qquad f \in L^1(m), \tag{A.9}$$

which is possible as $\eta : L^1(m) \to \mathbb{C}$ is continuous and linear. So, we have

$$|\nu_\eta(E)| = |\langle \chi_E, \eta \rangle| \leqslant p(m)_1(\chi_E) = p(m)(E), \qquad E \in \Sigma, \tag{A.10}$$

by (A.7) and (A.9) with $f := \chi_E$. Let $E_n \downarrow \emptyset$ in Σ. Then (A.10), with E_n in place of E for $n \in \mathbb{N}$, gives $\lim_{n\to\infty} \nu_\eta(E_n) = 0$ because $\lim_{n\to\infty} p(m)(E_n) = 0$, [16, Lemma II.1.3], [18, Theorem 1.3]. Thus, ν_η is σ-additive.

(ii) If $E \in \mathscr{N}_0(m)$, then $\Sigma \cap E \subseteq \mathscr{N}_0(m)$ and hence, $\chi_{F \cap E} \in \mathscr{N}(m)$ for $F \in \Sigma$ via (A.7) with $F \cap E$ in place of E, which implies that

$$\mathscr{N}_0(m) \subseteq \mathscr{N}_0(\nu_\eta) \tag{A.11}$$

as η vanishes on $\mathscr{N}(m)$.
Via [24, Lemma 2.7(i)] we can find Σ-simple functions $s_n : \Omega \to \mathbb{C}$ for $n \in \mathbb{N}$ such that $|s_n| \leqslant |f|$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = f$ pointwise outside of an m-null set $E(0)$ and such that

$$\lim_{n\to\infty} s_n = f \tag{A.12}$$

in the mean convergence topology. We may assume the $E(0) = \emptyset$, that is, $\lim_{n\to\infty} s_n = f$ pointwise on Ω, because we can identify s_n with $s_n \chi_{\Omega \setminus E(0)}$

and f with $f\chi_{\Omega\setminus E(0)}$ as elements of $L^1(m)$. Now, given $E \in \Sigma$, we have from (A.12) that $\lim_{n\to\infty} s_n\chi_E = f\chi_E$ for the mean convergence topology in $L^1(m)$ and hence, the continuity of η yields

$$\lim_{n\to\infty}\int_E s_n\,d\nu_\eta = \lim_{n\to\infty}\langle s_n\chi_E,\eta\rangle = \langle f\chi_E,\eta\rangle$$

because each s_n for $n \in \mathbb{N}$ is clearly ν_η-integrable and satisfies $\int_E s_n\,d\nu_\eta = \langle s_n\chi_E,\eta\rangle$. So, it follows from [18, Lemma 2.3] that f is ν_η-integrable and $\int_\Omega f\,d\nu_\eta = \langle f,\eta\rangle$.

(iii) Let $p \in \mathscr{P}(X)$ be as in (A.9). Via Lemma A.2, select $x_p^* \in X^*$ satisfying (A.3). Then (A.10) implies, with $\mu := |\langle m, x_p^*\rangle|$, that $\lim_{\mu(E)\to 0}\nu_\eta(E) = 0$. That is, $\nu_\eta << \mu$ and hence,

$$\mathscr{N}_0(\langle m, x_p^*\rangle) \subseteq \mathscr{N}_0(\nu_\eta). \tag{A.13}$$

Since $\langle m, x_p^*\rangle$ is truly continuous with respect to ι by assumption, there is a sequence $\{E_n\}_{n=1}^\infty$ in Σ such that $\iota(E_n) < \infty$ for each $n \in \mathbb{N}$ and such that $(\Omega \setminus \bigcup_{n=1}^\infty E_n)$ is $\langle m, x_p^*\rangle$-null; apply Lemma 2.1(i) with $\xi := \langle m, x_p^*\rangle$. So, $(\Omega \setminus \bigcup_{n=1}^\infty E_n)$ is also ν_η-null by (A.13). Again by Lemma 2.1(i), now with $\xi := \nu_\eta$, the measure ν_η is truly continuous with respect to ι.
The remaining part of (iii) follows via Lemma 2.1(i) with $\xi := \nu_\eta$. ■

Assume, for the moment, that the lcHs X is sequentially complete and that $\iota : \Sigma \to [0,\infty]$ is a localizable measure such that $\langle m, x^*\rangle$ is truly continuous with respect to ι for every $x^* \in X^*$. Then $\mathscr{N}_0(\iota) \subseteq \mathscr{N}_0(m)$ via (2.4). Next, every individual function $g \in L^\infty(\iota)$ is m-integrable. Indeed, choose a positive number $M > 0$ and a ι-null set $E \in \Sigma$ such that $|g(\omega)| \leqslant M$ for every $\omega \in (\Omega \setminus E)$. Then $E \in \mathscr{N}_0(m)$ as $\mathscr{N}_0(\iota) \subseteq \mathscr{N}_0(m)$ and hence, $g\chi_E \in \mathscr{N}(m)$. So, $g = g\chi_{\Omega\setminus E} + g\chi_E \in L^1(m)$ because the bounded function $g\chi_{\Omega\setminus E}$ is m-integrable, [16, Lemma II.3.1], [18, p.161].

Let $\Psi : L^\infty(\iota) \to L^1(m)$ be the natural map which assigns to each $g \in L^\infty(\iota)$ the m-integrable function g in $L^1(m)$. Recalling that $L^\infty(\iota)$ and $L^1(m)$ are the quotient spaces modulo ι-null and m-null functions, respectively, we need to ensure that Ψ is well defined. This can be seen once we observe that, if g_1 and g_2 are two individual functions in $L^\infty(\iota)$ such that $g_1 = g_2$ pointwise ι-a.e., then the m-integrable functions g_1 and g_2 coincide pointwise outside of an m-null set because $\mathscr{N}_0(\iota) \subseteq \mathscr{N}_0(m)$.

Let $\iota : \Sigma \to [0,\infty]$ be a localizable measure. Then the canonical map J_ι from $L^\infty(\iota)$ to $(L^1(\iota))^*$ is a bijective, isometric isomorphism, [9, Theorem 243G(b)], and so the weak* topology $\sigma(L^\infty(\iota), L^1(\iota))$ is well defined on $L^\infty(\iota)$.

Lemma A.5. *Let $m : \Sigma \to X$ be a vector measure, with X a sequentially complete lcHs, and $\iota : \Sigma \to [0,\infty]$ be a localizable measure such that $\langle m, x^*\rangle$ is truly continuous with respect to ι for every $x^* \in X^*$.*

(i) *The linear map $\Psi : L^\infty(\iota) \to L^1(m)$ is continuous with respect to the weak* topology $\sigma(L^\infty(\iota), L^1(\iota))$ on $L^\infty(\iota)$ and the weak topology $\sigma(L^1(m), (L^1(m))^*)$ on $L^1(m)$.*

(ii) *The image* $\Psi(\mathbb{B}[L^\infty(\iota)])$ *of the closed unit ball* $\mathbb{B}[L^\infty(\iota)]$ *is weakly compact in* $L^1(m)$.

(iii) *The vector measure* m *is closed.*

Proof.

(i) Let $\eta \in (L^1(m))^*$. Then, it follows from Lemma A.4 above that

$$\langle \Psi(g), \eta \rangle = \int_\Omega g \, d\nu_\eta = \int_\Omega g\psi_\eta \, d\iota = \langle \psi_\eta, g \rangle, \qquad g \in L^\infty(\iota),$$

with $\psi_\eta \in L^1(\iota)$. This implies (i) as $L^\infty(\iota) = (L^1(\iota))^*$ and because the seminorms generating $\sigma(L^\infty(\iota), L^1(\iota))$ are given by

$$g \mapsto |\langle g, \psi \rangle| = \left| \int_\Omega g\psi \, d\iota \right|, \qquad g \in L^\infty(\iota),$$

for each $\psi \in L^1(\iota)$, and those generating $\sigma(L^1(m), (L^1(m))^*)$ are given by

$$h \mapsto |\langle h, \eta \rangle|, \qquad h \in L^1(m),$$

for each $\eta \in (L^1(m))^*$.

(ii) This is a consequence of both part (i) and the fact that $\mathbb{B}[L^\infty(\iota)]$ is weak* compact in $L^\infty(\iota)$ by Alaoglu's Theorem.

(iii) Since $\{\chi_E : E \in \Sigma\} \subseteq \mathbb{B}[L^\infty(\iota)]$ and $\Sigma(m) = \Psi(\{\chi_E : E \in \Sigma\})$ in $L^1(m)$, part (iii) follows from (ii) and Lemma A.3(iii). ∎

Proof of Theorem 1. Given $\xi^* \in (\tilde{X})^*$, we have $\langle J_X \circ m, \xi^* \rangle = \langle m, \xi^* \circ J_X \rangle$ on Σ. So, $\langle J_X \circ m, \xi^* \rangle$ is truly continuous with respect to ι because so is $\langle m, \xi^* \circ J_X \rangle$ by assumption (as $\xi^* \circ J_X \in X^*$). This allows us to apply Lemma A.5 with $\tilde{X}$ in place of X and $J_X \circ m$ in place of m to deduce that $J_X \circ m : \Sigma \to \tilde{X}$ is a closed vector measure. So, m is also closed by Lemma A.1(iv). ∎

B. Relevant examples

Theorem 1 has its origins in Theorem IV.7.3 of [16]. But, as noted in Section 1, this latter result is incorrect because of the use of a Radon-Nikodým Theorem which is not applicable to the localizable measures being used in [16]; see the following paragraph. Our Theorem 1 is an analogous result but, with stronger assumptions, which turn out to be genuinely necessary. To be precise, let (Ω, Σ) be a measurable space. In order to be able to distinguish the two notions, throughout this Appendix B we will call our localizable measures (as defined in Section 2) F-localizable, whereas those in Assertion K-1 will be called K-localizable. Every F-localizable measure is clearly K-localizable; see Section 1. The converse is *not* valid, in general; see Example B.1 below.

In [16, p.10] it is stated that the class of K-localizable measures coincides with that of the localizable measures in the sense of [30, Definition 2.6]. This is incorrect and arises because the measure spaces (defined on certain rings of sets) and measurable sets considered in [30] (see Definitions 2.1 and 2.4 there) are different to those considered in [16]. To see this let Ω be any uncountable set, Σ be the σ-algebra of all countable-cocountable subsets of Ω and $\iota : \Sigma \to [0,\infty]$ be the counting measure. Then $\mathscr{N}_0(\iota) = \{\emptyset\}$ and so $\Sigma/\mathscr{N}_0(\iota) \simeq \Sigma$ is *not* a complete B.a. Observe that $\Sigma_f := \{A \in \Sigma : \iota(A) < \infty\}$ is a conditional ring of sets (consisting of all the finite subsets of Ω). The restriction ι_f of ι to Σ_f is clearly a measure (on Ω) in the sense of [30, Definition 2.1]. It is routine to check that every subset of Ω is measurable in the sense of [30, Definition 2.2]; denote this family of measurable sets by $\tilde{\Sigma}_f$ (i.e., $\tilde{\Sigma}_f = 2^\Omega$ for this example). If we extend ι_f to the set function $\tilde{\iota}_f : \tilde{\Sigma}_f \to [0,\infty]$ by

$$\tilde{\iota}_f(K) := \sup\{\iota_f(E) : E \in \Sigma_f, E \subseteq K\}, \qquad K \in \tilde{\Sigma}_f,$$

then $\tilde{\iota}_f$ is σ-additive on $\tilde{\Sigma}_f$, [30, Theorem 2.1]. Observe that $\Sigma_f \subseteq \tilde{\Sigma}_f$ and $\Sigma_f \subseteq \Sigma$. For this example $\tilde{\iota}_f$ is precisely the counting measure on 2^Ω and so the B.a. $\tilde{\Sigma}_f/\mathscr{N}_0(\tilde{\iota}_f) \simeq 2^\Omega$ is complete, that is, $(\Omega, \Sigma_f, \iota_f)$ is localizable in the sense of Definition 2.6 in [30]. Note that $\Sigma \subseteq \tilde{\Sigma}_f$ *properly* and that the B.a. $\Sigma/\mathscr{N}_0(\iota)$ is not complete whereas $\tilde{\Sigma}_f/\mathscr{N}_0(\tilde{\iota}_f)$ is complete. To see that ι is not K-localizable, first observe that $f \in L^1(\iota)$ if and only if $\{\omega \in \Omega : f(\omega) \neq 0\}$ is a countable set and $\|f\|_{L^1(\iota)} = \sum_{\omega\in\Omega} |f(\omega)| < \infty$. Let Λ be any *non-measurable* subset of Ω, in which case $\chi_\Lambda \notin L^\infty(\iota)$. Define ξ by

$$\langle f, \xi\rangle := \sum_{\omega\in\Omega} f(\omega)\chi_\Lambda(\omega) = \int_\Omega f\chi_\Lambda \, d\iota, \qquad f \in L^1(\iota).$$

The inequality $|\langle f,\xi\rangle| \leqslant \|f\|_{L^1(\iota)}$ for $f \in L^1(\iota)$ shows that ξ is a continuous linear functional on $L^1(\iota)$. But, ξ does *not* belong to the range of J_ι. Hence, ι is *not* K-localizable. The following example illustrates that Assertion K-1 is invalid.

Example B.1. Let $\Omega := [0,1]$ and Σ be the Borel σ-algebra of Ω. Then $\Sigma \subsetneqq 2^\Omega$. Let X be the complete lcHs $\mathbb{C}^\Omega$ equipped with the pointwise convergence topology. Then the set function $m : E \mapsto \chi_E$ on Σ is an X-valued vector measure with $\mathscr{N}_0(m) = \{\emptyset\}$.

(i) Define a scalar measure $\iota_1 : \Sigma \to [0,\infty]$ by $\iota_1(E) := \infty$ if $E \neq \emptyset$ and by $\iota_1(\emptyset) := 0$. Then $L^1(\iota_1) = \{0\}$, which implies that ι_1 is K-localizable. However, ι_1 fails to be F-localizable as it is not semifinite. Since $\mathscr{N}_0(\iota_1) = \{\emptyset\}$, it is clear that the B.a. $\Sigma/\mathscr{N}_0(\iota_1) \simeq \Sigma$ also fails to be complete. Accordingly, the K-localizability of ι_1 is *not* equivalent to $\Sigma/\mathscr{N}_0(\iota_1)$ being complete, as is claimed to be the case on p. 10 of [14].

(ii) Given $x^* \in X^*$, we surely have $\langle m, x^*\rangle << \iota_1$. We claim that $\langle m, x^*\rangle$ is *not* truly continuous with respect to ι_1 whenever $x^* \in X^* \setminus \{0\}$. Indeed, for such an x^* there exists a non-empty finite set $F \subseteq \Omega$ and scalars α_ω, for $\omega \in F$, such that $x^* = \sum_{\omega\in F} \alpha_\omega \chi_{\{\omega\}}$ and hence, $\langle m, x^*\rangle = \sum_{\omega\in F} \alpha_\omega \delta_\omega$. This easily implies the stated claim.

(iii) Via Lemma 2.2 the vector measure $m : \Sigma \to X$ is *not* closed, because $\mathscr{N}_0(m) = \{\emptyset\}$ and so the B.a. $\Sigma/\mathscr{N}_0(m) \simeq \Sigma$ fails to be complete. So, even though the assumptions of Assertion K-1 are satisfied, the conclusion is not.

(iv) There also exist K-localizable measures (other than ι_1) which exhibit the same features but whose L^1-space is *non-trivial.* For example, let $\Lambda := \{\frac{1}{n} : n \in \mathbb{N}\}$ and define ι_2 by

$$\iota_2(E) := \iota_1(E \setminus \Lambda) + \sum_{\omega \in \Lambda} \delta_\omega(E), \qquad E \in \Sigma.$$

Then a function $f : \Omega \to \mathbb{C}$ is ι_2-integrable if and only if it is Σ-measurable and satisfies $f(\omega) = 0$ for $\omega \notin \Lambda$ with $\|f\|_1 = \sum_{\omega\in\Lambda} |f(\omega)| < \infty$. Moreover, $g \in L^\infty(\iota_2)$ if and only if g is Σ-measurable and $\sup_{\omega\in\Omega} |g(\omega)| < \infty$. It then follows routinely that if $\xi \in (L^1(\iota_2))^*$, then $g := \sum_{\omega\in\Lambda} \langle \chi_{\{\omega\}}, \xi\rangle\, \chi_{\{\omega\}}$ belongs to $L^\infty(\iota_2)$ and satisfies

$$\langle f, \xi\rangle = \int_\Omega fg\, d\iota_2 = \sum_{\omega\in\Lambda} \langle \chi_{\{\omega\}}, \xi\rangle f(\omega), \qquad f \in L^1(\iota_2).$$

This shows that J_{ι_2} is surjective (see Section 1), that is, ι_2 is K-localizable. Moreover, $L^1(\iota_2)$ is isometrically isomorphic to the sequence space ℓ^1 and so, is surely non-trivial. Since ι_2 is not semifinite it is not F-localizable. In addition, $\mathscr{N}_0(\iota_2) = \{\emptyset\}$ implies that $\langle m, x^*\rangle << \iota_2$ for every $x^* \in X^*$. As in part (ii) it follows that if $x^* \in X^*$ satisfies $x^*(\omega) \neq 0$ for some $\omega \in \Omega\setminus\Lambda$, then $\langle m, x^*\rangle$ is not truly continuous with respect to ι_2. Of course, m is still not a closed measure!

We also point out that, in Theorem 1, it is not possible to weaken the assumption of F-localizability of ι to its semifiniteness (we still maintain true continuity).

Example B.2. Let $m : \Sigma \to X$ be the vector measure in Example B.1.

(i) Let $\iota_3 : \Sigma \to [0, \infty]$ denote the counting measure, which is clearly semifinite. Of course, $\mathscr{N}_0(\iota_3) = \{\emptyset\}$. Hence, the B.a. $\Sigma/\mathscr{N}_0(\iota_3) \simeq \Sigma$ is not complete, that is, ι_3 is not F-localizable. Although $\langle m, x^*\rangle$ is truly continuous with respect to ι_3 for all $x^* \in X^*$, the vector measure m is not closed. Since the canonical map J_{ι_3} is injective (as ι_3 is semifinite) and ι_3 is not F-localizable, we know from Section 1 that J_{ι_3} is not surjective, that is, $(L^1(\iota_3))^*$ is genuinely larger than $L^\infty(\iota_3)$. This can also be seen directly. Let g be *any* scalar function on Ω satisfying $\sup_{\omega\in\Omega} |g(\omega)| < \infty$ such that g is *not* Σ-measurable. In particular, $g \notin L^\infty(\iota_3)$. Define the linear functional ξ by

$$\langle f, \xi\rangle := \sum_{\omega\in\Omega} f(\omega)g(\omega) = \int_\Omega fg\, d\iota_3, \qquad f \in L^1(\iota_3).$$

The inequality $|\langle f, \xi\rangle| \leqslant \big(\sup_{w\in\Omega} |g(w)|\big)\|f\|_{L^1(\iota_3)}$ for $f \in L^1(\iota_3)$, shows that ξ is continuous on $L^1(\iota_3)$ but ξ does *not* belong to the range of J_{ι_3}.

(ii) The non-closed vector measure m can be extended to a closed vector measure on the larger σ-algebra 2^Ω. Indeed, let $\bar{m} : 2^\Omega \to X$ be the set function $E \mapsto \chi_E$ on 2^Ω, which is an extension of m. By ι_4 we denote the counting measure on 2^Ω, which is an extension of ι_3 and still satisfies $\mathscr{N}_0(\iota_4) = \{\emptyset\}$. Since ι_4 is decomposable, it is also F-localizable. For every $x^* \in X^*$, the measure $\langle \bar{m}, x^* \rangle$ is truly continuous with respect to ι_4. Hence, by Theorem 1 (or Theorem 2) applied to "m"$:= \bar{m}$ and $\iota := \iota_4$ it follows that $\bar{m}$ is a closed vector measure.

C. Proof of Proposition 2.4

Given is a continuous linear map S from X into a (complex) lcHs Z. It was noted in Appendix A that $S \circ m : \Sigma \to Z$ is again a vector measure. The linear map S admits a unique continuous linear extension $\tilde{S} : \tilde{X} \to \tilde{Z}$, which can be proved as in [17, §23,1.(4)]. Then we have $\tilde{S} \circ J_X = J_Z \circ S$ as an equality between continuous linear maps from X into $\tilde{Z}$.

Lemma C.1. *If the vector measure $m : \Sigma \to X$ is closed, then so is the vector measure $S \circ m : \Sigma \to Z$.*

Proof. Every function integrable with respect to the vector measure $J_X \circ m : \Sigma \to \tilde{X}$ is necessarily integrable with respect to the vector measure $\tilde{S} \circ (J_X \circ m)$, [24, Lemma 2.8(ii)]. So, via [24, Lemma 2.8 and Remark 2.9], the canonical map $[\tilde{S}]_{J_X \circ m}$ which assigns to $f \in L^1(J_X \circ m)$ the same function $f \in L^1(\tilde{S} \circ (J_X \circ m))$ is continuous and linear from $L^1(J_X \circ m)$ into $L^1(\tilde{S} \circ (J_X \circ m))$ with respect to the mean convergence topologies.

Now, since m is closed, it follows from Lemma A.3(iii) that the subset $\Sigma(J_X \circ m)$ is relatively weakly compact in $L^1(J_X \circ m)$. So, its image $[\tilde{S}]_{J_X \circ m}(\Sigma(J_X \circ m))$ is also relatively weakly compact in $L^1(\tilde{S} \circ (J_X \circ m))$ because $[\tilde{S}]_{J_X \circ m}$ is weakly continuous, [17, §20,4.(5)]. It is clear from the definition of $[\tilde{S}]_{J_X \circ m}$ that the relatively weakly compact set $[\tilde{S}]_{J_X \circ m}(\Sigma(J_X \circ m))$ equals $\Sigma(\tilde{S} \circ J_X \circ m)$ and hence, the vector measure $\tilde{S} \circ J_X \circ m : \Sigma \to Z$ is closed by Lemma A.3(ii) with $\tilde{S} \circ J_X \circ m$ in place of m. Via the identity $\tilde{S} \circ J_X = J_Z \circ S$, we have $\tilde{S} \circ J_X \circ m = J_Z \circ (S \circ m)$, so that $J_Z \circ (S \circ m)$ is a closed vector measure. Apply Lemma A.1(iv) with $(S \circ m)$ in place of m and Z in place of X to conclude that $S \circ m$ is a closed vector measure. ■

Proof of Proposition 2.4. The 'if' portion has been verified in the proof of [26, Proposition 2] without referring to [15, Corollary 13] or [16, Theorem IV.7.3]. So, it suffices to prove the 'only if' portion. To this end, assume that m is a closed vector measure. Let $i_\sigma : X \to X_{\sigma(X,X^*)}$ be the identity map. Now apply Lemma C.1 with $Z := X_{\sigma(X,X^*)}$ and $S := i_\sigma$ to deduce that $i_\sigma \circ m : \Sigma \to X_{\sigma(X,X^*)}$ is a closed vector measure. On the other hand, $m_\sigma = i_\sigma \circ m$ by definition. So, m_σ is closed. ■

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