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ON THE SEPARABLE QUOTIENT PROBLEM FOR BANACH SPACES

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To the memory of our Friend
Professor Paweł Domański

Abstract: While the classic *separable quotient problem* remains open, we survey general results related to this problem and examine the existence of infinite-dimensional separable quotients in some Banach spaces of vector-valued functions, linear operators and vector measures. Most of the presented results are consequences of known facts, some of them relative to the presence of complemented copies of the classic sequence spaces c_0 and ℓ_p, for $1 \leqslant p \leqslant \infty$. Also recent results of Argyros, Dodos, Kanellopoulos [1] and Śliwa [64] are provided. This makes our presentation supplementary to a previous survey (1997) due to Mujica.

Keywords: Banach space, barrelled space, separable quotient, vector-valued function space, linear operator space, vector measure space, tensor product, Radon-Nikodým property.

1. Introduction

A famous unsolved problem of Functional Analysis (posed by S. Mazur in 1932) asks:

Problem 1. *Does every infinite-dimensional Banach space have a separable infinite-dimensional quotient* (*have SQ*)*?*

A nice application of the open mapping theorem shows that an infinite-dimensional Banach space X has SQ if and only if X is mapped onto a separable Banach space under a continuous linear map.

The first comments about Problem 1 are mentioned in [45] and [53]. It is already well known that all infinite-dimensional reflexive, even all infinite-dimensional weakly compactly generated Banach spaces (WCG for short) have SQ, result generalized in Theorem 1 of [65]. In [37, Theorem IV.1(i)] Johnson and Rosenthal

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proved that every infinite-dimensional separable Banach space admits an infinite-dimensional quotient with a Schauder basis. The latter result provides another (equivalent) reformulation of Problem 1.

Problem 2. *Does every infinite-dimensional Banach space admit an infinite-dimensional quotient with a Schauder basis?*

Théorème 4 on page 124 of Banach's monograph and Theorem 2 of [36] prove that if Y is a separable closed subspace of a Banach space X and the quotient X/Y has SQ, then Y is quasi-complemented in X. This provides another equivalent condition to Problem 1.

Problem 3 (Rosenthal [53, p. 188, Remark 2]). *Does every infinite-dimensional Banach space X contain a closed quasi-complemented infinite-dimensional separable subspace Y?*

Although Problem 1 remains open for Banach spaces, a corresponding question of whether every infinite-dimensional non-normed metrizable and complete locally convex space (i.e., a Fréchet space) X admits SQ has been already solved. Indeed, if X is a non-normed locally convex Fréchet space, a result of Eidelheit [20] ensures that X has a quotient isomorphic to $\mathbb{K}^{\mathbb{N}}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

In [16] Drewnowski posed a more general question: Does every infinite-dimensional metrizable and complete topological vector space X contain a closed subspace Y such that the dimension of the quotient X/Y is the continuum (in short, $\dim(X/Y) = \mathfrak{c}$)? The same paper contains an observation stating that every infinite-dimensional Fréchet locally convex space admits a quotient of dimension $\mathfrak{c}$. Drewnowski's problem was solved by M. Popov (see [52]). He showed that for $0 < p < 1$ the space $X := L_p([0,1]^{2^{\mathfrak{c}}})$ does not admit a proper closed subspace Y such that $\dim(X/Y) \leqslant \mathfrak{c}$. Consequently, X does not have SQ.

The organization of the present paper goes as follows. The second section gathers general selected results about the separable quotient problem and some classic results on Banach spaces containing copies of sequence spaces, providing as well some straightforward consequences. Recent results of Argyros, Dodos, Kanellopoulos [1] and Śliwa [64] are also provided.

In the third section we exhibit how some types of weak*-compactness of the dual unit ball of a Banach space X can be used to get quotients isomorphic to c_0 or ℓ_2. The next section contains three classic results about complete tensor products of Banach spaces and their applications to the separable quotient problem.

The last two sections are devoted to examine the existence of SQ for many concrete classes of 'big' Banach spaces, for example, Banach spaces of vector-valued functions, bounded linear operators and vector measures. This line of research has been also continued in a more general setting for the class of topological vector spaces, in particular for spaces $C(L)$ of real-valued continuous functions endowed with the pointwise and compact-open topology, respectively (see [39], [40] and [41]).

This paper supplements Mujica's survey article [50] by collecting together some results not mentioned there. We add new facts and nice consequences of older facts. Some imply the existence of complemented copies of the classic spaces c_0 and ℓ_p $(1 \leqslant p \leqslant \infty)$. We hope this will prove useful for researchers in the area.

2. A few results for general Banach spaces

Let us start with the following remarkable concrete result of Agryros, Dodos and Kanellopoulos [1] related to Problem 1. They proved (using Ramsey theory) that for every separable Banach space X with non-separable dual, the space X^{**} contains an unconditional family of size $|X^{**}|$. As an application they proved

Theorem 4 (Argyros-Dodos-Kanellopoulos). *Every infinite-dimensional dual Banach space has SQ.*

Corollary 5. *The space $\mathcal{L}(X,Y)$ of bounded linear operators between Banach spaces X and Y equipped with the operator norm has SQ provided $Y \neq \{\mathbf{0}\}$.*

Indeed, this follows from the fact that X^* is complemented in $\mathcal{L}(X,Y)$, see Theorem 32 below for details.

Let us select a few more equivalences to Problem 1. The equivalence of (2) and (3) below, even for Hausdorff locally convex spaces, is an obvious consequence of the bipolar theorem (see also [40], [60] and [61]). The equivalence of (1) and (4) is due to Saxon-Wilansky [63]. Recall that a locally convex space E is called *barrelled* if every barrel (an absolutely convex closed and absorbing set in E) is a neighborhood of zero. We refer also to [5] and [37] for some partial results related to the next theorem.

Theorem 6 (Saxon-Wilansky). *The following assertions are equivalent for an infinite-dimensional Banach space X.*

(1) *X contains a dense non-barrelled linear subspace.*
(2) *X admits a strictly increasing sequence of closed subspaces of X whose union is dense in X.*
(3) *X^* admits a strictly decreasing sequence of weak*-closed subspaces whose intersection consists only of the zero element.*
(4) *X has SQ.*

Proof. We prove only the equivalence between (2) and (4) (which also holds for every locally convex space X).

(4) $\Rightarrow$ (2) Note that separable Banach spaces have property (2), which is preserved by pre-images of surjective continuous (and open) linear operators.

(2) $\Rightarrow$ (4) Let $\{X_n : n \in \mathbb{N}\}$ be such a sequence. For each $n \in \mathbb{N}$ choose $x_n \in X_{n+1} \setminus X_n$ and $x_n^* \in X^*$ such that $x_n^* x_n = 1$ with x_n^* vanishing on X_n. Since, by induction, $X_{n+1} \subseteq \operatorname{span}\{x_1, \ldots, x_n\} + \bigcap_{k=1}^{\infty} \ker x_k^*$, we conclude that $\operatorname{span}\{x_n : n \in \mathbb{N}\} + \bigcap_{n=1}^{\infty} \ker x_n^*$ is dense in X. Clearly, then X/Y is separable (and infinite-dimensional) for $Y := \bigcap_{n=1}^{\infty} \ker x_n^*$. ■

In particular, every Banach space whose weak*-dual is separable has SQ. Theorem 6 applies to show that every infinite-dimensional WCG Banach space has SQ. Indeed, if X is reflexive we apply Theorem 4. If X is not reflexive, choose a weakly compact absolutely convex set K in X such that $\overline{\operatorname{span}(K)} = X$. Since K is a barrel of $Y := \operatorname{span}(K)$ and X is not reflexive, Y is a dense non-barrelled linear subspace of X.

The class of WCG Banach spaces, introduced in [4], provides a quite successful generalization of reflexive and separable spaces. As proved in [4] there are many bounded projection operators with separable ranges on such spaces, so many separable complemented subspaces exist. For example, if X is WCG and Y is a separable subspace, there exists a separable closed subspace Z with $Y \subseteq Z \subseteq X$ together with a contractive projection. This shows that every infinite-dimensional WCG Banach space X admits many separable complemented subspaces, so X has many SQ.

The Josefson-Nissenzweig theorem states that the dual of any infinite-dimensional Banach space contains a *normal sequence*, i. e., a normalized weak*-null sequence [51].

Recall (*cf.* [64]) that a sequence $\{y_n^*\}$ in the sphere $S(X^*)$ of X^* is *strongly normal* if the subspace $\{x \in X : \sum_{n=1}^{\infty} |y_n^* x| < \infty\}$ is dense in X. Clearly every strongly normal sequence is normal. Having in mind Theorem 8 below, the following question is of interest.

Problem 7 (Śliwa). *Does every normal sequence in X^* contain a strongly normal subsequence?*

By [64, Theorem 1], every strongly normal sequence in X^* contains a subsequence $\{y_n\}$ which is a Schauder basic sequence in the weak*-topology, i. e., $\{y_n\}$ is a Schauder basis in its closed linear span in the weak*-topology. Conversely, every normalized Schauder basic sequence in (X^*, w^*) is strongly normal [64, Proposition 1].

The following theorem from [64] exhibits a connection between these concepts.

Theorem 8 (Śliwa). *Let X be an infinite-dimensional Banach space. The following conditions are equivalent:*

(1) *X has SQ.*
(2) *X^* has a strongly normal sequence.*
(3) *X^* has a basic sequence in the weak* topology.*

Proof. We prove only (1) $\Leftrightarrow$ (2). The equivalence between (2) and (3) follows from the remark above.

(1) $\Rightarrow$ (2) Let Y be a closed subspace in X and $\{x_n\}_{n=1}^{\infty}$ a linearly independent sequence whose linear span Z is transverse to Y, with $Y + Z$ dense in X. For each $n \in \mathbb{N}$, choose $y_n^* \in S(X^*)$ such that $y_n^* x_n \neq 0$ and y_n^* vanishes on $Y + \operatorname{span}\{x_j : 1 \leqslant j < n\}$. Since $\{y_n^* x\}_{n=1}^{\infty}$ is eventually zero for every $x \in Y + Z$, the sequence $\{y_n^*\}_{n=1}^{\infty}$ is strongly normal.

(2) ⇒ (1) By [64, Theorem 1] (see also [37, Theorem III.1 and Remark III.1]) some sequence $\{y_n^*\}_{n=1}^\infty$ in X^* is a Schauder basis for its closed linear span $\overline{\operatorname{span}}\,\{y_n^* : n \in \mathbb{N}\}$ in (X^*, w^*). By definition of Schauder basis, the sequence $\{\overline{\operatorname{span}}\,\{y_n^* : n \geqslant k\}\}_{k=1}^\infty$ satisfies (3) of Theorem 6, from which the conclusion follows. ■

A slightly stronger property than condition (2) of Theorem 8 is stated below. As it is shown in the proof, this property turns out to be equivalent to the existence in X^* of a basic sequence equivalent to the unit vector basis of c_0.

Proposition 9. *A Banach space X has a quotient isomorphic to ℓ_1 if and only if X^* contains a normal sequence $\{y_n^*\}_{n=1}^\infty$ such that $\sum_{n=1}^\infty |y_n^* x| < \infty$ for all $x \in X$.*

Proof. If there is a bounded linear operator Q from X onto ℓ_1, its adjoint map fixes a sequence $\{x_n^*\}_{n=1}^\infty$ in X^* such that the formal series $\sum_{n=1}^\infty x_n^*$ is weak* unconditionally Cauchy and $\inf_{n\in\mathbb{N}} \|x_n^*\| > 0$. Setting $y_n^* := \|x_n^*\|^{-1} x_n^*$ for each $n \in \mathbb{N}$, the sequence $\{y_n^*\}_{n=1}^\infty$ is as required. Conversely, if there is a normal sequence $\{y_n^*\}_{n=1}^\infty$ like that of the statement, it defines a weak* Cauchy series in X^*. Since the series $\sum_{n=1}^\infty y_n^*$ does not converge in X^*, according to [14, Chapter V, Corollary 11] the space X^* must contain a copy of ℓ_∞. Thus X has a complemented copy of ℓ_1 by [14, Chapter V, Theorem 10]. ■

For a large class of Banach spaces, Problem 7 has a positive answer.

Theorem 10 (Śliwa). *If X is an infinite-dimensional WCG Banach space, every normal sequence in X^* contains a strongly normal subsequence.*

An interesting consequence of Theorem 6 is that 'small' Banach spaces always have *SQ*. We present another proof, different from the one presented in [62, Theorem 3]. Ours depends on the concept of strongly normal sequence.

Corollary 11 (Saxon–Sánchez Ruiz). *If the density character $d(X)$ of a Banach space X satisfies $\aleph_0 \leqslant d(X) < \mathfrak{b}$, then X has SQ.*

Recall that the *density character* of a Banach space X is the smallest cardinal of the dense subsets of X. The *bounding cardinal* $\mathfrak{b}$ is referred to as the minimum size for an unbounded subset of the preordered space $(\mathbb{N}^\mathbb{N}, \leqslant^*)$, where $\alpha \leqslant^* \beta$ stands for the *eventual dominance preorder*, defined so that $\alpha \leqslant^* \beta$ if the set $\{n \in \mathbb{N} : \alpha(n) > \beta(n)\}$ is finite. So we have $\mathfrak{b} := \inf\{|F| : F \subseteq \mathbb{N}^\mathbb{N}, \forall \alpha \in \mathbb{N}^\mathbb{N}\ \exists \beta \in F \text{ with } \alpha <^* \beta\}$. It is well known that $\mathfrak{b}$ is a regular cardinal and $\aleph_0 < \mathfrak{b} \leqslant \mathfrak{c}$. It is consistent that $\mathfrak{b} = \mathfrak{c} > \aleph_1$; indeed, Martin's Axiom implies that $\mathfrak{b} = \mathfrak{c}$.

Proof of Corollary 11. Assume X has a dense subset D of cardinality less than $\mathfrak{b}$. We show that X^* has a *strongly normal sequence* and then we apply Theorem 8. Choose a normalized weak*-null sequence $\{y_n^*\}$ in X^*. For $x \in D$ choose $\alpha_x \in \mathbb{N}^\mathbb{N}$ such that for each $n \in \mathbb{N}$ and every $k \geqslant \alpha_x(n)$ one has $|y_k^* x| < 2^{-n}$. Then $\sum_n |y_{\beta(n)}^* x| < \infty$ if $\alpha_x \leqslant^* \beta$. Finally select $\gamma \in \mathbb{N}^\mathbb{N}$ with $\alpha_x \leqslant^* \gamma$ for each $x \in D$. Then the sequence $\{y_{\gamma(n)}^*\}$ is strongly normal and Theorem 8 applies. ■

For 'large' Banach spaces we note the following interesting result [68].

Theorem 12 (Todorcevic). *Under Martin's maximal axiom every Banach space X of density character $\aleph_1$ has a quotient space with an uncountable monotone Schauder basis, and thus X has SQ.*

Another line of research related to Problem 1 deals with those Banach spaces which contain complemented copies of certain sequence spaces (see next section). An important useful result is found in Mujica's survey paper [50, Theorem 4.1]. First, we need the following result due to Rosenthal (see [53, Corollary 1.6, Proposition 1.2]).

Lemma 13. *Let X be a Banach space such that X^* contains an infinite-dimensional reflexive subspace Y. Then X has a quotient isomorphic to Y^*. Consequently X has SQ.*

Proof. Let $Q : X \to Y^*$ be defined by $Qx(y) = y(x)$ for $y \in Y$ and $x \in X$. Let $j : Y \to X^*$ and $\phi_X : X \to X^{**}$ be the inclusion maps. Clearly $Q = j^* \circ \phi_X$ and $Q^* = \phi_X^* \circ j^{**}$. Since Y is reflexive, Q^* is an embedding map and consequently Q is surjective. ■

Theorem 14 (Mujica). *If X is a Banach space that contains an isomorphic copy of ℓ_1, then X has a quotient isomorphic to ℓ_2.*

Proof. If X contains a copy of ℓ_1, the dual space X^* contains a copy of $L_1[0,1]$, see [14]. It is well known that the space $L_1[0,1]$ contains a copy of ℓ_2. We apply Lemma 13. ■

Let us recall that from classic Rosenthal-Dor's ℓ_1-dichotomy [14, Chapter 11] one easily gets the following general result.

Theorem 15. *If X is a non-reflexive weakly sequentially complete Banach space, then X contains an isomorphic copy of ℓ_1.*

The previous results suggest also the following

Problem 16. *Describe a possibly large class of non-reflexive Banach spaces X not containing an isomorphic copy of ℓ_1 and having SQ.*

In light of [35, Corollary 1] we may summarize this section with

Corollary 17. *Let X be an infinite-dimensional Banach space. Assume that either X or X^* contains an isomorphic copy of c_0, or either X or X^* contains an isomorphic copy of ℓ_1. Then X has SQ.*

It is noteworthy that there exists an infinite-dimensional separable Banach space X such that neither X nor X^* contains a copy of c_0, ℓ_1, or an infinite-dimensional reflexive subspace (see [33]). How much more difficult might it be to produce a Banach space that does not have SQ.

We refer to [32] for several results (and many references) concerning Banach spaces X not containing an isomorphic copy of ℓ_1.

From now onwards, unless otherwise stated, X is an infinite-dimensional Banach space over the field $\mathbb{K}$ of the real or complex numbers. Every measurable space (Ω, Σ), as well as every measure space (Ω, Σ, μ), is assumed to be nontrivial, i. e., there are in Σ infinitely many pairwise disjoints sets (of finite positive measure). If either X contains or does not contain an isomorphic copy of a Banach space Z we shall frequently write $X \supset Z$ or $X \not\supset Z$, respectively.

3. Weak* compactness of B_{X^*} and separable quotients

The next result shows that the existence of a concrete infinite-dimensional separable quotient of a Banach space depends on the type of weak*-compactness of the dual unit ball. We note the following

Theorem 18. *Let X be a Banach space and let B_{X^*} (weak*) be the dual unit ball equipped with the weak*- topology.*

(1) *If B_{X^*} (weak*) is not sequentially compact, then X has a separable quotient which is either isomorphic to c_0 or to ℓ_2.*

(2) *If B_{X^*} (weak*) is sequentially compact, then X has a copy of c_0 if and only if it has a complemented copy of c_0.*

Proof (Sketch). For statement (1), if B_{X^*} (weak*) is not sequentially compact, according to the classic Hagler-Johnson theorem [35, Corollary 1], X either has a quotient isomorphic to c_0 or X contains a copy of ℓ_1. The latter implies, via Theorem 14, that X has a quotient isomorphic to ℓ_2.

Statement (2) follows from [21], where the Gelfand-Phillips property is used. For a direct proof we refer the reader to [26, Theorem 4.1]. We provide a brief account of the argument. Let $\{x_n\}$ be a normalized basic sequence in X equivalent to the unit vector basis $\{e_n\}$ of c_0 and let $\{x_n^*\}$ denote the sequence of coordinate functionals of $\{x_n\}$ extended to X via Hahn-Banach's theorem. If $K > 0$ is the basis constant of $\{x_n\}$ then $\|x_n^*\| \leqslant 2K$, so that $x_n^* \in 2KB_{X^*}$ for every $n \in \mathbb{N}$. Since B_{X^*} (weak*) is sequentially compact, there is a subsequence $\{z_n^*\}$ of $\{x_n^*\}$ that converges to a point $z^* \in X^*$ under the weak*-topology. Let $\{z_n\}$ be the corresponding subsequence of $\{x_n\}$, still equivalent to the unit vector basis of c_0, and let F be the closed linear span of $\{z_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ define the linear functional $u_n : X \to \mathbb{K}$ by $u_n(x) = (z_n^* - z^*)x$, so that $|u_n(x)| \leqslant 4K \|x\|$ for each $n \in \mathbb{N}$. Since $u_n(x) \to 0$ for all $x \in X$, the linear operator $P : X \to F$ given by $Px = \sum_{n=1}^{\infty} u_n(x) z_n$ is well defined. As the formal series $\sum_{n=1}^{\infty} z_n$ is weakly unconditionally Cauchy, there is $C > 0$ such that $\|Px\| \leqslant 4CK \|x\|$. Now the fact that $z_n^* y \to 0$ for each $y \in F$ means that $z^* \in F^{\perp}$, which implies that $Pz_j = z_j$ for each $j \in \mathbb{N}$. Thus P is a bounded linear projection operator from X onto F. ■

We used the second part of Hagler-Johnson's classic [35, Corollary 1] to prove Theorem 18 (1). Let us re-phrase the first part of H-J: *If* $X^* \supset \ell_1$ *and* $X \not\supset \ell_1$*, then* X *has a quotient isomorphic to* c_0. The first part implies the second one, since, according to Diestel [14, p. 226], *if* B_{X^*} (weak*) *is not sequentially compact, then* $X^* \supset \ell_1$.

It is well-known that $\ell_\infty \supset \ell_1(\mathbb{R}) \supset \ell_1$. Therefore, if $X \supset \ell_\infty$ then, by Theorem 14, the Banach space X has a quotient isomorphic to ℓ_2. Useful characterizations of Banach spaces containing a copy of ℓ_∞ can be found in the classic paper [54]. It is also shown in [66] that ℓ_∞ is a quotient of a Banach space X if and only if B_{X^*} contains a weak*-homeomorphic copy of $\beta\mathbb{N}$. Hence such a space X has in particular a separable quotient isomorphic to ℓ_2.

The class of Banach spaces for which B_{X^*} (weak*) is sequentially compact is rich. This happens, for example, if X is a WCG Banach space. Of course, no WCG Banach space contains a copy of ℓ_∞. Another class of Banach spaces with weak* sequentially compact dual balls is the class of *Asplund* spaces. Note that the second statement of Theorem 18 applies in particular to each Banach space whose weak*-dual unit ball is Corson compact (a fact first observed in [49]) since, as is well-known, each Corson compact set is Fréchet-Urysohn. So, one has the following corollary, where a Banach space X is called *weakly Lindelöf determined* (WLD, for short) if there is a set $M \subseteq X$ with $\overline{\operatorname{span}(M)} = X$ enjoying the property that for each $x^* \in X^*$ the set $\{x \in M : x^*x \neq 0\}$ is countable (see Section 19.12 in [42]).

Corollary 19. *If* X *is a WLD Banach space, then* X *contains a complemented copy of* c_0 *if and only if it contains a copy of* c_0.

Proof. If X is a WLD Banach space, the dual unit ball B_{X^*} (weak*) of X is Corson compact (see [2, Proposition 1.2]), so the second statement of Theorem 18 applies. ■

If K is an infinite Gul'ko compact space, then $C(K)$ is weakly countable determined (see [3]), hence WLD. Since $C(K)$ has plenty of copies of c_0, it must have many complemented copies of c_0. It must be pointed out that if K is Corson compact then $C(K)$ need not be WLD. On the other hand, if a Banach space $X \supset c_0$ then $X^* \supset \ell_1$, so H-J and Theorem 14 ensure that X has c_0 as a quotient (a general characterization of Banach spaces containing a copy of c_0 is provided in [55]).

Corollary 20. *If a Banach space* X *contains a copy of* c_0*, then* X *has a quotient isomorphic to either* c_0 *or* ℓ_2.

Corollary 21 (cf. [45] and [53]). *If* K *is an infinite compact Hausdorff space, then* $C(K)$ *always has a quotient isomorphic to* c_0 *or* ℓ_2. *If* K *is scattered, then* c_0 *embeds in* $C(K)$ *complementably.*

Proof. The first statement is clear. The second is due to the fact that $C(K)$ is an Asplund space (see [34, Theorem 296]). ■

An extension of the previous corollary to all barrelled spaces $C_k(X)$ with the compact-open topology has been obtained in [40].

4. Separable quotients in tensor products

We quote three classic results about the existence of copies of c_0, ℓ_∞ and ℓ_1 in injective and projective tensor products which will be frequently used henceforth and provide infinite-dimensional separable quotients in $X \widehat{\otimes}_\pi Y$. We complement these classic facts with other results of our own. In the following theorem c_{00} stands for the linear subspace of c_0 consisting of all those sequences of finite range.

Theorem 22 (cf. [30, Theorem 2.3]). *Let X be an infinite-dimensional normed space and let Y be a Hausdorff locally convex space. If $Y \supset c_{00}$ then $X \widehat{\otimes}_\varepsilon Y$ contains a complemented subspace isomorphic to c_0.*

In particular, if X and Y are infinite-dimensional Banach spaces and $X \supset c_0$ or $Y \supset c_0$, then $X \widehat{\otimes}_\varepsilon Y$ contains a complemented copy of c_0, (*cf.* [59]). On the other hand, if either $X \supset \ell_\infty$ or $Y \supset \ell_\infty$ then $X \widehat{\otimes}_\varepsilon Y \supset \ell_\infty$ and consequently $X \widehat{\otimes}_\varepsilon Y$ also has a separable quotient isomorphic to ℓ_2. If $X \widehat{\otimes}_\varepsilon Y \supset \ell_\infty$, the converse statement also holds, as the next theorem asserts.

Theorem 23 (cf. [18, Corollary 2]). *Let X and Y be Banach spaces. $X \widehat{\otimes}_\varepsilon Y \supset \ell_\infty$ if and only if $X \supset \ell_\infty$ or $Y \supset \ell_\infty$.*

This also implies that if $X \widehat{\otimes}_\varepsilon Y \supset \ell_\infty$ then c_0 embeds complementably in $X \widehat{\otimes}_\varepsilon Y$. Concerning projective tensor products, we have the following well-known fact.

Theorem 24 (cf. [8, Corollary 2.6]). *Let X and Y be Banach spaces. If both $X \supset \ell_1$ and $Y \supset \ell_1$, then $X \widehat{\otimes}_\pi Y$ has a complemented subspace isomorphic to ℓ_1.*

Next we observe that if $X \widehat{\otimes}_\varepsilon Y$ is not a quotient of $X \widehat{\otimes}_\pi Y$, then $X \widehat{\otimes}_\varepsilon Y$ has *SQ*.

Theorem 25. *Let $J : X \otimes_\pi Y \to X \otimes_\varepsilon Y$ be the identity map and consider the continuous linear extension $\widetilde{J} : X \widehat{\otimes}_\pi Y \to X \widehat{\otimes}_\varepsilon Y$. If $\widetilde{J}$ is not a quotient map, then $X \widehat{\otimes}_\varepsilon Y$ has SQ.*

Proof. Observe that $X \otimes_\varepsilon Y \subseteq \operatorname{Im} \widetilde{J} \subseteq X \widehat{\otimes}_\varepsilon Y$. There are two cases.

Assume first that $X \otimes_\varepsilon Y$ is a barrelled space. In this case, since $X \otimes_\varepsilon Y$ is dense in $\operatorname{Im} \widetilde{J}$, we note that the range space $\operatorname{Im} \widetilde{J}$ is a barrelled subspace of $X \widehat{\otimes}_\varepsilon Y$. As the graph of $\widetilde{J}$ is closed in $(X \widehat{\otimes}_\pi Y) \times (X \widehat{\otimes}_\varepsilon Y)$ and $\operatorname{Im} \widetilde{J}$ is barrelled, it follows from [69, Theorem 19] that $\operatorname{Im} \widetilde{J}$ is a closed subspace of $X \widehat{\otimes}_\varepsilon Y$. Of course, this means that $\operatorname{Im} \widetilde{J} = X \widehat{\otimes}_\varepsilon Y$. Hence, the open map theorem shows that $\widetilde{J}$ is an open map from $X \widehat{\otimes}_\pi Y$ onto $X \widehat{\otimes}_\varepsilon Y$, so that $X \widehat{\otimes}_\varepsilon Y$ is a quotient of $X \widehat{\otimes}_\pi Y$.

Assume now that $X \otimes_\varepsilon Y$ is not barrelled. In this case $X \otimes_\varepsilon Y$ is a nonbarrelled dense subspace of the Banach space $X \widehat{\otimes}_\varepsilon Y$, so we may apply Theorem 6 to get that $X \widehat{\otimes}_\varepsilon Y$ has *SQ*. ■

Recall that the dual of $X \otimes_\pi Y$ coincides with the space of bounded linear operators from X into Y^*, whereas the dual of $X \otimes_\epsilon Y$ may be identified with the subspace of those operators which are integral, see [57, Section 3.5].

Proposition 26. *Let X and Y be Banach spaces. If X has the bounded approximation property and there is a bounded linear operator $T : X \to Y^*$ which is not integral, then $X \widehat{\otimes}_\varepsilon Y$ has SQ.*

Proof. Since there exists a bounded not integral linear operator between X and Y^*, the π-topology and ε-topology do not coincide on $X \otimes Y$, see [57]. Assume $X \otimes_\varepsilon Y$ is barrelled. Since X has the bounded approximation property, [6, Theorem] applies to get that $X \otimes_\varepsilon Y = X \otimes_\pi Y$, which contradicts the assumption that $(X \otimes_\varepsilon Y)^* \neq (X \otimes_\pi Y)^*$. Thus $X \otimes_\varepsilon Y$ must be a nonbarrelled dense linear subspace of $X \widehat{\otimes}_\varepsilon Y$, which according to Theorem 6 ensures that $X \widehat{\otimes}_\varepsilon Y$ has SQ. ■

For the next theorem, recall that a Banach space X is called *weakly countably determined* (WCD for short) if X (weak) is a Lindelöf Σ-space.

Theorem 27. *Let X and Y be WCD Banach spaces. If $X \widehat{\otimes}_\varepsilon Y \supset c_0$, then c_0 embeds complementably in $X \widehat{\otimes}_\varepsilon Y$.*

Proof. Since both X and Y are WCD Banach spaces, their dual unit balls B_{X^*} (weak*) and B_{X^*} (weak*) are Gul'ko compact. Since the countable product of Gul'ko compact spaces is Gul'ko compact, the product space $K := B_{X^*}$ (weak*) $\times$ B_{X^*} (weak*) is Gul'ko compact. Consequently $C(K)$ is a WCD Banach space, which implies that its weak*-dual unit ball $B_{C(K)^*}$ is Gul'ko compact. In particular $B_{C(K)^*}$ (weak*) is angelic and consequently sequentially compact. Let Z stand for the isometric copy of $X \widehat{\otimes}_\varepsilon Y$ in $C(K)$ and P for the isomorphic copy of c_0 in Z. From the proof of the second statement of Theorem 18 it follows that $C(K)$ has a complemented copy Q of c_0 *contained* in P. This implies that Z, hence $X \widehat{\otimes}_\varepsilon Y$, contains a complemented copy Q of c_0. ■

5. Separable quotients in spaces of vector-valued functions

If (Ω, Σ, μ) is a nontrivial arbitrary measure space, we denote by $L_p(\mu, X)$, $1 \leqslant p \leqslant \infty$, the Banach space of all X-valued p-Bochner μ-integrable (μ-essentially bounded when $p = \infty$) classes of functions equipped with its usual norm. If K is an infinite compact Hausdorff space, then $C(K, X)$ stands for the Banach space of all continuous functions $f : K \to X$ equipped with the supremum norm. By $B(\Sigma, X)$ we represent the Banach space of those bounded functions $f : \Omega \to X$ that are the uniform limit of a sequence of Σ-simple and X-valued functions, equipped with the supremum norm. The space of *all* X-valued bounded functions $f : \Omega \to X$ endowed with the supremum norm is written as $\ell_\infty(\Omega, X)$. Clearly $\ell_\infty(X) = \ell_\infty(\mathbb{N}, X)$. By $\ell_\infty(\Sigma)$ we denote the completion of the space $\ell_0^\infty(\Sigma)$ of scalar-valued Σ-simple functions, endowed with the supremum norm.

On the other hand, if (Ω, Σ, μ) is a (complete) finite measure space we represent by $P_1(\mu, X)$ the normed space consisting of all those [classes of] strongly μ-measurable X-valued Pettis integrable functions f defined on Ω provided with the semivariation norm

$$\|f\|_{P_1(\mu,X)} = \sup\left\{\int_\Omega |x^* f(\omega)|\, d\mu(\omega) : x^* \in X^*,\ \|x^*\| \leqslant 1\right\}.$$

As is well known, in general $P_1(\mu, X)$ is not a Banach space if X is infinite-dimensional, but it is always a barrelled space (see [19, Theorem 2] and [29, Remark 10.5.5]).

Our first result collects together a number of statements concerning Banach spaces of vector-valued functions related to the existence of separable quotients. Most of them can be easily derived from the well known facts about the existence of complemented copies of c_0 and ℓ_p for $1 \leqslant p \leqslant \infty$. We denote by $ca^+(\Sigma)$ the set of positive and finite measures on Σ.

Theorem 28. *The following statements on spaces of vector-valued functions hold.*

1. $C(K, X)$ *always has a complemented copy of* c_0.
2. $C(K, X)$ *has a quotient isomorphic to* ℓ_1 *if and only if* X *has* ℓ_1 *as a quotient.*
3. $L_p(\mu, X)$, *with* $1 \leqslant p < \infty$, *has a complemented copy of* ℓ_p. *In particular, the vector sequence space* $\ell_p(X)$ *has a complemented copy of* ℓ_p.
4. $L_p(\mu, X)$, *with* $1 < p < \infty$, *has a quotient isomorphic to* ℓ_1 *if and only if* X *has* ℓ_1 *as a quotient. In particular* $\ell_p(X)$ *has a quotient isomorphic to* ℓ_1 *if and only if the same happens to* X.
5. $L_\infty(\mu, X)$ *has a quotient isomorphic to* ℓ_2. *Hence, so does* $\ell_\infty(X)$.
6. *If* μ *is purely atomic and* $1 \leqslant p < \infty$, *then* $L_p(\mu, X)$ *has a complemented copy of* c_0 *if and only if* X *has a complemented copy of* c_0. *In particular, the space* $\ell_p(X)$ *has a complemented copy of* c_0 *if and only if so does* X.
7. *If* μ *is not purely atomic and* $1 \leqslant p < \infty$, *then* $L_p(\mu, X)$ *has complemented copy of* c_0 *if* $X \supset c_0$.
8. *If* $\mu \in ca^+(\Sigma)$ *is purely atomic and* $1 < p < \infty$, *then* $L_p(\mu, X)$ *has a quotient isomorphic to* c_0 *if and only if* X *contains a quotient isomorphic to* c_0.
9. *If* $\mu \in ca^+(\Sigma)$ *is not purely atomic and* $1 < p < \infty$, *then* $L_p(\mu, X)$ *has a quotient isomorphic to* c_0 *if and only if* X *contains a quotient isomorphic to* c_0 *or* $X \supset \ell_1$.
10. *If* μ *is* σ*-finite, then* $L_\infty(\mu, X)$ *has a quotient isomorphic to* ℓ_1 *if and only if* $\ell_\infty(X)$ *has* ℓ_1 *as a quotient.*
11. $B(\Sigma, X)$ *has a complemented copy of* c_0 *and a quotient isomorphic to* ℓ_2.
12. $\ell_\infty(\Omega, X)$ *has a quotient isomorphic to* ℓ_2.
13. *If the cardinality of* Ω *is less than the first real-valued measurable cardinal, then* $\ell_\infty(\Omega, X)$ *has a complemented copy of* c_0 *if and only if* X *enjoys the same property. In particular,* $\ell_\infty(X)$ *contains a complemented copy of* c_0 *if and only if* X *enjoys the same property.*
14. $c_0(X)$ *has a complemented copy of* c_0.
15. $\ell_\infty(X)^*$ *has a quotient isomorphic to* ℓ_1.

Proof. Let us proceed with the proofs of the statements.

1. This well-know fact can be found in [9, Theorem] and [30, Corollary 2.5] (or in [10, Theorem 3.2.1]).
2. This is because $C(K, X)$ contains a complemented copy of ℓ_1 if and only if X contains a complemented copy of ℓ_1 (see [58] or [10, Theorem 3.1.4]).

3. If $1 \leqslant p < \infty$, each $L_p(\mu, X)$ space contains a norm one complemented isometric copy of ℓ_p (see [10, Proposition 1.4.1]). For the second affirmation note that if (Ω, Σ, μ) is a σ-finite purely atomic measure space, then $\ell_p(X) = L_p(\mu, X)$ isometrically.
4. If $1 < p < \infty$ then $L_p(\mu, X)$ contains a complemented copy of ℓ_1 if and only if X does (see [48] or [10, Theorem 4.1.2]).
5. The space ℓ_∞ is isometrically embedded in $L_\infty(\mu)$, which is in turn isometric to a norm one complemented subspace of $L_\infty(\mu, X)$.
6. If μ is purely atomic, then $L_p(\mu, X)$ contains a complemented copy of c_0 if and only if X has the same property. This fact, discovered by F. Bombal in [7], can also be seen in [10, Theorem 4.3.1].
7. If μ is not purely atomic and $1 \leqslant p < \infty$, according to [22], the mere fact that $X \supset c_0$ implies that $L_p(\mu, X)$ contains a complemented copy of c_0.
8. If (Ω, Σ, μ) is a purely atomic finite measure space and $1 < p < \infty$, the statement corresponds to the first statement of [13, Theorem 1.1].
9. If (Ω, Σ, μ) is a not purely atomic finite measure space and $1 < p < \infty$, the statement corresponds to the second statement of [13, Theorem 1.1].
10. If (Ω, Σ, μ) is a σ-finite measure space, the existence of a complemented copy of ℓ_1 in $L_\infty(\mu, X)$ is related to the local theory of Banach spaces, a fact discovered by S. Díaz in [12]. The statement, as formulated above, can be found in [10, Theorem 5.2.3].
11. Since $\ell_0^\infty(\Sigma, X) = \ell_0^\infty(\Sigma) \otimes_\varepsilon X$ and X is infinite-dimensional, then $\ell_0^\infty(\Sigma, X)$ is not barrelled by virtue of Freniche's classic theorem (see [30, Corollary 1.5]). Since $\ell_0^\infty(\Sigma, X)$ is a nonbarrelled dense subspace of $B(\Sigma, X)$, Theorem 6 guarantees that $B(\Sigma, X)$ indeed has SQ. However, we can be more precise. Since $B(\Sigma, X) = \ell_0^\infty(\Sigma) \widehat{\otimes}_\varepsilon X$ and $\ell_0^\infty(\Sigma) \supset c_{00}$ due to the nontriviality of the σ-algebra Σ, Theorem 22 implies that $B(\Sigma, X)$ contains a complemented copy of c_0. On the other hand, since ℓ_∞ is isometrically embedded in $B(\Sigma, X)$, it turns out that ℓ_2 is a quotient of $B(\Sigma, X)$.
12. Clearly $\ell_\infty(\Omega, X) \supset \ell_\infty(\Omega) \supset \ell_\infty$ since the set Ω is infinite.
13. This property can be found in [46].
14. Just note that $c_0(X) = c_0 \widehat{\otimes}_\varepsilon X$, so we may apply Theorem 22.
15. It suffices to note that $\ell_1(X^*)$ is linearly isometric to a complemented subspace of $\ell_\infty(X)^*$ (see [10, Section 5.1]). ■

Remark 29. $L_p(\mu, X)$ if $1 \leqslant p < \infty$, as well as $C(K, X)$, need not contain a copy of ℓ_∞. By [47, Theorem], one has that $L_p(\mu, X) \supset \ell_\infty$ if and only if $X \supset \ell_\infty$, whereas $C(K, X) \supset \ell_\infty$ if and only if $C(K) \supset \ell_\infty$ or $X \supset \ell_\infty$, as shown in [18, Corollary 3].

Remark 30. Complemented copies of c_0 in $L_\infty(\mu, X)$. If (Ω, Σ, μ) is a σ-finite measure, according to [11, Theorem 1] a necessary condition for the space $L_\infty(\mu, X)$ to contain a complemented copy of c_0 is that $X \supset c_0$. The same happens with the space $\ell_\infty(\Omega, X)$ (see [26, Theorem 2.1 and Corollary 2.3]).

Theorem 31. *The following statements on the space* $\widehat{P_1(\mu, X)}$ *hold.*

1. *If the finite measure space* (Ω, Σ, μ) *is not purely atomic, the Banach space* $\widehat{P_1(\mu, X)}$ *has SQ.*
2. *If the range of the positive finite measure* μ *is infinite and* $X \supset c_0$ *then* $\widehat{P_1(\mu, X)}$ *has a complemented copy of* c_0.

Proof. Observe that $L_1(\mu) \widehat{\otimes}_\pi X = L_1(\mu, X)$ and $\widehat{P_1(\mu, X)} = L_1(\mu) \widehat{\otimes}_\varepsilon X$ isometrically. On the other hand, from the algebraic viewpoint $L_1(\mu, X)$ is a linear subspace of $P_1(\mu, X)$, which is dense under the norm of $P_1(\mu, X)$. If

$$J : L_1(\mu) \otimes_\pi X \to L_1(\mu) \otimes_\varepsilon X$$

is the identity map, R a linear isometry from $L_1(\mu, X)$ onto $L_1(\mu) \widehat{\otimes}_\pi X$ and S a linear isometry from $L_1(\mu) \widehat{\otimes}_\varepsilon X$ onto $\widehat{P_1(\mu, X)}$, the mapping

$$S \circ \widetilde{J} \circ R : L_1(\mu, X) \to \widehat{P_1(\mu, X)},$$

where $\widetilde{J}$ denotes the (unique) continuous linear extension of J to $L_1(\mu) \widehat{\otimes}_\pi X$, coincides with the natural inclusion map T of $L_1(\mu, X)$ into $P_1(\mu, X)$ over the dense subspace of $L_1(\mu, X)$ consisting of the X-valued (classes of) μ-simple functions, which implies that $S \circ \widetilde{J} \circ R = T$. Since X is infinite-dimensional and μ is not purely atomic, the space $P_1(\mu, X)$ is not complete [67]. So necessarily we have that $\operatorname{Im} T \neq \widehat{P_1(\mu, X)}$. This implies in particular that $\operatorname{Im} \widetilde{J} \neq L_1(\mu) \widehat{\otimes}_\pi X$. According to Theorem 25, this means that $\widehat{P_1(\mu, X)} = L_1(\mu) \widehat{\otimes}_\varepsilon X$ has a separable quotient.

The proof of the second statement can be found in [31, Corollary 2]. ■

6. Separable quotients in spaces of linear operators

If Y is also a Banach space, let us denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators $T : X \to Y$ equipped with the operator norm $\|T\|$. By $\mathcal{K}(X, Y)$ we represent the closed linear subspace of $\mathcal{L}(X, Y)$ consisting of all those compact operators. We designate by $\mathcal{L}_{w^*}(X^*, Y)$ the closed linear subspace of $\mathcal{L}(X^*, Y)$ formed by all weak*-weakly continuous operators and by $\mathcal{K}_{w^*}(X^*, Y)$ the closed linear subspace of $\mathcal{K}(X^*, Y)$ consisting of all weak*-weakly continuous operators. The closed subspace of $\mathcal{L}(X, Y)$ consisting of weakly compact linear operators is denoted by $\mathcal{W}(X, Y)$. It is worthwhile to mention that $\mathcal{L}_{w^*}(X^*, Y) = \mathcal{L}_{w^*}(Y^*, X)$ isometrically, as well as $\mathcal{K}_{w^*}(X^*, Y) = \mathcal{K}_{w^*}(Y^*, X)$, by means of the linear mapping $T \mapsto T^*$. The Banach space of nuclear operators $T : X \to Y$ equipped with the so-called nuclear norm $\|T\|_N$ is denoted by $\mathcal{N}(X, Y)$. Let us recall that $\|T\| \leqslant \|T\|_N$. Classic references for this section are the monographs [38] and [44].

The first statement of Theorem 32 answers a question of Prof. T. Dobrowolski posed during the 31st Summer Conference on Topology and its Applications at Leicester (2016).

Theorem 32. *The following conditions on $\mathcal{L}(X,Y)$ hold.*

1. *If $Y \neq \{\mathbf{0}\}$, then $\mathcal{L}(X,Y)$ always has SQ.*
2. *If $X^* \supset c_0$ or $Y \supset c_0$, then $\mathcal{L}(X,Y)$ has a quotient isomorphic to ℓ_2.*
3. *If $X^* \supset \ell_q$ and $Y \supset \ell_p$, with $1 \leqslant p < \infty$ and $1/p + 1/q = 1$, then $\mathcal{L}(X,Y)$ has a quotient isomorphic to ℓ_2.*

Proof. Let us prove each of these statements.

1. First observe that X^* is complemented in $\mathcal{L}(X,Y)$. Indeed, choose $y_0 \in Y$ with $\|y_0\| = 1$ and apply the Hahn-Banach theorem to get $y_0^* \in Y^*$ such that $\|y_0^*\| = 1$ and $y_0^* y_0 = 1$. The map $\varphi : X^* \to \mathcal{L}(X,Y)$ defined by $(\varphi x^*)(x) = x^* x \cdot y_0$ for every $x \in X$ is a linear isometry into $\mathcal{L}(X,Y)$ (see [44, 39.1.(2')]), and the operator $P : \mathcal{L}(X,Y) \to \mathcal{L}(X,Y)$ given by $PT = \varphi(y_0^* \circ T)$ is a norm one linear projection operator from $\mathcal{L}(X,Y)$ onto $\operatorname{Im}\varphi$. Hence X^* is linearly isometric to a norm one complemented linear subspace of $\mathcal{L}(X,Y)$. Since X^* is a dual Banach space, it has SQ by Theorem 4. Hence the operator space $\mathcal{L}(X,Y)$ has SQ.
2. Since $X^* \otimes_\varepsilon Y$ is isometrically embedded in $\mathcal{L}(X,Y)$ and both X^* and Y are isometrically embedded in $X^* \otimes_\varepsilon Y$, if either $X^* \supset c_0$ or $Y \supset c_0$, then $\mathcal{L}(X,Y) \supset c_0$. In this case, according to [25, Corollary 1], $\mathcal{L}(X,Y)$ contains an isomorphic copy of ℓ_∞. This ensures that $\mathcal{L}(X,Y)$ has a separable quotient isomorphic to ℓ_2.
3. If $\{e_n : n \in \mathbb{N}\}$ is the unit vector basis of ℓ_p, define $T_n : \ell_p \to \ell_p$ by $T_n\xi = \xi_n e_n$ for each $n \in \mathbb{N}$. Since

$$\left\|\sum_{i=1}^n a_i T_i\right\| = \sup_{\|\xi\|_p \leqslant 1} \left(\sum_{i=1}^n |a_i \xi_i|^p\right)^{1/p} \leqslant \sup_{1 \leqslant i \leqslant n} |a_i|$$

for any scalars $a_1, \ldots, a_n$, we can see that $\{T_n : n \in \mathbb{N}\}$ is a basic sequence in $\mathcal{K}(\ell_p, \ell_p)$ equivalent to the unit vector basis of c_0. Recall that in general $E^* \widehat{\otimes}_\varepsilon F = \mathcal{K}(E,F)$ for Banach spaces E and F whenever E^* has the approximation property. Hence if $1/p + 1/q = 1$, then

$$\ell_q \widehat{\otimes}_\varepsilon \ell_p = \ell_p^* \widehat{\otimes}_\varepsilon \ell_p = \mathcal{K}(\ell_p, \ell_p)$$

isometrically. So we have $\ell_q \widehat{\otimes}_\varepsilon \ell_p \supset c_0$. As in addition $\ell_q \widehat{\otimes}_\varepsilon \ell_p$ is isomorphically embedded in $X^* \widehat{\otimes}_\varepsilon Y$, which in turn is isometrically embedded in $\mathcal{L}(X,Y)$, we have that $\mathcal{L}(X,Y) \supset c_0$. So, we use again [25, Corollary 1] to conclude that $\mathcal{L}(X,Y) \supset \ell_\infty$. Thus $\mathcal{L}(X,Y)$ has a quotient isomorphic to ℓ_2. ■

The Banach space $\mathcal{L}(X,Y)$ need not contain a copy of ℓ_∞ in order to have SQ, as the following example shows.

Let $1 < p, q < \infty$ with conjugated indices p', q', i. e., $1/p + 1/p' = 1/q + 1/q' = 1$.

Example 33. If $p > q'$ then $\mathcal{L}(\ell_p, \ell_{q'})$ does not contain an isomorphic copy of c_0.

Proof. Recall that in general $\mathcal{L}(X, Y^*) = (X \widehat{\otimes}_\pi Y)^*$ isometrically for arbitrary Banach spaces X and Y (see for instance [57, Section 2.2]), hence the fact that $\ell_{q'}^* = \ell_q$ assures that $\mathcal{L}(\ell_p, \ell_{q'}) = (\ell_p \widehat{\otimes}_\pi \ell_q)^*$ isometrically. Now let us assume by way of contradiction that $\mathcal{L}(\ell_p, \ell_{q'}) \supset c_0$, which implies that $\ell_p \widehat{\otimes}_\pi \ell_q$ contains a complemented copy of ℓ_1 (see [14, Chapter 5, Theorem 10]). Since $p > q'$, according to [57, Corollary 4.24] or [15, Chapter 8, Corollary 5], the space $\ell_p \widehat{\otimes}_\pi \ell_q$ is reflexive, which contradicts the fact that it has a quotient isomorphic to the nonreflexive space ℓ_1. So we must conclude that $\mathcal{L}(\ell_p, \ell_{q'}) \not\supset c_0$.

On the other hand, since $\mathcal{L}(\ell_p, \ell_{q'})$ is a dual Banach space, Theorem 4 shows that $\mathcal{L}(\ell_p, \ell_{q'})$ has SQ. Alternatively, we can also apply the first statement of Theorem 32. ■

Proposition 34. *If X^* has the approximation property, the Banach space $\mathcal{N}(X, Y)$ of nuclear operators has SQ.*

Proof. Since X^* enjoys the approximation property, it follows that $\mathcal{N}(X, Y) = X^* \widehat{\otimes}_\pi Y$ isometrically. Hence X^* is linearly isometric to a complemented subspace of $\mathcal{N}(X, Y)$. Since X^*, as a dual Banach space, has SQ, the transitivity of the quotient map yields that $\mathcal{N}(X, Y)$ has SQ. ■

Theorem 35. *The following statements hold.*

1. *If $X \supset c_0$ and $Y \supset c_0$, then $\mathcal{L}_{w^*}(X^*, Y)$ has a quotient isomorphic to ℓ_2.*
2. *If X has a separable quotient isomorphic to ℓ_1, then $\mathcal{L}_{w^*}(X^*, Y)$ enjoys the same property.*
3. *If (Ω, Σ, μ) is an arbitrary measure space and $Y \neq \{\mathbf{0}\}$, then $\mathcal{L}_{w^*}(L_\infty(\mu), Y)$ has a quotient isomorphic to ℓ_1.*
4. *If $X^* \supset c_0$ or $Y \supset \ell_\infty$, then $\mathcal{K}(X, Y)$ has a quotient isomorphic to ℓ_2.*
5. *If either $X^* \supset c_0$ or $Y \supset c_0$, then $\mathcal{K}(X, Y)$ contains a complemented copy of c_0.*
6. *If $X \supset \ell_\infty$ or $Y \supset \ell_\infty$, then $\mathcal{K}_{w^*}(X^*, Y)$ has a quotient isomorphic to ℓ_2.*
7. *The space $\mathcal{W}(X, Y)$ always has SQ.*
8. *If $X \supset c_0$ and $Y \supset c_0$, then $\mathcal{W}(X, Y)$ contains a complemented copy of c_0.*

Proof. Copies of ℓ_∞ often suffice, since $\ell_\infty \supset \ell_1$ and then Theorem 14 applies.

1. By [27, Theorem 1.5] if $X \supset c_0$ and $Y \supset c_0$ then $\mathcal{L}_{w^*}(X^*, Y) \supset \ell_\infty$.
2. Choose $y_0 \in Y$ with $\|y_0\| = 1$ and select $y_0^* \in Y^*$ such that $\|y_0^*\| = 1$ and $y_0^* y_0 = 1$. The map $\psi : X \to \mathcal{L}_{w^*}(X^*, Y)$ given by $\psi(x)(x^*) = x^* x \cdot y_0$, for $x^* \in X^*$, is well-defined and if $x_d^* \to x^*$ under the weak*-topology of X^* then $\psi(x)(x_d^*) \to \psi(x)(x^*)$ weakly in Y, so that ψ embeds X isometrically in $\mathcal{L}_{w^*}(X^*, Y)$. On the other hand, the operator $Q : \mathcal{L}_{w^*}(X^*, Y) \to \mathcal{L}_{w^*}(X^*, Y)$ given by $QT = \psi(y_0^* \circ T)$, which is also well-defined since $y_0^* \circ T \in X$ whenever T is weak*-weakly continuous, is a bounded linear projection operator from $\mathcal{L}_{w^*}(X^*, Y)$ onto $\operatorname{Im} \psi$. Since we are assuming that ℓ_1 is a quotient of X, it follows that ℓ_1 is also isomorphic to a quotient of $\mathcal{L}_{w^*}(X^*, Y)$.

3. This statement is a consequence of the previous one, since ℓ_1 embeds complementably in $L_1(\mu)$.
4. According to [43], if $X^* \supset c_0$ or $Y \supset \ell_\infty$, then $\mathcal{K}(X,Y) \supset \ell_\infty$.
5. This property has been shown in [56, Corollary 1].
6. The map $\psi : X \to \mathcal{L}_{w^*}(X^*, Y)$ defined above by $\psi(x)(x^*) = x^*x \cdot y_0$, for every $x^* \in X^*$, yields a finite-rank (hence compact) operator $\psi(x)$, so that $\operatorname{Im}\psi \subseteq \mathcal{K}_{w^*}(X^*, Y)$. On the other hand, if $x_0 \in X$ with $\|x_0\| = 1$ and $x_0^* \in X^*$ verifies that $\|x_0^*\| = 1$ and $x_0^* x_0 = 1$, the map $\phi : Y \to \mathcal{K}_{w^*}(X^*, Y)$ given by $\phi(y)(x) = x_0^* x \cdot y$, for every $x \in X$, is a linear isometry from Y into $\mathcal{K}_{w^*}(X^*, Y)$. Hence X and Y are isometrically embedded in $\mathcal{K}_{w^*}(X^*, Y)$.
7. Just note that $\mathcal{W}(X,Y) = \mathcal{L}_{w^*}(X^{**}, Y)$ isometrically. Since X^* is complementably embedded in $\mathcal{L}_{w^*}(X^{**}, Y)$, the conclusion follows from Theorem 4.
8. According to [28, Theorem 2.5], under those conditions the space $\mathcal{W}(X,Y)$ contains a complemented copy of c_0. ■

Remark 36. If neither X nor Y contains a copy of c_0, then $\mathcal{L}_{w^*}(X^*, Y)$ cannot contain a complemented copy of c_0 as observed in [23].

The following result sharpens the first statement of Theorem 31.

Corollary 37. *If (Ω, Σ, μ) is a finite measure space, $\widehat{P_1(\mu, X)}$ has a quotient isomorphic to ℓ_1.*

Proof. This follows from the second statement of the previous theorem together with the fact that $\widehat{P_1(\mu, X)} = \mathcal{L}_{w^*}(L_\infty(\mu), X)$ (see [15, Chapter 8, Theorem 5]). ■

Remark 38. The space $P_1(\mu, X)$ need not contain a copy of ℓ_∞. It can be easily shown that $P_1(\mu, X)$ embeds isometrically in the space $\mathcal{K}_{w^*}\left(ca(\Sigma)^*, X\right)$, where $ca(\Sigma)$ denotes the Banach space of scalar-valued countably additive measures equipped with the variation norm. Since $ca(\Sigma) \not\supset \ell_\infty$, it follows from [18, Theorem] that $P_1(\mu, X) \supset \ell_\infty$ if and only if $X \supset \ell_\infty$.

7. Separable quotients in spaces of vector measures

In this section we denote by $ba(\Sigma, X)$ the Banach space of all X-valued bounded finitely additive measures $F : \Sigma \to X$ provided with the semivariation norm $\|F\|$. The closed linear subspace of $ba(\Sigma, X)$ consisting of countably additive measures is represented by $ca(\Sigma, X)$, while $cca(\Sigma, X)$ stands for the (closed) linear subspace of $ca(\Sigma, X)$ of all measures with relatively compact range. It can be easily shown that $ca(\Sigma, X) = \mathcal{L}_{w^*}\left(ca(\Sigma)^*, X\right)$ isometrically. We also designate by $bvca(\Sigma, X)$ the Banach space of all X-valued countably additive measures $F : \Sigma \to X$ of bounded variation equipped with the variation norm $|F|$. Finally, following [57, page 107], we denote by $\mathcal{M}_1(\Sigma, X)$ the closed linear subspace of $bvca(\Sigma, X)$ consisting of all those $F \in bvca(\Sigma, X)$ that have the so-called *Radon-Nikodým property*, i. e., such that for each $\lambda \in ca^+(\Sigma)$ with $F \ll \lambda$ there exists $f \in L_1(\lambda, X)$ with $F(E) = \int_E f \, d\lambda$ for every $E \in \Sigma$. For this section, our main references are [15] and [57].

Theorem 39. *The following statements hold. In the first case X need not be infinite-dimensional.*

1. *If $X \neq \{\mathbf{0}\}$, then $ba(\Sigma, X)$ always has SQ.*
2. *If $X \supset c_0$, then $ba(\Sigma, X)$ has a quotient isomorphic to ℓ_2.*
3. *If $X \supset c_0$ but $X \not\supset \ell_\infty$, then $ba(\Sigma, X)$ has a complemented copy of c_0.*
4. *If Σ admits no atomless probability measure, then $ca(\Sigma, X)$ has a quotient isomorphic to ℓ_1.*
5. *If $X \supset c_0$ and Σ admits a nonzero atomless $\lambda \in ca^+(\Sigma)$, then $ca(\Sigma, X)$ has a quotient isomorphic to ℓ_2.*
6. *If there exists some $F \in cca(\Sigma, X)$ of unbounded variation, then $cca(\Sigma, X)$ has SQ.*
7. *If $X \supset c_0$, then $cca(\Sigma, X)$ contains a complemented copy of c_0.*
8. *If $X \supset \ell_1$, then $\mathcal{M}_1(\Sigma, X)$ has a quotient isomorphic to ℓ_1.*

Proof. In cases 2 and 3 it would suffice to show that the corresponding Banach space contains an isomorphic copy of ℓ_∞.

1. This happens because $ba(\Sigma, X) = \mathcal{L}(\ell_\infty(\Sigma), X)$ isometrically. Since $\ell_\infty(\Sigma)$ is infinite-dimensional by virtue of the nontriviality of the σ-algebra Σ, the statement follows from the first statement of Theorem 32.
2. By point 2 of Theorem 32, if $X \supset c_0$ then $\mathcal{L}(\ell_\infty(\Sigma), X)$ has a quotient isomorphic to ℓ_2. The statement follows from the fact that $ba(\Sigma, X) = \mathcal{L}(\ell_\infty(\Sigma), X)$.
3. $ba(\Sigma, X)$ has a complemented copy of c_0 by virtue of [28, Corollary 3.2].
4. If the nontrivial σ-algebra Σ admits no atomless probability measure, it can be shown that $ca(\Sigma, X)$ is linearly isometric to $\ell_1(\Gamma, X)$ for some infinite set Γ. Since $\ell_1(\Gamma, X) = L_1(\mu, X)$, where μ is the counting measure on 2^Γ, the conclusion follows from the third statement of Theorem 28.
5. Since $ca(\Sigma) \widehat{\otimes}_\varepsilon X = cca(\Sigma, X)$ isometrically, if $X \supset c_0$ then $cca(\Sigma, X) \supset c_0$ and hence $ca(\Sigma, X) \supset c_0$. If Σ admits a nonzero atomless $\lambda \in ca^+(\Sigma)$, then one has $ca(\Sigma, X) \supset \ell_\infty$ by virtue of [17, Theorem 1].
6. Observe that $cca(\Sigma, X) = ca(\Sigma) \widehat{\otimes}_\varepsilon X$ and $\mathcal{M}_1(\Sigma, X) = ca(\Sigma) \widehat{\otimes}_\pi X$ isometrically (see [57, Theorem 5.22]) but, at the same time, from the algebraic point of view, $\mathcal{M}_1(\Sigma, X)$ is a linear subspace of $cca(\Sigma, X)$ since every Bochner indefinite integral has a relatively compact range, [15, Chapter II, Corollary 9 (c)]. If there exists some $F \in cca(\Sigma, X)$ of unbounded variation, then $\mathcal{M}_1(\Sigma, X) \neq cca(\Sigma, X)$, so the statement follows from Theorem 25.
7. Since $X \supset c_0$ and $ca(\Sigma)$ is infinite-dimensional, then $X \widehat{\otimes}_\varepsilon ca(\Sigma)$ contains a complemented copy of c_0 by [30, Theorem 2.3].
8. Since $\mathcal{M}_1(\Sigma, X) = ca(\Sigma) \widehat{\otimes}_\pi X$ and $ca(\Sigma) \supset \ell_1$, if $X \supset \ell_1$ then $\mathcal{M}_1(\Sigma, X)$ has a quotient isomorphic to ℓ_1 as follows from Theorem 24. ■

Remark 40. If $\omega \in \Omega$ and $E(\Sigma, X)$ is either $ba(\Sigma, X)$, $ca(\Sigma, X)$ or $bvca(\Sigma, X)$, the map $P_\omega : E(\Sigma, X) \to E(\Sigma, X)$ defined by $P_\omega(F) = F(\Omega)\,\delta_\omega$ is a bounded linear projection operator onto the copy $\{x\,\delta_\omega : x \in X\}$ of X in $E(\Sigma, X)$. Hence, if X has a separable quotient isomorphic to Z, then $E(\Sigma, X)$ also has a separable quotient isomorphic to Z.

Remark 41. $cca(\Sigma, X)$ may not have a copy of ℓ_∞. Since $cca(\Sigma, X) = \mathcal{K}_{w^*}\left(ca(\Sigma)^*, X\right)$, according to [18, Theorem or Corollary 4], $cca(\Sigma, X) \supset \ell_\infty$ if and only if $X \supset \ell_\infty$.

Remark 42. Concerning the space $\mathcal{M}_1(\Sigma, X)$, it is worthwhile to mention that it follows from [24, Theorem] that $\mathcal{M}_1(\Sigma, X) \supset \ell_\infty$ if and only if $X \supset \ell_\infty$.

References

[1] S. A. Argyros, P. Dodos, and V. Kanellopoulos, *Unconditional families in Banach spaces*, Math. Ann. **341** (2008), 15–38
[2] S. Argyros and S. Mercourakis, *On weakly Lindelöf Banach spaces*, Rocky Mount. J. Math. **23** (1993), 395–446.
[3] S. Argyros and S. Negrepontis, *On weakly $\mathcal{K}$-countably determined spaces of continuous functions*, Proc. Amer. Math. Soc. **87** (1983), 731–736.
[4] D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. **88** (1968), 33–46.
[5] G. Bennet and N. J. Kalton, *Inclusion theorems for k-spaces*, Canad. J. Math. **25** (1973), 511–524.
[6] J. Bonet, *Una nota sobre la coincidencia de topologías en productos tensoriales*, Rev. R. Acad. Cienc. Exactas Fis. Nat. (Esp.) **81** (1987), 87–89.
[7] F. Bombal, *Distinguished subsets in vector sequence spaces*, Progress in Functional Analysis, Proceedings of the Peñíscola Meeting on occasion of 60th birthday of M. Valdivia (Eds. J. Bonet et al.), Elsevier 1992, 293–306.
[8] F. Bombal, Fernández-Unzueta, M. and Villanueva, I., *Local structure and copies of c_0 and ℓ_1 in the tensor product of Banach spaces*, Bol. Soc. Mat. Mex. **10** (2004), 1–8.
[9] P. Cembranos, *$C(K, E)$ contains a complemented copy of c_0*. Proc. Amer. Math. Soc. **91** (1984), 556–558.
[10] P. Cembranos and J. Mendoza, *Banach Spaces of Vector-Valued Functions*, Lecture Notes in Math. **1676**, Springer, Berlin Heidelberg, 1997.
[11] S. Díaz, *Complemented copies of c_0 in $L_\infty(\mu, X)$*, Proc. Amer. Math. Soc. **120** (1994), 1167–1172.
[12] S. Díaz, *Complemented copies of ℓ_1 in $L_\infty(\mu, X)$*, Rocky Mountain J. Math. **27** (1997), 779–784.
[13] S. Díaz and G. Schlüchtermann, *Quotients of vector-valued function spaces*, Math. Proc. Cambridge Philos. Soc. **126** (1999), 109–116.
[14] J. Diestel, *Sequences and series in Banach spaces*, GTM **92**. Springer-Verlag. New York Berlin Heidelberg Tokyo, 1984.
[15] J. Diestel and J. J. Jr. Uhl, *Vector Measures*, Math. Surveys and Monographs **15**, AMS, Providence, 1977.
[16] L. Drewnowski, *A solution to a problem of De Wilde and Tsirulnikov*, Manuscripta Math. **37** (1982), 61–64.
[17] L. Drewnowski, *When does $ca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞?*, Proc. Amer. Math. Soc. **109** (1990), 747–752.

[18] L. Drewnowski, *Copies of ℓ_∞ in an operator space*, Math. Proc. Cambridge Philos. Soc. **108** (1990), 523–526.

[19] L. Drewnowski, Florencio, M. and Paúl, P. J., *The space of Pettis integrable functions is barrelled*, Proc. Amer. Math. Soc. **114** (1992), 687–694.

[20] M. Eidelheit, *Zur Theorie der Systeme linearer Gliechunmgen*, Studia Math. **6** (1936), 139–148.

[21] G. Emmanuele, *On complemented copies of c_0*. Extracta Math. **3** (1988), 98–100.

[22] G. Emmanuele, *On complemented copies of c_0 in $L_p(X)$, $1 \leqslant p < \infty$*, Proc. Amer. Math. Soc. **104** (1988), 785–786.

[23] G. Emmanuele, *On complemented copies of c_0 in spaces of operators II*, Comment. Math. Univ. Carolin. **35** (1994), 259–261.

[24] J. C. Ferrando, *When does $bvca(\Sigma, X)$ contain a copy of ℓ_∞?*, Math. Scand. **74** (1994), 271–274.

[25] J. C. Ferrando, *Copies of c_0 in certain vector-valued function Banach spaces*, Math. Scand. **77** (1995), 148–152.

[26] J. C. Ferrando, *Complemented copies of c_0 in the vector-valued bounded function space*, J. Math. Anal. Appl. **239** (1999), 419–426.

[27] J. C. Ferrando, *On copies of c_0 and ℓ_∞ in $L(X^*, Y)$*, Bull. Belg. Math. Soc. Simon Stevin **9** (2002), 259–264.

[28] J. C. Ferrando, *Complemented copies of c_0 in spaces of operators*, Acta Math. Hungar. **90** (2003), 57–61.

[29] J. C. Ferrando, M. López-Pellicer, and L. M. Sánchez Ruiz, *Metrizable barrelled spaces*, Pitman Research Notes in Math. Series **332**, Longman, Essex, 1995.

[30] F. J. Freniche, *Barrelledness of the space of vector valued and simple functions*, Math. Ann. **267** (1984), 479–486.

[31] F. J. Freniche, *Embedding c_0 in the space of Pettis integrable functions*, Quaest. Math. **21** (1998), 261–267.

[32] S. Gabriyelyan, J. Kąkol, and G. Plebanek, *The Ascoli property for function spaces and the weak topology of Banach and Fréchet spaces* Studia Math. **233** (2016), 119–139.

[33] W. T. Gowers, *A Banach space not containing c_0, ℓ_1 or a reflexive subspace*, Trans. Amer. Math. Soc. **344** (1994), 407–420.

[34] P. Habala, P. Hájek, and V. Zizler, *Introduction to Banach spaces II*, MATFYZPRESS, University Karlovy, 1996.

[35] J. Hagler and W. B. Johnson, *On Banach spaces whose dual balls are not weak*-sequentially compact*, Israel J. Math. **28** (1977), 325–330.

[36] W. B. Johnson, *On quasi-complements*, Pacific J. Math. **48** (1973), 113–117.

[37] W. B. Johnson and H. P. Rosenthal, *On weak*-basic sequences and their applications to the study of Banach spaces*, Studia Math. **43** (1975), 166–168.

[38] H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart, 1981.

[39] J. Kąkol and W. Śliwa, *Remarks concerning the separable quotient problem*, Note Mat. **13** (1993), 277–282.

[40] J. Kąkol, S. A. Saxon, and A. Tood, *Barrelled spaces with(out) separable quotients*, Bull. Aust. Math. Soc. **90** (2014), 295–303.

[41] J. Kąkol and S. A. Saxon, *Separable quotients in $C_c(X)$, $C_p(X)$ and their duals*, Proc. Amer. Math. Soc. **145** (2017), 3829–3841.

[42] J. Kąkol, W. Kubiś, and M. López-Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Springer, Developments in Math. **24**, New York Dordrecht Heidelberg, 2011.

[43] N. J. Kalton, *Spaces of compact operators*, Math. Ann. **208** (1974), 267–278.

[44] G. Köthe, *Topological Vector Spaces II*, Springer-Verlag, New York Heidelberg Berlin, 1979.

[45] H. E. Lacey, *Separable quotients of Banach spaces*, An. Acad. Brasil. Ciènc. **44** (1972), 185–189.

[46] D. Leung and F. Räbiger, *Complemented copies of c_0 in ℓ_∞-sums of Banach spaces*, Illinois J. Math. **34** (1990), 52–58.

[47] J. Mendoza, *Copies of ℓ_∞ in $L_p(\mu, X)$*, Proc. Amer. Math. Soc. **109** (1990), 125–127.

[48] J. Mendoza, *Complemented copies of ℓ_1 in $L_p(\mu, X)$*, Math. Proc. Cambridge Philos. Soc. **111** (1992), 531–534.

[49] A. Moltó, *On a theorem of Sobczyk*, Bull. Austral. Math. Soc. **43** (1991), 123–130.

[50] J. Mujica, *Separable quotients of Banach spaces*, Rev. Mat. Complut. **10** (1997), 299–330.

[51] A. Nissenzweig, *w^* sequential convergence*, Israel J. Math., **22** (1975), 266–272.

[52] M. M. Popov, *Codimension of subspaces of $L_p(\mu)$ for $p < 1$*, Funct. Anal. Appl. **18** (1984), 168–170

[53] H. P. Rosenthal, *On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L_p(\mu)$ to $L_r(\nu)$*, J. Funct. Anal. **4** (1969), 176–214.

[54] H. P. Rosenthal, *On relative disjoint families of measures, with some application to Banach space theory*, Studia Math. **37** (1979), 13–36.

[55] H. P. Rosenthal, *A characterization of Banach spaces containing c_0*, J. Amer. Math. Soc. **7** (1994), 707–748.

[56] R. A. Ryan, *Complemented copies of c_0 in spaces of compact operators*, Math. Proc. R. Ir. Acad. **91**A (1991), 239–241.

[57] R. A. Ryan, *Introduction to Tensor Products of Banach spaces*, Springer, SMM, London, 2002.

[58] E. Saab and S. Saab, *A stability property of a class of Banach spaces not containing a complemented copy of ℓ_1*, Proc. Amer. Math. Soc. **84** (1982), 44–46.

[59] E. Saab and S. Saab, *On complemented copies of c_0 in injective tensor products*, Contemp. Math. **52** (1986), 131–135.

[60] S. A. Saxon, *(LF)-spaces with more-than-separable quotients*, J. Math. Anal. Appl. **434** (2016), 12–19.

[61] S. A. Saxon and L. M. Sánchez Ruiz, *Reinventing weak barrelledness*, J. Convex Anal. **24** (2017), 707–762.

[62] S. A. Saxon and L. M. Sánchez Ruiz, *Barrelled countable enlargements and the bounding cardinal*, J. Lond. Math. Soc. **53** (1996), 158–166.

[63] S. A. Saxon and A. Wilansky, *The equivalence of some Banach space problems*, Colloq. Math. **37** (1977), 217–226.

[64] W. Śliwa, *The separable quotient problem and the strongly normal sequences*, J. Math. Soc. Japan **64** (2012), 387–397.

[65] W. Śliwa and M. Wójtowicz, *Separable Quotients of Locally Convex Spaces*, Bull. Pol. Acad. Sci. Math., **43** (1995), 175–185.

[66] M. Talagrand, *Sur les espaces de Banach contenant* $\ell_1(\tau)$. Israel J. Math. **40** (1981), 324–330.

[67] E. Thomas, *The Lebesgue-Nikodým theorem for vector-valued Radon measures*, Memoirs of the AMS **139** (1974).

[68] S. Todorcevic, *Biorthogonal systems and quotient spaces via Baire category methods*, Math. Ann. **335** (2006), 687–715.

[69] M. Valdivia, *Sobre el teorema de la gráfica cerrada*, Collect. Math. **22** (1971), 51–72.

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