Functiones et Approximatio
59.2 (2018), 175–189
doi: 10.7169/facm/1708

OPERATORS ON FOCK-TYPE AND WEIGHTED SPACES OF ENTIRE FUNCTIONS

Oscar Blasco

Dedicated to the memory
of Paweł Domański

Abstract: Let us denote by $\mathcal{F}_1^\phi(\mathbb{C})$ the space of entire functions $f \in \mathcal{H}(\mathbb{C})$ such that $\int_{\mathbb{C}} |f(z)|e^{-\phi(|z|)}dm(z) < \infty$, where $\phi : (0,\infty) \to \mathbb{R}^+$ is assumed to be continuous and non-decreasing. Also given a continuous non-increasing function $v : (0,\infty) \to \mathbb{R}^+$ and a complex Banach space X, we write $H_v^\infty(\mathbb{C}, X)$ for the space of X-valued entire functions F such that $\sup_{z\in\mathbb{C}} v(z)\|F(z)\| < \infty$. We find a very general class of weights ϕ and v for which the space of bounded operators $\mathcal{L}(\mathcal{F}_1^\phi(\mathbb{C}), X)$ can be identified with $H_v^\infty(\mathbb{C}, X)$.

Keywords: weighted spaces, entire function, Fock-type spaces.

1. Introduction

Given a complex Banach space $(X, \|\cdot\|)$, let $\mathcal{P}(X)$ and $\mathcal{H}(\mathbb{C}, X)$ stand for the spaces of polynomials and entire functions with values in X respectively. We write $\mathcal{P}$ and $\mathcal{H}(\mathbb{C})$ in the case $X = \mathbb{C}$ and use the notation $u_n(z) = z^n$ for $n \in \mathbb{N} \cup \{0\}$. For a radial weight we understand a continuous non-increasing function $v : \mathbb{C} \to \mathbb{R}^+$ such that $v(z) = v(|z|) > 0$ for $z \in \mathbb{C} \setminus \{0\}$. Given a complex Banach space X and a radial weight we write $H_v^\infty(\mathbb{C}, X)$ for the space of entire functions $F \in \mathcal{H}(\mathbb{C}, X)$ such that

$$\|F\|_{v,X} := \sup_{z\in\mathbb{C}} v(|z|)\|F(z)\| < \infty. \tag{1}$$

As usual $H_v^0(\mathbb{C}, X)$ denotes the subspace of functions such that

$$\lim_{|z|\to\infty} v(|z|)\|F(z)\| = 0. \tag{2}$$

We use the notation $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$ in the case $X = \mathbb{C}$. For convenience we shall write $v(z) = e^{-\phi(|z|)}$ for $z \neq 0$ for some continuous non-decreasing function

Partially supported by Proyecto MTM2014-53009-P (MINECO Spain).
2010 Mathematics Subject Classification: primary: 30H20; secondary: 46E40

$\phi : (0,\infty) \to \mathbb{R}$. Since we want $\mathcal{P}(X) \subset H^\infty_v(\mathbb{C}, X)$ we consider only continuous non-increasing weights satisfying

$$M_k(\phi) = \sup_{r>0} r^k e^{-\phi(r)} < \infty, \qquad \forall k \in \mathbb{N}. \tag{3}$$

We denote such a class by $\mathbb{W}$ and write $\phi \in \mathbb{W}$ whenever (3) is satisfied. The basic examples of weights in $\mathbb{W}$ to keep in mind, to be denoted by $\varphi_{\alpha,\beta,\gamma}$ where $\alpha, \beta > 0$ and $\gamma \geqslant 0$, are defined by

$$\varphi_{\alpha,\beta,\gamma}(r) = \beta r^\alpha - \gamma \log r \in \mathbb{W} \tag{4}$$

that is $e^{-\varphi_{\alpha,\beta,\gamma}(r)} = r^\gamma e^{-\beta r^\alpha}$. The reader is referred to [3] for more examples and a detailed study of this class.

The space $H^\infty_v(\mathbb{C})$ can be regarded as the case $p = \infty$ in the scale of Fock-type spaces $\mathcal{F}^\phi_p(\mathbb{C})$. For each $1 \leqslant p < \infty$ and each continuous function $\phi : (0,\infty) \to \mathbb{R}^+$ we define the Fock-type space $\mathcal{F}^\phi_p(\mathbb{C})$ consisting in those entire functions $f \in \mathcal{H}(\mathbb{C})$ such that

$$\|f\|_{\mathcal{F}^\phi_p} = \Big(\int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} dm(z) \Big)^{1/p} < \infty.$$

The particular case $\phi(z) = \frac{|z|^2}{2}$ are the classical Fock spaces, sometimes called Bargmann spaces, (see [2, 3, 16, 25, 29]).

There are a number of results concerning the boundedness of classical operators acting on Fock-type spaces and weighted spaces of entire functions. The reader is referred to [8, 14, 9] for multiplication operators, to [6, 9, 13, 17] for integration/derivation operators, to [19] for interpolating sequences, to [7, 11, 26, 27, 28] for composition operators, to [1, 24] for weighted composition operators, to [18, 21, 23] for Toeplitz operators, to [22] for Hankel operators or to [3, 5, 9, 12, 13] for Volterra-type operators among many others.

The classes of weights where the results in the previous papers can be applied varies according to the problem into consideration, usually some differentiability assumption of ϕ is assumed (see the class $\mathcal{I}$ in [13] or [9, Proposition 3.2]) or sometimes ϕ is taken to be subharmonic and satisfying the doubling condition on the measure $\Delta\phi$ (see [19, 21]). In this paper we shall try to keep the weight as general as possible. A priori we simply assume that $\phi \in \mathbb{W}$, which it is actually equivalent to $\mathcal{P} \subset H^0_v(\mathbb{C})$ or $\mathcal{P} \subset \cap_{p\geqslant 1} \mathcal{F}^p_\phi(\mathbb{C})$. We shall follow the ideas originated in [3] when dealing with the Volterra operator. Namely for each weight in $\mathbb{W}$ we denote by $C_k(\phi)$ the sequence $\frac{1}{2\pi}\|u_k\|_{\mathcal{F}^1_\phi(\mathbb{C})}$ and use it to define the kernel

$$K_\phi(z) = \sum_{k=0}^\infty C_{2k}(\phi)^{-1} z^k, \qquad z \in \mathbb{C}. \tag{5}$$

We shall see that $K_{2\phi}(\bar{z}w) = K^z(w)$ for all $z, w \in \mathbb{C}$ where $K^z \in \mathcal{F}^2_\phi(\mathbb{C})$ is the reproducing kernel of the Hilbert space $\mathcal{F}^2_\phi(\mathbb{C})$ with the inner product

$$\langle f, g\rangle_{2\phi} = \int_{\mathbb{C}} f(w)\overline{g(w)} e^{-2\phi(w)} dA(w),$$

i.e. is the unique functions K^z such that $f(z) = \langle f, K^z\rangle$ for every $f \in \mathcal{F}^2_\phi(\mathbb{C})$.

In this paper we introduce the class $\mathbb{AW}$ of weights $\phi \in \mathbb{W}$ for which there exists another $\psi \in \mathbb{W}$ (to be written $\psi \in \mathbb{W}_\phi$) satisfying

$$\sup_{z\in\mathbb{C}} e^{-\phi(|z|)} \int_{\mathbb{C}} |K_{\phi+\psi}(\bar{z}w)| e^{-\psi(|w|)} dm(w) < \infty. \tag{6}$$

For weights in this class we shall show that the boundedness of operators acting on $\mathcal{F}^1_\phi(\mathbb{C})$ or $H^0_v(\mathbb{C})$ is determined by the behaviour of a single vector-valued function given by the action of the operator on the kernel. In particular for such weights we can explicitly describe the dual of $\mathcal{F}^1_\phi(\mathbb{C})$ and $H^0_v(\mathbb{C})$.

The paper is divided into three sections. The first one is devoted to studying some properties of the sequences $C_k = \frac{1}{2\pi}\|u_k\|_{\mathcal{F}^\phi_1}$ and $M_k = \|u_k\|_{H^\infty_v}$ and some connections between them. In the second section we introduce the class $\mathbb{AW}$ of weights mentioned above. We show that $\varphi_{\alpha,\beta,\gamma} \in \mathbb{AW}$ whenever $\alpha \in \mathbb{N}$, $\beta > 0$ and $\gamma \in \mathbb{N} \cup \{0\}$ or also that any subharmonic ϕ with a doubling measure $\Delta\phi$ belongs to this class. The main results are contained in Section 4 where we prove that if $v = e^{-\phi}$ with $\psi \in \mathbb{W}_\phi$ then we can identify the space of bounded operators $\mathcal{L}(\mathcal{F}^1_\psi(\mathbb{C}), X)$ with $H^\infty_v(\mathbb{C}, X)$ (see Theorem 4.2 below) and also $\mathcal{L}(H^0_v(\mathbb{C}), X)$ with the space of functions $F \in \mathcal{H}(\mathbb{C}, X)$ such that $\langle x^*, F\rangle \in \mathcal{F}^1_\psi(\mathbb{C})$ for any $x^* \in X^*$ (see Theorem 4.5 below).

2. Sequences associated to weights

Following [3, Definition 2.3] we shall use a notation for the norm of the monomials in $\mathcal{F}^\phi_1(\mathbb{C})$ and $H^\infty_v(\mathbb{C})$, namely for each $\phi \in \mathbb{W}$ and $k \in \mathbb{N} \cup \{0\}$ we write

$$C_k := C_k(\phi) = \int_0^\infty r^{k+1} e^{-\phi(r)} dr = \frac{1}{2\pi}\|u_k\|_{\mathcal{F}^\phi_1}. \tag{7}$$

and

$$M_k := M_k(\phi) = \sup_{0<r<\infty} r^k e^{-\phi(r)} = \|u_k\|_{H^\infty_v}. \tag{8}$$

Note that (3) actually gives $\lim_{r\to\infty} r^k e^{-\phi(r)} = 0$ for all k which allows us to give the following definition.

Definition 2.1. Let $\phi \in \mathbb{W}$ and $k \in \mathbb{N} \cup \{0\}$ We define

$$R_k := R_k(\phi) = \max\{r \geqslant 0 : M_k = r^k e^{-\phi(r)}\}$$

In particular $M_k = R_k^k e^{-\phi(R_k)}$ and $M_k > s^k e^{-\phi(s)}$ for $s > R_k$.

As mentioned in [3, Example 2.1] for each $\beta, \alpha > 0$ and $\gamma \geqslant 0$

$$C_k(\varphi_{\alpha,\beta,\gamma}) = \frac{\beta^{-\frac{k+2+\gamma}{\alpha}}}{\alpha}\Gamma(\frac{k+2+\gamma}{\alpha}) \tag{9}$$

and

$$M_k(\varphi_{\alpha,\beta,\gamma}) = (\alpha\beta)^{-\frac{k+\gamma}{\alpha}} (k+\gamma)^{\frac{k+\gamma}{\alpha}} e^{-\frac{k+\gamma}{\alpha}}. \tag{10}$$

It is elementary to see that

$$R_k(\varphi_{\alpha,\beta,\gamma}) = \Big(\frac{k+\gamma}{\alpha\beta}\Big)^{1/\alpha}. \tag{11}$$

Let us also point out (see [3, Lemma 2.4]) that for $\phi \in \mathbb{W}$ any of the sequences $(C_k^{1/k})_k$, $(\frac{C_{k+1}}{C_k})_k$, $(M_k^{1/k})_k$ and $(\frac{M_{k+1}}{M_k})_k$ increases to ∞ as $k \to \infty$.

Moreover $\frac{C_{k+j}}{C_k} \leqslant \Big(\frac{C_{k+m}}{C_k}\Big)^{\frac{j}{m}}$ and $\frac{M_{k+j}}{M_k} \leqslant \Big(\frac{M_{k+m}}{M_k}\Big)^{\frac{j}{m}}$ for any $0 \leqslant j \leqslant m$.

Let us find some relations between the values M_k, C_k and R_k. Clearly one has that for $\phi, \psi \in \mathbb{W}$ and $k_1, k_2 \in \mathbb{N} \cup \{0\}$

$$C_{k_1+k_2}(\phi+\psi) \leqslant \min\{C_{k_1}(\phi)M_{k_2}(\psi), C_{k_2}(\phi)M_{k_1}(\psi)\}. \tag{12}$$

Proposition 2.1. *Let $\phi \in \mathbb{W}$ and $k \in \mathbb{N}$. Then*

$$\frac{R_k}{k+2} \leqslant \frac{M_{k+1}}{M_k} \leqslant R_{k+1}. \tag{13}$$

Proof. Clearly

$$M_{k+1} = R_{k+1}R_{k+1}^k e^{-\phi(R_{k+1})} \leqslant R_{k+1}M_k.$$

On the other hand, since $M_k(\phi) = R_k^k e^{-\phi(R_k)}$ we have

$$\frac{R_k^2}{k+2}M_k \leqslant \int_0^{R_k} r^{k+1}e^{-\phi(r)}dr \leqslant M_{k+1}R_k$$

and we obtain (13). ■

Proposition 2.2. *Let $\phi \in \mathbb{W}$ and $k \in \mathbb{N}$. Then*

$$\frac{1}{k+2}M_{k+2} \leqslant C_k \leqslant \frac{2\sqrt{M_2}}{k}\sqrt{M_{2k+2}}. \tag{14}$$

Proof. For each $R > 0$

$$\int_0^\infty r^{k+1}e^{-\phi(r)}dr \geqslant \left(\int_0^R r^{k+1}dr\right)e^{-\phi(R)} = \frac{R^{k+2}}{k+2}e^{-\phi(R)}.$$

This gives the first inequality.

Let $0 < s < \infty$ and write

$$\begin{aligned} C_k &= \int_0^s r^{k-1}r^2e^{-\phi(r)}dr + \int_s^\infty r^{k+1}e^{-\phi(r)}dr \\ &\leqslant \frac{s^k}{k}M_2 + \int_s^\infty r^{2k+2}e^{-\phi(r)}\frac{dr}{r^{k+1}} \\ &\leqslant \frac{s^k}{k}M_2 + M_{2k+2}\int_s^\infty \frac{dr}{r^{k+1}} \leqslant \frac{s^k}{k}M_2 + \frac{M_{2k+2}}{k}s^{-k}. \end{aligned}$$

Now select $s = \Big(\frac{M_{2k+2}}{M_2}\Big)^{1/2k}$ to obtain $kC_k \leqslant 2\sqrt{M_2M_{2k+2}}$ and then (14) is shown. ■

We need the following lemma to improve the upper estimates.

Lemma 2.3. *Let $k, j \in \mathbb{N}$ and $A, B > 0$. Then*

$$\min_{0<s<\infty}(As^k + Bs^{-j}) = (k+j)\Big(\frac{A}{j}\Big)^{\frac{j}{k+j}}\Big(\frac{B}{k}\Big)^{\frac{k}{k+j}}.$$

Proof. If $f(s) = As^k + Bs^{-j}$ then $f'(s) = kAs^{k-1} - jBs^{-(j+1)}$. Therefore the minimum is attained at $s^{k+j} = \frac{jB}{Ak}$. This gives the result. ∎

Proposition 2.4. *Let $\phi \in \mathbb{W}$ and $3 \leqslant j \leqslant k$. Then*

$$C_k \leqslant \frac{2j}{j^2-4} M_{k-j}^{\frac{j-2}{2j}} M_{k+j}^{\frac{j+2}{2j}}. \tag{15}$$

In particular, for $k \geqslant 3$,

$$\begin{aligned} C_k &\leqslant \frac{2k}{k^2-4} M_0^{\frac{k-2}{2k}} M_{2k}^{\frac{k+2}{2k}}, \\ C_k &\leqslant \frac{6}{5}\sqrt[6]{M_{k-3}M_{k+3}^5}. \end{aligned}$$

Proof. Let $0 < s < \infty$ and write, for $j \geqslant 1$

$$\begin{aligned} C_k &= \int_0^s r^{k-j} r^{1+j} e^{-\phi(r)} dr + \int_s^\infty r^{k+1} e^{-\phi(r)} dr \\ &\leqslant \frac{s^{2+j}}{2+j} M_{k-j} + \int_s^\infty r^{k+j} e^{-\phi(r)} \frac{dr}{r^{j-1}} \\ &\leqslant \frac{s^{2+j}}{2+j} M_{k-j} + M_{k+j} \int_s^\infty \frac{dr}{r^{j-1}} \\ &\leqslant \frac{M_{k-j}}{2+j} s^{2+j} + \frac{M_{k+j}}{j-2} s^{-j+2}. \end{aligned}$$

Hence

$$C_k \leqslant \min_{0<s<\infty}\left(\frac{M_{k-j}}{j+2}s^{j+2} + \frac{M_{k+j}}{j-2}s^{-(j-2)}\right), \qquad 3 \leqslant j \leqslant k.$$

Now apply Lemma 2.3 to conclude

$$C_k \leqslant 2j\Big(\frac{M_{k-j}}{(j+2)(j-2)}\Big)^{\frac{j-2}{2j}}\Big(\frac{M_{k+j}}{(j-2)(j+2)}\Big)^{\frac{j+2}{2j}}.$$

The estimates correspond to $j = k$ and $j = 3$ respectively and the proof is complete. ∎

3. A new class of weights

For each $\varphi \in \mathbb{W}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{C})$ and $g(z) = \sum_{n=0}^{m} b_n z^n \in \mathcal{P}$ we shall write

$$\langle f, g\rangle_\varphi = \frac{1}{2\pi}\int_{\mathbb{C}} f(w)\overline{g(w)}e^{-\varphi(|w|)}dm(w) = \sum_{n=0}^{m} a_n \bar{b}_n C_{2n}(\varphi). \tag{16}$$

and we shall denote by K^z the reproducing kernel of $\mathcal{F}^2_\varphi(\mathbb{C})$, i.e. $f(z) = \langle f, K^z\rangle_{2\varphi}$ for $z \in \mathbb{C}$ and $f \in \mathcal{F}^2_\varphi(\mathbb{C})$.

On the other hand, since $(C_k^{1/k})_k$ increases to ∞ one obtains (see [3]) that

$$K_\phi(z) = \sum_{k=0}^{\infty} C_{2k}(\phi)^{-1} z^k \in \mathcal{H}(\mathbb{C}). \tag{17}$$

Both processes are related one each other as the following result shows. For convenience we shall use the notation $f_z(w) = f(\bar{z}w)$ whenever $f \in \mathcal{H}(\mathbb{C})$ and $z, w \in \mathbb{C}$.

Proposition 3.1. *Let $\varphi \in \mathbb{W}$. Then $K^z = (K_{2\varphi})_z$.*

Proof. Since for each $\phi \in \mathbb{W}$ and $k \in \mathbb{N} \cup \{0\}$ one can write

$$\langle u_k, (K_\phi)_z\rangle_\phi = C_{2k}^{-1}(\phi)\frac{1}{2\pi}\int_{\mathbb{C}} |w|^{2k} e^{-\phi(|w|)}dm(w))z^k = z^k = u_k(z),$$

then for each $g \in \mathcal{P}$ one has

$$g(z) = \langle g, (K_\phi)_z\rangle_\phi = \frac{1}{2\pi}\int_{\mathbb{C}} K_\phi(z\bar{w})g(w)e^{-\phi(|w|)}dm(w). \tag{18}$$

The result follows selecting $\phi = 2\varphi$ from the uniqueness of the reproducing kernel. ∎

Definition 3.1. Let $\phi \in \mathbb{W}$. We shall say that $v(z) = e^{-\phi(|z|)}$ is an admissible weight, to be denoted $\phi \in \mathbb{AW}$, whenever there exists $\psi \in \mathbb{W}$ and $C > 0$ such that

$$\|(K_{\phi+\psi})_z\|_{\mathcal{F}_1^\psi} \leqslant Ce^{\phi(|z|)}, \qquad z \in \mathbb{C}. \tag{19}$$

We shall write $\mathbb{W}_\phi$ for the set of weights $\psi \in \mathbb{W}$ such that the couple (ϕ, ψ) satisfies (19).

Example 3.1. Let $\phi(r) = \varphi_{2,\frac{1}{2},0}(r) = \frac{r^2}{2}$. Then $\phi \in \mathbb{AW}$. Moreover $\phi \in \mathbb{W}_\phi$.

Indeed, recall that $C_{2k}(2\phi) = \frac{k!}{2}$ and $K_{2\phi}(z) = 2e^z$. Hence

$$\|(K_{2\phi})_z\|_{\mathcal{F}_1^\phi} = 2\int_{\mathbb{C}} e^{Re(z\bar{w})}e^{-\frac{|w|^2}{2}}dm(w) = Ce^{-\phi(|z|)}.$$

We denote by $\tilde{v}$ the associate weight (see [10]) defined by $\tilde{v}(z) = \|\delta_z\|^{-1}_{(H_v^\infty)^*}$ where $\|\delta_z\|_{(H_v^\infty)^*} = \sup\{|f(z)| : \|f\|_v \leqslant 1\}$. It is known that $H_v^\infty(\mathbb{C}) = H_{\tilde{v}}^\infty(\mathbb{C})$. Also recall that in general $v \leqslant \tilde{v}$ and a radial weight is called *essential* whenever there exists $C_0 > 0$ such that $\tilde{v}(z) \leqslant C_0 v(z)$ for all $z \in \mathbb{C}$.

Proposition 3.2. *If $v(z) = e^{-\phi(|z|)}$ is an essential admissible weight then $v^{-1}(z) \approx \|(K_{\phi+\psi})_z\|_{\mathcal{F}_1^\psi}$ for some $\psi \in \mathbb{W}$.*

Proof. Assume that $\psi \in \mathbb{W}_\phi$. Then using (18) when applied to $\phi = \varphi + \psi$ one easily shows that

$$|f(z)| \leqslant \|(K_{\varphi+\psi})_z\|_{\mathcal{F}_1^\psi} \|f\|_v, \qquad f \in \mathcal{P}.$$

Hence $\|\delta_z\|_{(H_v^\infty)^*} \leqslant \|(K_{\varphi+\psi})_z\|_{\mathcal{F}_1^\psi}$. This shows that $\tilde{v}^{-1}(z) \leqslant \|(K_{\varphi+\psi})_z\|_{\mathcal{F}_1^\psi}$. In particular

$$v^{-1}(z) \leqslant C_0 \tilde{v}^{-1}(z) \leqslant C_0 \|(K_{\varphi+\psi})_z\|_{\mathcal{F}_1^\psi}.$$

The other inequality follows since $\psi \in \mathbb{W}_\phi$ due to (19). ■

Remark 3.1. Let $\phi \in \mathbb{W}$ such that there exists $\psi \in \mathbb{W}$ satisfying

$$\sum_{k=0}^\infty \frac{C_k(\psi)}{C_{2k}(\phi+\psi)} r^k \leqslant C e^{-\phi(r)}, \qquad r > 0 \tag{20}$$

then $\phi \in \mathbb{AW}$ and $\psi \in \mathbb{W}_\phi$.

Indeed, from (20) it follows that

$$\|(K_{\phi+\psi})_z\|_{\mathcal{F}_1^\psi} \leqslant \sum_{k=0}^\infty \frac{\|u_k\|_{\mathcal{F}_1^\psi}}{C_{2k}(\phi+\psi)} |z|^k \leqslant C e^{-\phi(|z|)}.$$

Let us show that $\varphi_{\alpha,\beta,\gamma} \in \mathbb{AW}$ at least for $\alpha, \gamma \in \mathbb{N}$. We need the following lemmas.

Lemma 3.3 (see [4, Lemma 1.7]). *Let $p \in \mathbb{N}$, $\psi(r) = r^{p-2} e^{-r^p}$ and write*

$$K_p(w) = \sum_{k=0}^\infty \frac{2^{\frac{2k}{p}+1}}{\Gamma(\frac{2k}{p}+1)} w^n.$$

Then there exists $C > 0$ such that $\|(K_p)_z\|_{\mathcal{F}_1^\psi} \leqslant C e^{|z|^p}$ for $z \in \mathbb{C}$.

Lemma 3.4. *Let $\alpha > 0$, $\phi \in \mathbb{W}$ and write $\phi_\alpha(r) = \phi(\alpha r)$. If $\phi \in \mathbb{AW}$ then $\phi_\alpha \in \mathbb{AW}$.*

Proof. Let $\psi \in \mathbb{W}_\phi$. We shall see that $\psi_\alpha \in \mathbb{W}_{\phi_\alpha}$. First observe that $C_k(\varphi_\alpha) = \alpha^{-(k+2)} C_k(\varphi)$ for any $\varphi \in \mathbb{W}$. Hence selecting $\psi_\alpha(r) = \psi(\alpha r)$ since $\phi_\alpha + \psi_\alpha = (\phi+\psi)_\alpha$ we conclude that

$$K_{\phi_\alpha+\psi_\alpha}(\bar{z}w) = \alpha^2 \sum_{k=0}^\infty \frac{(\alpha\bar{z})^k (\alpha w)^k}{C_{2k}(\phi+\psi)} = \alpha^2 (K_{\phi+\psi})_{\alpha z}(\alpha w).$$

Hence, making the change of variable $\alpha w = w'$,

$$\begin{aligned}\|(K_{\phi_\alpha+\psi_\alpha})_z\|_{\mathcal{F}_1^{\psi_\alpha}} &= \int_{\mathbb{C}} \alpha^2 |(K_{\phi+\psi})_{\alpha z}(\alpha w)| e^{-\psi_\alpha(w)} dm(w) \\ &= \int_{\mathbb{C}} |(K_{\phi+\psi})_{\alpha z}(w')| e^{-\psi(w')} dm(w') \\ &= \|(K_{\phi+\psi})_{\alpha z}\|_{\mathcal{F}_1^{\psi}} \\ &\leqslant C e^{-\phi(\alpha|z|)} = C e^{-\phi_\alpha(|z|)}.\end{aligned}$$

The proof is complete. ■

Lemma 3.5. *Let $m \in \mathbb{N}$, $\phi \in \mathbb{W}$ and write*

$$\phi_{(m)}(r) = \phi(r) - m \log r, \qquad r > 0.$$

If $\phi \in \mathbb{AW}$ then $\phi_{(m)} \in \mathbb{AW}$.

Proof. Assume that $\psi \in \mathbb{W}_\phi$. Let us show that $\psi_{(m)} \in \mathbb{W}_{\phi_{(m)}}$. Clearly $C_k(\phi_{(m)}) = C_{k+m}(\phi)$. Notice also that $(\phi + \psi)_{(2m)} = \phi_{(m)} + \psi_{(m)}$ and then

$$C_{2n}(\phi_{(m)} + \psi_{(m)}) = C_{2n+2m}(\phi + \psi).$$

Therefore

$$\begin{aligned}\bar{z}^m u_m \sum_{n=0}^{\infty} \frac{\bar{z}^n u_n}{C_{2n}(\phi_{(m)} + \psi_{(m)})} &= \sum_{n=0}^{\infty} \frac{\bar{z}^{n+m} u_{n+m}}{C_{2n+2m}(\phi+\psi)} \\ &= \sum_{n=m}^{\infty} \frac{\bar{z}^n u_n}{C_{2n}(\phi+\psi)} \\ &= (K_{\phi+\psi})_z - \sum_{n=0}^{m-1} \frac{\bar{z}^n u_n}{C_{2n}(\phi+\psi)}.\end{aligned}$$

Using now $\|f\|_{\mathcal{F}_1^{\psi_{(m)}}} = \|u_m f\|_{\mathcal{F}_1^{\psi}}$ for all $m \in \mathbb{N}$ we conclude that

$$\begin{aligned}\|(K_{\phi_{(m)}+\psi_{(m)}})_z\|_{\mathcal{F}_1^{\psi_{(m)}}} &= |z|^{-m} \|z^m u_m (K_{\phi_{(m)}+\psi_{(m)}})_z\|_{\mathcal{F}_1^{\psi}} \\ &\leqslant |z|^{-m} \|R_m (K_{\phi+\psi})_z\|_{\mathcal{F}_1^{\psi}},\end{aligned}$$

where $R_m g = g - P_{m-1} g$ for $g \in \mathcal{H}(\mathbb{C})$. Therefore using the estimate $\|R_m g\|_{\mathcal{F}_1^{\psi}} \leqslant C(m+1)\|g\|_{\mathcal{F}_1^{\psi}}$ for any $m \in \mathbb{N}$ (see the argument in [3, Lemma 3.1]) and the fact $\psi \in \mathbb{W}_\phi$ one obtains the estimate $\|(K_{\phi_{(m)}+\psi_{(m)}})_z\|_{\mathcal{F}_1^{\psi_{(m)}}} \leqslant C e^{\phi_{(m)}(|z|)}$. The proof is then complete. ■

Proposition 3.6. *Let $\alpha \in \mathbb{N}$, $\beta > 0$, $\gamma \in \mathbb{N} \cup \{0\}$ and, as in the introduction,*

$$\varphi_{\alpha,\beta,\gamma}(r) = \beta r^\alpha - \gamma \log r.$$

Then $\varphi_{\alpha,\beta,\gamma} \in \mathbb{AW}$.

Proof. Since $\varphi_{\alpha,\beta,\gamma}(r) = \varphi_{\alpha,1,\gamma}(\beta^{1/\alpha}r)$, due to Lemmas 3.4 and 3.5 it suffices to show the result in the case $\beta = 1$ and $\gamma = 0$. Write $\phi(r) = \varphi_{\alpha,1,0}(r)$ and select $\psi = \varphi_{\alpha,1,\alpha-2}$. Let us show that $\psi \in \mathbb{W}_\phi$. Note that $e^{-(\phi(r)+\psi(r))} = r^{\alpha-2}e^{-2r^\alpha}$ and then

$$C_{2k}(\phi+\psi) = \int_0^\infty r^{2k}r^{\alpha-1}e^{-2r^\alpha}\,dr = \frac{\Gamma(\frac{2k}{\alpha}+1)}{2^{\frac{2k}{\alpha}+1}}.$$

This gives that $(K_{\phi+\psi})_z = \sum_{k=0}^\infty \frac{2^{\frac{2k}{\alpha}+1}}{\Gamma(\frac{2k}{\alpha}+1)} z^k u_k = (K_p)_z$ and applying Lemma 3.3 one obtains (19). ■

Other cases used in the literature are also admissible weights. Recall that if ϕ be a subharmonic function on $\mathbb{C}$ then in particular $\Delta\phi$ defines a locally finite Borel measure. Let us denote by $\mathbb{S}$ the collection of subharmonic functions on $\mathbb{C}$ such that $\mu = \Delta\phi$ satisfies the doubling condition, i.e.

$$0 < \mu(D(z,2r)) \leqslant C\mu(D(z,r)) < \infty \tag{21}$$

for any $z \in \mathbb{C}$ and $r > 0$, where $D(z,r) = \{w \in \mathbb{C} : |w-z| < r\}$

Basic results on pointwise estimates for the reproducing kernel of the space $\mathcal{F}^2_\phi(\mathbb{C})$ where achieved by Marzo and Ortega-Cerdá in [20]. Making use of such estimates Oliver and Pascuas [21] proved the following lemma.

Lemma 3.7 (see [21, Lemma 2.8]). *Let $\phi \in \mathbb{S}$. There exists $C > 0$ such that*

$$\int_{\mathbb{C}} |K^z(w)|e^{-\phi(w)}dm(w) \leqslant Ce^{\phi(z)}$$

Proposition 3.8. *If $\phi \in \mathbb{S}$ then $\phi \in \mathbb{AW}$. Moreover $\phi \in \mathbb{W}_\phi$.*

Proof. Taking into account Proposition 3.1 one has that $K^z = (K_{2\phi})_z$. Hence selecting $\psi = \phi$ and applying Lemma 3.7 one has

$$\|(K_{2\phi})_z\|_{\mathcal{F}^1_\phi} \leqslant Ce^{\phi(z)}$$

and the result follows. ■

4. Vector-valued functions and operators

In this section we shall characterize the boundedness of operators with domain in $\mathcal{F}^1_\psi(\mathbb{C})$ or $H^0_v(\mathbb{C})$ for weights in the class $\mathbb{AW}$. For such a purpose we shall consider the following vector-valued function.

Definition 4.1. Let ϕ and $\psi \in \mathbb{W}$, X be a complex Banach space and $T : \mathcal{P} \to X$ be a linear map. Set $x_n = T(u_n)$ for $n \geqslant 0$ and write

$$F_{T,\phi,\psi}(z) = \sum_{n=0}^\infty \frac{x_n}{C_{2n}(\phi+\psi)} z^n$$

for the formal power series with values in X.

Under some assumptions on the weights ϕ and $\psi \in \mathbb{W}$ and the operator T we shall see that $F_{T,\phi,\psi} \in \mathcal{H}(\mathbb{C}, X)$ and it will be used to describe the boundedness of T in certain spaces. We need the following lemma.

Lemma 4.1. *If $\psi \in \mathbb{W}_\phi$ then the map $z \to (K_{\phi+\psi})_{\bar{z}}$ belongs to $\mathcal{H}(\mathbb{C}, \mathcal{F}_1^\psi(\mathbb{C}))$.*

Proof. The condition (19) guarantees that $(K_{\phi+\psi})_z \in \mathcal{F}_1^\psi(\mathbb{C})$ for any $z \in \mathbb{C}$. Let us show that the series $\sum_{k=0}^\infty C_{2k}(\phi+\psi)^{-1} u_k z^k$ converges in $\mathcal{F}_\psi^1(\mathbb{C})$ for any $z \in \mathbb{C}$. Invoking Hardy's inequality for functions in $H^1(\mathbb{D})$ (see [15]) we have that for $f \in \mathcal{H}(\mathbb{C})$ with $f(z) = \sum_{k=0}^\infty a_k z^k$

$$\sum_{k=0}^\infty \frac{|a_k|}{k+1} r^k \leqslant C M_1(f, r), \qquad r > 0. \tag{22}$$

Therefore, applying (22) to $(K_{\phi+\psi})_z$ and integrating with respect to $re^{-\psi(r)}dr$, we obtain

$$\begin{aligned}\sum_{k=0}^\infty \frac{C_k(\psi)|z|^k}{(k+1)C_{2k}(\phi+\psi)} &\leqslant C \int_0^\infty M_1((K_{\phi+\psi})_z, r) r e^{-\psi(r)} dr \\ &= C\|(K_{\phi+\psi})_z\|_{\mathcal{F}_1^\psi} \\ &\leqslant C' e^{\phi(z)} < \infty\end{aligned}$$

and consequently the series $\sum_{k=0}^\infty \frac{z^{k+1}}{(k+1)C_{2k}(\phi+\psi)} u_k$ converges absolutely in $\mathcal{F}_1^\psi(\mathbb{C})$ and defines a $\mathcal{F}_1^\psi(\mathbb{C})$-valued entire function. Now the result follows from Weierstrass theorem since also its derivative $\sum_{k=0}^\infty \frac{z^k}{C_{2k}(\phi+\psi)} u_k$ defines a $\mathcal{F}_1^\psi(\mathbb{C})$-valued entire function. ■

Theorem 4.2. *Let $v = e^{-\phi}$ with $\phi \in \mathbb{AW}$ and $\psi \in \mathbb{W}_\phi$, let X be a complex Banach space and let $T : \mathcal{P} \to X$ be a linear map. Then T extends to a bounded linear operator from $\mathcal{F}_1^\psi(\mathbb{C})$ into X if and only if $F_{T,\phi,\psi}(z) \in H_v^\infty(\mathbb{C}, X)$. Moreover $\|T\| \approx \|F_{T,\phi,\psi}\|_{v,X}$.*

Proof. Assume that $T : \mathcal{F}_1^\psi(\mathbb{C}) \to X$ is a bounded operator. Using that $\psi \in \mathbb{W}_\phi$, one has

$$\|\sum_{n=0}^\infty \frac{u_n}{C_{2n}(\phi+\psi)} z^n\|_{\mathcal{F}_1^\psi} \leqslant C e^{\phi(|z|)}$$

for each $z \in \mathbb{C}$. Hence, due to the continuity of T, one obtains that $F_{T,\phi,\psi}(z) = T((K_{\phi+\psi})_{\bar{z}})$ for any $z \in \mathbb{C}$. Invoking Lemma 4.1 one obtains that $F_{T,\phi,\psi}$ is well defined and holomorphic from $\mathbb{C}$ into X. Moreover it belongs to $H_v^\infty(\mathbb{C}, X)$ and $\|F_{T,\phi,\psi}\|_{v,X} \leqslant C\|T\|$.

Assume now $F_{T,\phi,\psi} \in H_v^\infty(\mathbb{C}, X)$. Due to the density of polynomials in $\mathcal{F}_1^\psi(\mathbb{C})$ it suffices to show that there exists $C' > 0$ such that $\|T(f)\| \leqslant C'\|f\|_{\mathcal{F}_1^\psi}$ for all $f \in \mathcal{P}$. Given $f = \sum_{k=0}^m a_k u_k \in \mathcal{P}$ we have that

$$C_{2k}(\phi+\psi) a_k = \int_{\mathbb{C}} f(w) \bar{w}^k e^{-\phi(|w|)} e^{-\psi(|w|)} dm(w)$$

which shows

$$T(f) = \sum_{k=1}^{m} a_k x_k = \int_{\mathbb{C}} f(w) F_{T,\phi,\psi}(\bar{w}) e^{-\phi(|w|)} e^{-\psi(|w|)} dm(w). \tag{23}$$

Hence

$$\|T(f)\| \leqslant \int_{\mathbb{C}} \|F_{T,\phi,\psi}(\bar{w})\| |f(w)| e^{-\phi(|w|)} e^{-\psi(|w|)} dm(w) \leqslant \|F_{T,\phi,\psi}\|_{v,X} \|f\|_{\mathcal{F}_1^{\psi}}.$$

This shows that $\|T\| \leqslant \|F_{T,\phi,\psi}\|_{v,X}$ and the proof is now complete. ■

Corollary 4.3. *Let $v = e^{-\phi}$ with $\phi \in \mathbb{AW}$ and let $\psi \in \mathbb{W}_\phi$. Then $H_v^\infty(\mathbb{C}) = (\mathcal{F}_1^{\psi}(\mathbb{C}))^*$ under the pairing $\langle \cdot, \cdot \rangle_{\phi+\psi}$.*

Our purpose now is to study operators with domain in $H_v^0(\mathbb{C})$. This time we need the following lemma.

Lemma 4.4. *If $\psi \in \mathbb{W}_\phi$ then the map $z \to (K_{\phi+\psi})_{\bar{z}}$ belongs to $\mathcal{H}(\mathbb{C}, H_v^0(\mathbb{C}))$.*

Proof. Let us see that $\sum_{n=0}^{\infty} \frac{u_n}{C_{2n}(\phi+\psi)} z^n$ is an absolutely convergent series with values in $H_v^0(\mathbb{C})$. Note that for each $n \geqslant 0$, by Corollary 4.3 we have that $\|u_n\|_v \approx \sup\{|\langle f, u_n\rangle_{\phi+\psi} : \|f\|_{\mathcal{F}_1^{\psi}} \leqslant 1\}$. Hence, using the notation $f(z) = \sum_{n=0}^{\infty} a_n(f) z^n$, we conclude that

$$\|u_n\|_v \approx \sup\{|a_n(f)| C_{2n}(\phi+\psi) : \|f\|_{\mathcal{F}_1^{\psi}} \leqslant 1\} \tag{24}$$

Now from the trivial estimate $|a_n(f)| r^n \leqslant M_1(f, r)$ and integrating with respect $re^{-\psi(r)}$ we obtain $|a_n(f)| C_n(\psi) \leqslant \|f\|_{\mathcal{F}^1_{\psi}}$, which combined with (24) implies

$$\|u_n\|_v \leqslant C \frac{C_{2n}(\phi+\psi)}{C_n(\psi)}. \tag{25}$$

This gives for each $r > 0$

$$\sum_{k=0}^{\infty} \frac{\|u_k\|_v}{C_{2k}(\phi+\psi)} r^k \leqslant C \sum_{k=0}^{\infty} C_k^{-1}(\psi) r^k < \infty,$$

due to the fact $\lim_{k\to\infty} C_k^{1/k}(\psi) = \infty$, and therefore $\sum_{n=0}^{\infty} \frac{u_n}{C_{2n}(\phi+\psi)} z^n$ is well defined and holomorphic from $\mathbb{C}$ into $H_v^0(\mathbb{C})$. ■

Definition 4.2. Let $\varphi \in \mathbb{W}$ and let X be a complex Banach space. We denote by $\mathcal{F}_{1,weak}^{\varphi}(\mathbb{C}, X)$ the space of holomorphic functions $F : \mathbb{C} \to X$ such that $F_{x^*} \in \mathcal{F}_1^{\varphi}(\mathbb{C})$ for all $x^* \in X^*$ where $F_{x^*}(z) = \langle F(z), x^*\rangle$. We write

$$\|F\|_{\mathcal{F}_{1,weak}^{\varphi}} = \sup\{\|F_{x^*}\|_{\mathcal{F}_1^{\varphi}} : \|x^*\| \leqslant 1\}.$$

Theorem 4.5. *Let $v = e^{-\phi}$ with $\phi \in \mathbb{AW}$ and $\psi \in \mathbb{W}_\phi$, let X be a complex Banach space and let $T : \mathcal{P} \to X$ be a linear map. Then T extends to a bounded linear operator from $H^0_v(\mathbb{C})$ into X if and only if $F_{T,\phi,\psi}(z) \in \mathcal{F}^\psi_{1,weak}(X)$. Moreover $\|T\| \approx \|F_{T,\phi,\psi}\|_{\mathcal{F}^\psi_{1,weak}}$.*

Proof. Assume that $T : H^0_v(\mathbb{C}) \to X$ is a bounded operator. As shown in Lemma 4.4 $(K_{\phi+\psi})_{\bar z} = \sum_{n=0}^\infty \frac{u_n}{C_{2n}(\phi+\psi)} z^n$ where the series is absolutely convergent in $H^0_v(\mathbb{C})$ for each $z \in \mathbb{C}$. Due to the continuity of T we obtain that the function $F_{T,\phi,\psi}(z) = T\Big((K_{\phi+\psi})_{\bar z}\Big)$ is well defined and holomorphic from $\mathbb{C}$ into X. Let us show that $F_{T,\phi,\psi} \in \mathcal{F}^\psi_{1,weak}(\mathbb{C}, X)$. For each $x^* \in X^*$ we consider $T_{x^*} \in (H^0_v(\mathbb{C}))^*$ defined by $T_{x^*}(f) = \langle T(f), x^*\rangle$. Since the mapping $g \to g(z)e^{-\phi(|z|)}$ produces an isometric embedding $H^0_v(\mathbb{C}) \subseteq C_0(\mathbb{C})$, then using Hahn-Banach we can find a Borel measure ν_{x^*} with $|\nu_{x^*}|(\mathbb{C}) = \|T_{x^*}\|$ such that $T_{x^*}(g) = \int_\mathbb{C} g(w)e^{-\phi(|w|)} d\nu_{x^*}(w)$ for any $g \in H^0_v(\mathbb{C})$. Define

$$f_{x^*}(z) = \frac{1}{2\pi} \int_\mathbb{C} K_{\phi+\psi}(zw) e^{-\phi(|w|)} d\nu_{x^*}(w).$$

Observe that

$$f_{x^*}(z) = T_{x^*}\Big((K_{\phi+\psi})_{\bar z}\Big) = \langle T\Big((K_{\phi+\psi})_{\bar z}\Big), x^*\rangle.$$

Hence $f_{x^*}(z) = \langle F_{T,\phi,\psi}(z), x^*\rangle$ for any $z \in \mathbb{C}$ and $x^* \in X^*$. Notice that

$$\begin{aligned}
\int_\mathbb{C} |f_{x^*}(z)| e^{-\psi(|z|)} dm(z) &\leqslant \frac{1}{2\pi} \int_\mathbb{C}\int_\mathbb{C} |K_{\phi+\psi}(zw)| e^{-\phi(w)} e^{-\psi(|z|)} d|\nu_{x^*}|(w) dm(z) \\
&\leqslant \frac{1}{2\pi} \int_\mathbb{C} \Big(\int_\mathbb{C} |K_{\phi+\psi}(zw)| e^{-\psi(|z|)} dm(z)\Big) e^{-\phi(w)} d|\nu_{x^*}|(w) \\
&\leqslant \frac{1}{2\pi} \int_\mathbb{C} \|(K_{\phi+\psi})_{\bar w}\|_{\mathcal{F}^\psi_1} e^{-\phi(w)} d|\nu_{x^*}|(w) \\
&\leqslant \frac{C}{2\pi} |\nu_{x^*}|(\mathbb{C}) \leqslant \frac{C}{2\pi} \|T\| \|x^*\|.
\end{aligned}$$

This gives that $F_{T,\phi,\psi} \in \mathcal{F}^\psi_{1,weak}(\mathbb{C}, X)$ and $\|F_{T,\phi,\psi}\|_{\mathcal{F}^\psi_{1,weak}} \leqslant C\|T\|$.

Assume now that $F_{T,\phi,\psi} \in \mathcal{F}^\psi_{1,weak}(\mathbb{C}, X)$. Since polynomials are dense in $H^0_v(\mathbb{C})$ it suffices to show $\|T(f)\| \leqslant C\|f\|_v$ for all $f \in \mathcal{P}$. Using (23) we have for all $f \in \mathcal{P}$ and $x^* \in X^*$

$$\langle T(f), x^*\rangle = \int_\mathbb{C} \langle F_{T,\phi,\psi}(\bar w), x^*\rangle f(w) e^{-\phi(|w|)} e^{-\psi(|w|)} dm(w).$$

Therefore

$$|\langle T(f), x^*\rangle| \leqslant \|F_{T,\phi,\psi}\|_{\mathcal{F}^\psi_{1,weak}} \|x^*\| \|f\|_v$$

and the proof is complete. ∎

A corollary from Theorem 4.5 is the following duality result.

Corollary 4.6. *Let $v = e^{-\phi}$ with $\phi \in \mathbb{AW}$ and let $\psi \in \mathbb{W}_\phi$. Then $\mathcal{F}_1^\psi(\mathbb{C}) = (H_v^0(\mathbb{C}))^*$ under the pairing $\langle \cdot, \cdot \rangle_{\phi+\psi}$.*

Corollary 4.7. *Let $v(z) = e^{-\phi(|z|)}$ with $\phi \in \mathbb{AW}$, let X be a complex Banach space and let $T : X \to H_v^0(\mathbb{C})$ be a bounded operator. Then there exists a sequence $(x_n^*)_{n\geqslant 0}$ in X^* such that $\sum_{n=1}^\infty \frac{x_n^*}{C_{2n}(\phi+\psi)} u_n \in H_v^\infty(\mathbb{C}, X^*)$ and*

$$T(x) = \sum_{n=0}^\infty \frac{\langle x_n^*, x\rangle}{C_{2n}(\phi + \psi)} u_n, \qquad x \in X. \tag{26}$$

Proof. From Corollary 4.6 we know that $(H_v^0(\mathbb{C}))^* = \mathcal{F}_1^\psi(\mathbb{C})$ for some $\psi \in \mathbb{W}_\phi$. Since the adjoint operator $T^* : \mathcal{F}_1^\psi(\mathbb{C}) \to X^*$ is continuous we can define $x_n^* = T^*(u_n)$ for $n \geqslant 0$ and apply Theorem 4.2 to conclude that $F_{T^*,\phi,\psi} \in H_v^\infty(\mathbb{C}, X^*)$. Let $f_x = T(x) \in H_v^0(\mathbb{C})$. To show that $f_x = \langle F_{T^*,\phi,\psi}, x\rangle$ for each $x \in X$ it suffices to observe that

$$a_n(f_x) = \frac{\langle x_n^*, x\rangle}{C_{2n}(\phi + \psi)}, \qquad n \geqslant 0. \tag{27}$$

This follows since

$$\langle x_n^*, x\rangle = \langle u_n, f_x\rangle_{\phi+\psi} = \int_{\mathbb{C}} f_x(w)\bar{w}^n e^{-\phi(w)-\psi(w)} dm(w) = a_n(f_x) C_{2n}(\phi + \psi).$$

The proof is then complete. ■

Acknowledgements. The author would like to thank J. Pau and E. Mangino for providing a number of important references that the author was not aware of and that helped to complete the paper.

References

[1] W. Al-Rawashdeh, *Compact weighted composition operators between generalized Fock spaces,* Missouri J. Math. Sci. **28**(1) (2016), 62–75.

[2] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform,* Communications in Pure and Applied Mathematics **24** (1961), 187–214.

[3] O. Blasco, *Boundedness of Volterra operator on spaces of entire functions*, Annales Acad. Scient. Fennicae Mathematica (2018), to appear.

[4] O. Blasco, A. Galbis, *On Taylor coefficient of entire functions integrable against exponential weights*, Math. Nach. **223** (2001), 5–21.

[5] J. Bonet, *The spectrum of Volterra operators on weighted spaces of entire functions*, Quart. J. Math. **66** (2015), 799–807.

[6] M. J. Beltran, J. Bonet, C. Fernandez, *Classical operators on weighted spaces of entire functions*, Proc. Amer. Math. Soc. **141**(12) (2013), 4293–4303.

[7] M. J. Beltran, M. C. Gómez-Collado, E. Jordá, D. Jornet *Mean ergodic composition operators on Banach spaces of holomorphic functions*, J. Funct. Anal. **270** (2016), 4369–4385.

[8] J. Bonet, P. Dománski, M. Lindström, *Pointwise multiplication operators on weighted Banach spaces of analytic functions*, Studia Math. **137** (1999), 177–194.

[9] J. Bonet, J. Taskinen, *A note on Volterra operators on weighted Banach spaces of entire functions*, Math. Nachr. **288** (2015), no. 11–12, 1216–1225.

[10] K. D. Bierstedt, J. Bonet, J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. **127** (1998), 137–168.

[11] B. Carswell, B. MacCluer, A. Schuster, *Composition operators on the Fock space,* Acta Sci. Math. (Szeged) **69** (2003), 871–887.

[12] O. Constantin, *A Volterra-type integration operator on Fock spaces*, Proc. Amer. Math. Soc. **140**(12) (2012), 4247–4257.

[13] O. Constantin, J. A. Peláez, *Integral operators, embedding theorems and a Littlewood–Paley formula on weighted Fock spaces*, J. Geom. Anal. **26**(2) (2016), 1109–1154.

[14] M. Contreras and A. G. Hernández-Diaz, *Weighted composition operators in weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. (Series A) **69** (2000), 41–60.

[15] P. Duren, *Theory of H^p spaces,* Pure and Applied Math V. **38**, Academic Press, 1970.

[16] V. Fock, *Konfigurationsraum und zweite Quantelung,* Z. Phys. **75** (1932).

[17] A. Harutyunyan, W. Lusky, *On the boundedness of the differentiation operator between weighted spaces of holomorphic functions*, Studia Math. **184** (2008), no. 3, 233–247, DOI 10.4064/sm184-3-3.

[18] J. Isralowitz, K. Zhu, *Toeplitz operators on the Fock space,* Int. Equ. Oper. Th. **66** (2010), no. 4, 593–611.

[19] J. Marzo, X. Massaneda, J. Ortega-Cerdà, *Interpolating and sampling sequences for entire functions*, Geom. Funct. Anal. **13**(4) (2003), 862–914.

[20] J. Marzo, J. Ortega-Cerdà, *Pointwise estimates for the Bergman kernel of the weighted Fock spaces*, J. Geom. Anal. **10**(4) (2009), 890–910.

[21] R. Oliver, D. Pascuas, *Toeplitz operators on doubling Fock spaces*, J. Math. Anal. Appl. **435** (2016), 1426–1457.

[22] K. Seip, E. H. Youssfi, *Hankel operators on Fock spaces and related Bergman kernel estimates,* J. Geom. Anal. **23** (2013), 170–201.

[23] A. P. Schuster, D. Varolin, *Toeplitz operators and Carleson measures on generalized Bargmann–Fock spaces*, Integr. Equ. Oper. Theory **72** (2012), 363–392.

[24] S. Stevic, *Weighted composition operators between Fock-type spaces in $\mathbb{C}^n$*, Applied Mathematics and Computation **215** (2009), 2750–2760.

[25] J. Tung, *Fock Spaces,* PhD dissertation, University of Michigan, 2005.

[26] S. Ueki, *Weighted composition operator on the Fock space,* Proc. Amer. Math. Soc. **135** (2007), 1405–1410.

[27] S. Ueki, *Weighted composition operators on the Bargmann-Fock spaces,* International Journal of Modern Mathematics **3** (2008), 231–243.

[28] S. Ueki, *Weighted composition operators on some function spaces of entire functions,* Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 343–353.

[29] K. Zhu, *Analysis on Fock Spaces,* Springer-Verlag, New York, 2012.

Address: Oscar Blasco: Departamento de Análisis Matemático, Universidad de Valencia, 46100 Burjassot, Valencia, Spain.

E-mail: oblasco@uv.es

Received: 2 October 2017; **revised:** 11 July 2018