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A SHORT ACCOUNT OF THE VALUES OF THE ZETA FUNCTION AT INTEGERS

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Abstract: We use methods of real analysis to continue the Riemann zeta function $\zeta(s)$ to all complex s, and to express the values at integers in terms of Bernoulli numbers, using only those infinite series for which we could write down an explicit estimate for the remainder after N terms. This paper is self-contained, apart from appeals to the uniqueness theorems for analytic continuation and for real power series, and, *verbis in Latinam translatis*, would be accessible to Euler.

Keywords: Riemann zeta function, analytic continuation, Bernoulli numbers, Bernoulli polynomials, Euler polynomials, generating functions, accelerated convergence, box spline.

1. Introduction

The zeta function was defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1}$$

for $s = 2, 3, 4, \ldots$. The series converges absolutely for real $s > 1$, and diverges for real $s \leqslant 1$. For real $s > 1$ let

$$\eta(s) = \left(1 - \frac{2}{2^s}\right)\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots. \tag{1.2}$$

The series on the right of (1.2) converges for real $s > 0$, and it gives an interpretation of $\zeta(s)$ for real s in $0 < s < 1$. Euler announced methods which interpreted $\zeta(s)$ for all real $s \leqslant 0$, with

$$\zeta(1-k) = -\frac{B_k}{k} \qquad \text{for} \quad k = 1, 2, \ldots, \tag{1.3}$$

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where B_k is the Bernoulli number. The values of $\zeta(k)$ for k an even positive integer involve powers of π and Bernoulli numbers. The values at odd integers $k = 3, 5, 7, \ldots$ and the derivatives at positive integers have no simple formula known, and presumably no simple formula exists.

The zeta function heads the class of Dirichlet series generating functions $F(s) = \sum a(n)/n^s$. For $\zeta(s)$ the sequence of coefficients $a(n)$ is constant, $a(n) = 1$. Several interesting generating functions are related to $\zeta(s)$: when $a(n) = n$, we get $\zeta(s-1)$, when $a(n) = n^2$, we get $\zeta(s-2)$, when $a(n) = \log n$, we get $-\zeta'(s)$, and when $a(n) = (-1)^{n-1}$, we get the series $\eta(s)$ in (1.2). In fact the series for $\zeta(s)$ has a multiplicative structure. When $a(n) = \mu(n)$, the Möbius function, we get $1/\zeta(s)$. When $a(n) = \phi(n)$, where $\phi(q)$ is 'Euler's counting function', the number of fractions a/q in their lowest terms with $0 \leqslant a \leqslant q-1$, we get $\zeta(s-1)/\zeta(s)$. The logarithm of the lowest common multiple of the first N integers can be written as the 'prime sum' $\psi(N) = \Lambda(1) + \cdots + \Lambda(N)$, where $\Lambda(n)$ is 0 except when n is a prime number p or a prime power p^r; in both cases $\Lambda(n) = \log p$. We get $\sum \Lambda(n)/n^s = -\zeta'(s)/\zeta(s)$, the relation that inspired Riemann's programme to prove the Prime Number Theorem.

Riemann's theory of functions of a complex variable showed us $\zeta(s)$ in full. The series (1.1) for $\zeta(s)$ gives a regular function for $\Re s > 1$, and the series (1.2) for $\eta(s)$ gives a regular function for $\Re s > 0$. Riemann found an integral that defines $\zeta(s)$ as a meromorphic function for all complex s, with a single pole at $s = 1$, of residue 1. The kernel function in the integral is a generating function for Bernoulli polynomials, which explains the values in (1.3). A symmetry in the kernel function leads to the functional equation for $\zeta(s)$, of the form

$$\zeta(1-s) = k(s)\zeta(s). \tag{1.4}$$

An interesting proof of the functional equation for $\zeta(s)$ ([7] Chapter 1, [12] Chapter 2 notes) uses the generating function $E(z, s)$ of the Epstein zeta functions of quadratic forms in two variables, called the non-holomorphic Eisenstein series on $SL(2, \mathbb{Z})$. The classifying parameter z is a point in the hyperbolic plane. For fixed s Laplace's equation $\nabla^2 E(z, s) = -\lambda E(z, s)$ holds pointwise in z with $\lambda = s(1-s)$. The same equation holds for $E(z, 1-s)$, and its growth properties lead to a relation of the form

$$E(z, 1-s) = K(s)E(z, s), \tag{1.5}$$

from which (1.4) follows by averaging over the point z.

In this note we give a direct proof of the zeta values in (1.3), which builds on ideas from Euler, and is strictly elementary ('low analytic'), using only real analysis. High analytic ('non-elementary') arguments are those requiring Fourier theory, complex function theory, or group representations as in (1.5).However we need the high analytic theorem that the function $\eta(s)$ has at most one entire continuation to the whole complex plane in order to give meaning to the values $\zeta(1-k)$ outside the convergence region.

For completeness we also derive the values at positive even integers

$$\zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!}. \tag{1.6}$$

Our proof of (1.6) is high analytic, using Fourier series for the Bernoulli polynomials. We sketch a (longer) real-analytic proof of (1.6) by expanding the hyperbolic function $\coth t = \operatorname{ch} t/\operatorname{sh} t$ in two different ways as a power series in t. The special value $\zeta(2)$ can be identified purely by number theory; we count the integer points in a four-dimensional sphere in two ways. The simplest real-analytic 'main term plus remainder term' argument is enough.

The proof of (1.3) falls into three parts.

Step One. Accelerated convergence for infinite series. We show that a method of Euler, which uses repeated differencing, is regular, that is, it gives the right sum for a convergent series.

Step Two. Simplifications for alternating series and for $\eta(s)$ in particular. For each positive integer R, there is a series $E(s)$ that converges uniformly in $|s| \leqslant R$, and equals $\eta(s)$ for $\Re s > 0$. The values of $E(s)$ at $s = 0, -1, -2, \ldots, -R$ are rational. Hence $\zeta(s)$ has a meromorphic continuation to the whole plane.

Step Three. Generating function identities for Bernoulli and Euler polynomials.

This paper arose from a study by the Cardiff Number Theory Group of polynomials pertinent to number theory. Our aim was to reconstruct Euler's argument for (1.3) and make it rigorous. We have not found the whole argument anywhere in the literature. Hasse [6] works with $\zeta(s)$, not with $\eta(s)$; the convergence is slower. He omits Step 3, referring to Worpitzky [13]. Sondow [11] gives Step 2 of the argument here. He refers Step 1 to Knopp [8] and Step 3 to Worpitzky. Goss [4] continues Euler's integral for the factorial function to all complex s by induction. He multiplies by $\eta(s)$, and then continues the product to all s. The values in (1.3) require an inverse Mellin transform. Ram Murty and Reece [10] integrate a periodic function which is essentially the first Bernoulli polynomial, then use induction, and then use properties of the Bernoulli polynomials to identify two sequences of rational numbers that satisfy the same recurrence.

The methods of Hasse, Goss, and Ram Murty and Reece apply to a wide class of summands which are monotone on the real axis, and they can be adapted to Dirichlet L-functions. Our method applies to series which have alternating sign on the real axis, whose absolute values are given by the values at integers of some monotone function. The only Dirichlet series which we can construct belong to moduli $q = 4m$ with $\chi(2m+1) = -1$. None of these methods seems to help to estimate the Riemann zeta function in its critical strip $0 < \Re s < 1$.

What would we tell Euler? We only write down infinite series for which we could give an explicit bound for the remainder after N terms.

2. An Euler summation method

Let $A = \{a_1, a_2, \ldots\}$ be a sequence of numbers, with partial sums $A_n = a_1 + a_2 + \cdots + a_n$. For x real and positive, put $t = 1/(1+x)$, $u = x/(1+x)$, so that $t + u = 1$. We construct a new sequence $B(x) = \{b_0(x), b_1(x), \ldots\}$ by

$$b_n(x) = \frac{1}{(1+x)^{n+1}} \sum_{m=0}^{n} {}_nC_m a_{m+1} x^{m+1}, \tag{2.1}$$

with partial sums

$$\begin{aligned} B_N(x) &= \sum_{n=0}^{N} b_n(x) = \sum_{n=0}^{N} \frac{1}{(1+x)^{n+1}} \sum_{m=0}^{n} {}_nC_m a_{m+1} x^{m+1} \\ &= \sum_{m=0}^{N} a_{m+1} x^{m+1} \sum_{n=m}^{N} {}_nC_m \frac{1}{(1+x)^{n+1}} \\ &= \sum_{m=0}^{N} a_{m+1} \left(\frac{x}{1+x}\right)^{m+1} g\left(m, N-m, \frac{1}{1+x}\right) \\ &= \sum_{m=0}^{N} a_{m+1} u^{m+1} g\left(m, N-m, t\right), \end{aligned} \tag{2.2}$$

where we write

$$g(m, n, t) = 1 + {}_{m+1}C_1 t + {}_{m+2}C_2 t^2 + \cdots + {}_{m+n}C_n t^n \tag{2.3}$$

for the sum of the first $n+1$ terms of the binomial series for $(1-t)^{-m-1}$. The coefficient of a_{m+1} in (2.2) lies between 0 and 1, and it is approximately 1 when N is large and m is small.

By Pascal's recurrence ${}_mC_{r-1} + {}_mC_r = {}_{m+1}C_r$, we have

$$\begin{aligned} &(1-t)g(m, n, t) \\ &= 1 + ({}_{m+1}C_1 - {}_mC_0)\, t + \cdots + ({}_{m+n}C_m - {}_{m+n-1}C_{n-1})\, t^n - {}_{m+n}C_n t^{n+1} \\ &= 1 + {}_mC_1 t + {}_{m+1}C_2 t^2 + \cdots + {}_{m+n-1}C_n t^n - {}_{m+n}C_n t^{n+1} \\ &= g(m-1, n, t) - {}_{m+n}C_n t^{n+1} = g(m-1, n, t) - {}_{m+n}C_m t^{n+1} \qquad (2.4) \\ &= g(m-1, n+1, t) - {}_{m+n+1}C_{n+1} t^{n+1} = g(m-1, n+1, t) - {}_{m+n+1}C_m t^{n+1}. \qquad (2.5) \end{aligned}$$

Since $0 < t < 1$, we see from (2.5) that the weight $u^{m+1} g(m, N-m, t)$ in (2.2) is positive and decreasing in m.

We iterate (2.4) as

$$\begin{aligned} u^m g(m, n, t) &= (1-t)^m g(m, n, t) \\ &= g(0, n, t) - t^{n+1} \left({}_{m+n}C_m u^{m-1} + {}_{m+n-1}C_{m-1} u^{m-2} + \cdots + {}_{n+1}C_1\right) \\ &= \frac{1 - t^{n+1}}{1-t} - t^{n+1} \left(\frac{g(n, m, u) - 1}{u}\right), \end{aligned}$$

so we get the delightful identity

$$u^{m+1}g(m,n,t) = 1 - t^{n+1}g(n,m,u). \tag{2.6}$$

We iterate (2.5) as

$$\begin{aligned} u^m g(m,n,t) &= (1-t)^m g(m,n,t) \\ &= g(0,m+n,t) - {}_{m+n+1}C_m t^{n+1}u^{m-1} \\ &\quad - {}_{m+n+1}C_{m-1}t^{n+2}u^{m-2} - \cdots - {}_{m+n+1}C_1 t^{m+n} \\ &= \frac{1-t^{m+n+1}}{1-t} - \frac{1}{u}\Big({}_{m+n+1}C_m t^{n+1}u^m \\ &\quad - {}_{m+n-1}C_{m-1}t^{n+2}u^{m-1} - \cdots - {}_{m+n+1}C_1 t^{m+n}u\Big), \end{aligned}$$

so that

$$u^{m+1}g(m,n,t) = 1 - \left(t^{m+n+1} + {}_{m+n+1}C_1 t^{m+n}u + \cdots + {}_{m+n+1}C_m t^{n+1}u^m\right). \tag{2.7}$$

From (2.6) and (2.7)

$$t^{n+1}g(n,m,u) = t^{m+n+1} + {}_{m+n+1}C_1 t^{m+n}u + \cdots + {}_{m+n+1}C_m t^{n+1}u^m, \tag{2.8}$$

a partial sum of the binomial expansion of $(t+u)^{m+n+1}$.

We can now discuss convergence. Suppose that the partial sum A_n tends to some limit α. We choose M large, and take $N > 2M$. Then by (2.2) and (2.6)

$$\begin{aligned} B_N(x) &= \sum_{m=0}^{M-1} a_{m+1}\left(1 - t^{N-m+1}g(N-m,m,u)\right) + \sum_{m=M}^{N} a_{m+1}u^{m+1}g(m,N-m,t) \\ &= A_M - E_1(M) + E_2(M), \end{aligned}$$

say. We have shown that $u^{m+1}g(m,N-m,t)$ lies between 0 and 1, and it decreases as m increases, so

$$\begin{aligned} |E_2(M)| &= \left|\sum_{m=M}^{N} a_{m+1}u^{m+1}g(m,N-m,t)\right| \\ &\leqslant \max_{M\leqslant m\leqslant N} u^{m+1}g(m,N-m,t) \max_{M\leqslant k\leqslant N} |a_k + a_{k+1} + \cdots + a_N| \\ &\leqslant \max_{M\leqslant k\leqslant N} |a_k + a_{k+1} + \cdots + a_N|, \end{aligned}$$

so $|E_2(M)| \to 0$ as $M \to \infty$ uniformly in N.

We use (2.8) with $m+n=N$, $m \leqslant M < n$ to obtain the bound

$$\begin{aligned} t^{n+1}g(N-m,m,u) &\leqslant t^{N-m+1}\max(t^m,u^m)(N+1)\max_{0\leqslant r\leqslant m} {}_{N+1}C_r \\ &\leqslant t^{N-m+1}(N+1)_{N+1}C_m \leqslant \frac{(N+1)^{m+1}}{m!}t^{N-m+1}. \end{aligned}$$

Hence

$$
\begin{aligned}
|E_1(M)| &= \left| \sum_{m=0}^{M-1} a_{m+1} t^{N-m+1} g(N-m, m, u) \right| \\
&\leqslant \left(\max_{1 \leqslant k \leqslant M} |a_k| \right) \sum_{m=0}^{M-1} \frac{(N+1)^{m+1}}{m!} t^{N-m+1} \\
&< \left(\max_{1 \leqslant k \leqslant M} |a_k| \right) \frac{M(N+1)^M}{(M-1)!} t^{N/2},
\end{aligned}
$$

which tends to zero as $N \to \infty$ with M fixed.

By picking M large enough, then picking $N > 2M$ large enough, we can make the partial sum $B_N(x)$ arbitrarily close to α. So Euler's summation method is regular, that is, it transforms convergent series $\sum a_n$ to convergent series $\sum b_n(x)$ with the same sum.

The Euler-Knopp summation method was deduced by Knopp [8] from two of Euler's many devices: the variable change between x and t appears in [2, Part 2, Chapter 1] and the balanced case $t = u = 1/2$ appears in [2, Part 2, Chapter 3], in [3], and no doubt also elsewhere. Ames [1] and Hardy [5, Section 1.3] consider only the balanced case.

3. Formally alternating series

Euler summation speeds the convergence of alternating series, where $a_n = (-1)^{n-1} P(n)$. In the balanced case $x = 1$, $t = u = 1/2$, (2.1) becomes

$$
b_n(1) = \frac{1}{2^{n+1}} \sum_{m=0}^{n} {}_nC_m (-1)^m P(m+1) = \frac{(-1)^n}{2^{n+1}} \Delta^n P(1), \tag{3.1}
$$

where Δ is the difference operator defined by $\Delta f(c) = f(c+1) - f(c)$. When $P(n)$ is the value at $x = n$ of an infinitely differentiable function $P(x)$, the n-th difference $\Delta^n P(x)$ becomes a box spline integral:

$$
\begin{aligned}
\Delta P(c) &= P(c+1) - P(c) = \int_0^1 P'(c+x)\,\mathrm{d}x, \\
\Delta^n P(c) &= \int_{x_1=0}^{1} \cdots \int_{x_n=0}^{1} P^{(n)}(c + x_1 + \cdots + x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n \\
&= \frac{1}{\sqrt{n}} \int_0^n k_n(t) P^{(n)}(c+t)\,\mathrm{d}t,
\end{aligned} \tag{3.2}
$$

where $k_n(t)$ is the $(n-1)$-dimensional volume of the intersection of the unit cube in n dimensions and the plane $\pi(t) : \quad x_1 + \cdots + x_n = t$. Let $K_n(t)$ be the n-dimensional volume of the part of the unit cube between the origin and the plane $\pi(t)$. Then

$$
K_n(x) = \frac{1}{\sqrt{n}} \int_0^x k_n(t)\,\mathrm{d}t \leqslant K_n(n) = 1, \tag{3.3}
$$

and

$$K_n(x) \leqslant \frac{x^n}{n!}, \qquad \text{with equality for } \; 0 \leqslant x \leqslant 1. \tag{3.4}$$

We consider the special case

$$P(x) = \frac{1}{x^s}, \qquad P^{(n)}(x) = (-1)^n \frac{s(s+1)\dots(s+n-1)}{x^{n+s}}$$

Suppose that $|s| \leqslant r$, where $r \geqslant 2$ is a positive integer. Then

$$|s(s+1)\dots(s+n-1)| \leqslant r(r+1)\dots(r+n-1) = \frac{(r+n-1)!}{(r-1)!}.$$

Let $\sigma = \Re s$. When $n + \sigma \geqslant 0$, then by (3.3) and (3.4), for $1 \leqslant m \leqslant n$

$$\begin{aligned}\left|\int_{m-1}^{m} \frac{k_n(t)}{(t+1)^{n+s}}\,\mathrm{d}t\right| &\leqslant \int_{m-1}^{m} \frac{k_n(t)}{(t+1)^{n+\sigma}}\,\mathrm{d}t \leqslant \frac{1}{m^{n+\sigma}} \int_{m-1}^{m} k_n(t)\,\mathrm{d}t \\ &\leqslant \frac{\sqrt{n}K_n(m)}{m^{n+\sigma}} \leqslant \frac{\sqrt{n}}{m^{\sigma} n!},\end{aligned}$$

so that

$$\left|\frac{1}{\sqrt{n}} \int_{1}^{n} \frac{k_n(t)}{(t+1)^{n+s}}\,\mathrm{d}t\right| \leqslant \sum_{m=1}^{n} \frac{1}{m^{\sigma} n!} \leqslant \frac{n \max(1, n^{-\sigma})}{n!}. \tag{3.5}$$

From (3.1), (3.2) and (3.5) we have

$$\begin{aligned}|b_n| &\leqslant \frac{1}{2^{n+1}} \cdot \frac{(r+n-1)!}{(r-1)!} \cdot \frac{\max(1, n^{-\sigma})}{(n-1)!} \\ &= \frac{n(n+1)\dots(n+r-1)\max(1, n^{-\sigma})}{2^{n+1}(r-1)!} \leqslant \frac{n^{r+1}(n+r-1)^{r-1}}{2^{n+1}(r-1)!} = B_n(r),\end{aligned}$$

say, and $\sum B_n(r)$ converges to a sum depending on the radius r.

Euler's summation process converges uniformly in the disc $|s| \leqslant r$ to a limit $\eta(s)$. Each term b_n is a regular function of the complex variable s, so $\eta(s)$ is regular for all s. For $\sigma = \Re s > 1$, $\eta(s)$ is the Dirichlet series in (1.2), with $\eta(s) = (1 - 2^{1-s})\zeta(s)$. So Euler's function $\eta(s)$ provides a meromorphic continuation of $\zeta(s)$, with possible poles where $1 - 2/2^s$ vanishes at $(1-s)\log 2 = 2m\pi i$ for non-zero integers m. However $\theta(s) = (1 - 3/3^s)\zeta(s)$ converges uniformly to a regular function of s for $\Re s \geqslant 1/2$, with a disjoint set of possible poles at $(1-s)\log 3 = 2n\pi i$. Thus the only pole of $\zeta(s)$ is at $s = 1$.

Let $k \geqslant 1$ be an integer. Then $\eta(1-k)$ is given by an absolutely convergent series whose n-th term is (3.1) with $P(x) = x^{k-1}$. The n-th differences $\Delta^n P(x)$ are identically zero for $n \geqslant k$, so $\eta(1-k)$ is a finite sum of k terms, each a rational number. Since (1.2) remains valid for all s, $\zeta(1-k)$ is also a rational number.

The Dirichlet L-functions $L(s,\chi)$ modulo $q = 4m$ with $\chi(2m+1) = -1$ have $\chi(n+2m) = -\chi(n)$. They are continued similarly, taking $P(x) = (x+\alpha)^{-s}$, where α runs through values $(2a+1)/2m$. The character $\chi(n)$ takes values in some cyclotomic field K, and the values of $L(s,\chi)$ for non-positive integers lie in K.

4. Bernoulli and Euler polynomials

We start from the definitions of Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ that use the difference operator Δ.

$$\begin{aligned} B_n(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1} \Delta^r x^n = \sum_{r=0}^{n} \frac{(-1)^r}{r+1} \Delta^r x^n \\ &= \sum_{r=0}^{n} \frac{(-1)^r}{r+1} \sum_{s=0}^{r} (-1)^{r-s} {}_rC_s (x+s)^n, \end{aligned} \tag{4.1}$$

$$\begin{aligned} E_n(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r} \Delta^r x^n = \sum_{r=0}^{n} \frac{(-1)^r}{2^r} \Delta^r x^n \\ &= \sum_{r=0}^{n} \frac{(-1)^r}{2^r} \sum_{s=0}^{r} (-1)^{r-s} {}_rC_s (x+s)^n. \end{aligned} \tag{4.2}$$

The Bernoulli polynomials are related to the Faulhaber polynomials that give the sum to x terms of the divergent series $1 + 2^n + 3^n + \ldots$, and to the power series for $\cot x$. By differentiation we see at once that

$$B_n'(x) = nB_{n-1}(x), \qquad E_n'(x) = nE_{n-1}(x). \tag{4.3}$$

The values $B_i = B_i(0)$ are called the Bernoulli numbers. By induction on (4.3) we see that

$$B_n(x) = \sum_{r=0}^{n} {}_nC_r B_{n-r} x^r.$$

Further properties of Bernoulli polynomials can be seen from the generating function $B(x,t)$, using the low analytic theorem that a power series in t that converges to 0 on some open interval $0 < t < \delta$ must have all coefficients zero. For $0 \leqslant x \leqslant 1$ and $0 < t < \log 2$ we can rearrange the absolutely convergent double series for $B(x,t)$:

$$
\begin{aligned}
B(x,t) &= \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1}\Delta^r x^n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1} \sum_{n=r}^{\infty} \Delta^r \frac{(xt)^n}{n!} \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1}\Delta^r \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1}\Delta^r e^{xt} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1}(e^t-1)^r e^{xt} \\
&= e^{xt}\cdot \frac{1}{e^t-1}\cdot \log(1+e^t-1) = \frac{te^{xt}}{e^t-1},
\end{aligned} \tag{4.4}
$$

since $\Delta^r x^n = 0$ for $n < r$. Differencing (4.4) gives $B(x+1,t) - B(x,t) = te^{xt}$, so

$$B_n(x+1) - B_n(x) = nx^{n-1}, \tag{4.5}$$

and the Faulhaber polynomial is $(B_{n+1}(x) - B_{n+1}(0))/(n+1)$. Replacing x by $1-x$ in (4.4) gives

$$\sum_{n=0}^{\infty} B_n(1-x)\frac{t^n}{n!} = B(1-x,t) = \frac{te^{t-xt}}{e^t-1} = -\frac{te^{-xt}}{e^{-t}-1} = \sum_{m=0}^{\infty} B_n(x)\frac{(-t)^n}{n!},$$

so

$$B_n(1-x) = (-1)^n B_n(x). \tag{4.6}$$

We see that by (4.5) that $B_n(1) - B_n(0) = 0$ for $n \neq 1$, 1 for $n = 1$, but by (4.6) $B_n(1) = (-1)^n B_n(0)$. So the Bernoulli number B_n is 0 for n odd, $n \geqslant 3$, but $B_1 = -1/2$.

Since $(e^t-1)/(e^{t/q}-1) = 1 + e^{t/q} + \cdots + e^{(q-1)t/q}$, we get

$$B_n(qx) = q^{n-1}\sum_{a=0}^{q-1} B_n\left(x + \frac{a}{q}\right), \tag{4.7}$$

the so-called multiplication formula.

Similarly the Euler polynomials have a generating function $E(x,t)$ for $0 \leqslant x \leqslant 1$ and $0 < t < \log 2$, which can be rearranged:

$$
\begin{aligned}
E(x,t) &= \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r}\Delta^r e^{xt} = \sum_{r=0}^{\infty} \left(\frac{-1}{2}\right)^r (e^t-1)^r e^{xt} \\
&= e^{xt}\cdot \frac{1}{1+(e^t-1)/2} = \frac{2e^{xt}}{e^t+1}.
\end{aligned} \tag{4.8}
$$

We note that

$$tE(x,t) = \frac{2te^{xt}(e^t-1)}{e^{2t}-1} = B\left(\frac{x+1}{2}, 2t\right) - B\left(\frac{x}{2}, 2t\right),$$

so comparing coefficients gives

$$\frac{E_{n-1}(x)}{(n-1)!} = \frac{2^n}{n!}\left(B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right). \tag{4.9}$$

So far we have shown that Euler summation gives the analytic continuation of the Dirichlet series $\eta(s)$, and that $\eta(1-k)$ is equal to a finite sum. We can now identify that finite sum:

$$\eta(1-k) = \sum_{n=0}^{k-1} \frac{(-1)^n}{2^{n+1}} \Delta^n x^{k-1}\Big|_{x=1} = \frac{1}{2} E_{k-1}(1) = \frac{2^{k-1}}{k} \left(B_k(1) - B_k\left(\frac{1}{2}\right)\right). \tag{4.10}$$

We can read off at once

$$\eta(0) = \frac{1}{2}, \qquad \zeta(0) = -\frac{1}{2}.$$

For $k \geqslant 2$ we use (4.7) with $q = 2$ (the so-called duplication formula) and $x = 0$ to get

$$B_k(0) = 2^{k-1}\left(B_k(0) + B_k\left(\frac{1}{2}\right)\right). \tag{4.11}$$

Since $B_k(1) = B_k(0) = B_k$ for $k \geqslant 2$, we substitute (4.11) in (4.10) to get

$$\eta(1-k) = \frac{1}{k}\left(2^{k-1}B_k - \left(B_k - 2^{k-1}B_k\right)\right) = \frac{(2^k-1)}{k} B_k,$$

which gives the values in (1.3).

5. Values at even positive integers

We work with the periodic functions $\overline{B}_n(x) = B_n(x-[x])$, $\overline{B}(x,t) = B(x-[x],t)$; we use $[x]$ for the greatest integer n with $n \leqslant x$. The function $\overline{B}(x,t)$ is defined for all real x and for $0 < t < \log 2$. These functions are periodic, but $\overline{B}_1(x)$ and $\overline{B}(x,t)$ are discontinuous at integers.

We compute the Fourier coefficients of $\overline{B}(x,t)$. We have for $h \neq 0$

$$\begin{aligned}\int_0^1 \overline{B}(x,t)e^{-2\pi i h x}\,\mathrm{d}x &= \int_0^1 \frac{te^{xt}}{e^t-1} e^{-2\pi i h x}\,\mathrm{d}x = \int_0^1 \frac{t}{e^t-1} e^{(t-2\pi i h)x}\,\mathrm{d}x \\ &= \frac{t}{e^t-1}\left[\frac{e^{(t-2\pi i h)x}}{t-2\pi i h}\right]_0^1 = \frac{t}{e^t-1}\cdot\frac{e^t-1}{t-2\pi i h} \\ &= \frac{1}{-2\pi i h}\cdot\frac{t}{1-t/2\pi i h} = -\sum_{n=1}^{\infty}\left(\frac{t}{2\pi i h}\right)^n.\end{aligned}$$

For $h = 0$ we have the simpler calculation

$$\int_0^1 \overline{B}(x,t)\,\mathrm{d}x = \int_0^1 \frac{te^{xt}}{e^t-1}\,\mathrm{d}x = \frac{1}{e^t-1}\left[e^{xt}\right]_0^1 = 1.$$

Hence for $n \geqslant 1$ the periodic function $\overline{B}_n(x)$ has Fourier series

$$\overline{B}_n(x) \sim -n! \sum_{h\neq 0} \frac{e^{2\pi i h n x}}{(2\pi i h)^n}.$$

For $n = 2k \geqslant 2$, $B_n(1) = B_n(0)$, so the Fourier series for $\overline{B}_n(x)$ converges at $x = 0$ to their common value B_n. We have

$$B_{2k} = \overline{B}_{2k}(0) = -\sum_{h=1}^{\infty} \frac{2(2k)!}{(2\pi i h)^{2k}} = (-1)^{k-1}\frac{2\zeta(2k)(2k)!}{(2\pi)^{2k}},$$

which gives the values in (1.6).

We sketch a real-analytic evaluation of the values $\zeta(2k)$ in (1.6) by way of the partial fractions for $\cot x$. Let $F(x)$ be defined on the interval $0 < x < \pi$ by the absolutely convergent series

$$F(x) = f_0(x) + \sum_{n=1}^{\infty} f_n(x) = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2x}{n^2\pi^2 - x^2}.$$

A careful calculation using partial fractions and regrouping terms without destroying the absolute convergence leads to the differential equation $F'(x)+F^2(x)+c^2 = 0$, where $c^2 = 6\zeta(2)/\pi^2$. The general solution [9] is

$$F(x) = c\tan(\alpha - cx).$$

Since $F(x)$ is asymptotic to $1/x$ as $x \to 0$, we must have $\alpha = \pi/2$. Since $F(x+\pi) = F(x)$, c must be an integer. The inequalities $9 < \pi^2 < 10$ and $1 < \zeta(2) < 2$ show that we can take $c = 1$, so $F(x) = \cot x$. The Hadamard product for $\sin x$ in logarithmic form now follows by integration.

We have shown along the way that $\zeta(2) = \pi^2/6$. To identify further values of $\zeta(2k)$, we expand $f_n(x)$ as a power series and interchange orders of summation in the resulting absolutely convergent double series to get

$$\cot x = \frac{1}{x} + \frac{2}{x}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{x}{n\pi}\right)^{2k} = \frac{1}{x} + \frac{2}{x}\sum_{k=1}^{\infty}\zeta(2k)\left(\frac{x}{\pi}\right)^{2k}.$$

The corresponding series for the hyperbolic function $\coth t = \operatorname{ch} t/\operatorname{sh} t$ has an extra factor $(-1)^k$. We can express $\coth t$ in terms of the generating function $B(x,t)$ for Bernoulli polynomials:

$$\coth t - 1 = \frac{\operatorname{ch} t - \operatorname{sh} t}{\operatorname{sh} t} = \frac{2e^{-t}}{e^t - e^{-t}} = \frac{2}{e^{2t} - 1} = \frac{1}{t}B(0, 2t).$$

The power series coefficients can now be read off in terms of Bernoulli numbers.

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