Functiones et Approximatio
60.1 (2019), 87–96
doi: 10.7169/facm/1700

A NOTE ON THE DIOPHANTINE EQUATION $2^{n-1}(2^n - 1) = x^3 + y^3 + z^3$

Maciej Ulas

Abstract: Motivated by a recent result of Farhi we show that for each $n \equiv \pm 1 \pmod 6$ the title Diophantine equation has at least two solutions in integers. As a consequence, we get that each (even) perfect number is a sum of three cubes of integers. Moreover, we present some computational results concerning the considered equation and state some questions and conjectures.

Keywords: perfect numbers, sums of three cubes.

1. Introduction

Let $\mathbb{N}$ and $\mathbb{N}_+$ denote the set of non-negative integers and positive integers respectively. Let $n \in \mathbb{N}_+$ and put $P_n = 2^{n-1}(2^n - 1)$. We say that N is a perfect number if it is the sum of its divisors. In other words, N is a perfect number if and only if $\sigma(N) = 2N$, where $\sigma(N) = \sum_{d|N} d$. We do not know whether there is an odd perfect number. On the other hand, as was proved by Euclid, if N is an even perfect number then $N = P_p$, where p and $2^p - 1$ are primes. An early state of research on perfect numbers is presented in the first chapter in Dickson's classical book [3]. We know that there are at least 49 even perfect numbers. The largest known corresponds to $p = 74207281$. One among many interesting properties of perfect numbers, is the property observed by Heath, that each even perfect number > 6 is a sum of consecutive odd cubes of positive integers. This observation motivated Farhi to ask what is the smallest number r such that each even perfect number > 6 is the sum of at most r cubes of non-negative integers. In [6], Farhi proved that $r = 5$ does the job. In fact, he observed that if $n \equiv 1 \pmod 6$, then P_n is the sum of three cubes of positive integers. This is simple consequence of the

The research of the author is partially supported by the grant of the Polish National Science Centre no. DEC-2017/01/X/ST1/00407.

2010 Mathematics Subject Classification: primary: 11A25; secondary: 11B13 classical polynomial identity

$$2t^6 - 1 = (t^2 + t - 1)^3 + (t^2 - t - 1)^3 + 1.$$

Indeed, multiplying it by t^6 and then taking $t = 2^n$ we immediately get the representation of P_{6n+1} as sum of three positive cubes. In case of $n \equiv 5 \pmod 6$ the number P_n is a sum of five positive cubes. It is important to note that P_n is not necessarily perfect in the proof presented by Farhi. Let us also note that perfect numbers corresponding to $p = 3, 5, 7, 13, 17$ can be represented as a sum of three cubes of positive integers. This observation motivated Farhi to state the conjecture saying that each perfect number is such a sum (Conjecture 2 in [6]). Unfortunately, we were unable to prove this statement. This is a good motivation to consider the Diophantine equation

$$P_n = x^3 + y^3 + z^3 \tag{1}$$

for fixed n, and asks about its solutions in (not necessarily positive) integers.

The question about the existence of integer solutions of the equation $N = x^3 + y^3 + z^3$ is a classical one. The equation has no solutions for $N \equiv \pm 4 \pmod 9$ and it is conjectured that there are infinitely many solutions otherwise. However, this conjecture is proved only for N being a cube or twice a cube (see for example [9]). It is clear that the number P_n is not a cube nor twice a cube and $P_n \not\equiv \pm 4 \pmod 9$ for all $n \in \mathbb{N}_+$. Thus, the question concerning the existence of integer solutions of the equation (1) is non-trivial. Moreover, let us note that a lot of effort was devoted to find integer solutions of the equation $N = x^3+y^3+z^3$ for relatively small positive values of N (say $N < 10^4$). The reason is a consequence of the method employed in the numerical searches, which essentially use the observation that N/x^3 is very small (and thus close to 0). This idea was introduced by Elkies in [4] and used in [5] (and the recent paper [7]). It is related to finding rational points near algebraic curves. If N is small, the curve of interests is given by the equation $X^3 + Y^3 = 1$. Some other methods were proposed by Bremner [2] and Beck et all [1]. In all these methods we are interested in finding *big* representations of N. However, it is not clear whether they can be used in the case of representation of P_n as sum of three cubes. Indeed, the sequence $(P_n)_{n\in\mathbb{N}_+}$ has exponential growth, and it is likely that for given n, the equation (1) may have solutions (x, y, z) satisfying $\max\{|x|, |y|, |z|\} = O(P_n^{1/3})$. Let us describe the content of the paper in some details.

In Section 2 we prove that for $n \equiv 1, 2, 4, 5 \pmod 6$ the Diophantine equation (1) has at least one solution in integers. Moreover, in the case of $n \equiv \pm 1 \pmod 6$ we show the existence of at least two solutions. We also prove that for each $n \in \mathbb{N}_+$ the number P_n can be represented as a sum of four cubes of integers. In Section 3 we propose a method which, for given n, allows us to compute all positive integer solutions of equation (1) (and some other). In particular, for each $n \leqslant 50$ a solution of (1) is found and the table of all non-negative solutions for $n \leqslant 40$ is presented. Moreover, we state some questions and conjectures which may stimulate further research.

2. The results

We have the following

Theorem 2.1. *If* $n \equiv 1 \pmod 3$ *or* $n \equiv 2 \pmod 6$ *then the Diophantine equation* (1) *has at least one solution in integers. Moreover, if* $n \equiv \pm 1 \pmod 6$ *then the Diophantine equation* (1) *has at least two solutions in integers.*

Proof. Our result is an immediate consequence of the following identities which hold for all $n \in \mathbb{N}_+$:

$$\begin{aligned}
P_{3n+1} &= (2^{2n})^3 + (2^{2n})^3 - (2^n)^3,\\
P_{6n+2} &= (2^{4n+1})^3 - (2^{2n})^3 - (2^{2n})^3,\\
P_{6n+1} &= (2^{n-2}(2^{3n+2}-21))^3 + (2^{n-2}(2^{3n+2}+21))^3 - (11\cdot 2^{2n-1})^3,\\
P_{6n+5} &= (2^n(2^{3(n+1)}+2^{2(n+1)}+1))^3\\
&\quad + (2^n(2^{3(n+1)}-2^{2(n+1)}-1))^3 - (2^{2(n+1)}(2^{2n+1}+1))^3\\
&= (2^{2n+1}(2^{2(n+1)}-2^{n+1}-1))^3\\
&\quad + (2^{2n+1}(2^{2(n+1)}+2^{n+1}-1))^3 - (2^{4n+3})^3.
\end{aligned}$$

Replacing n by $2n$ in the first equality we get the second solution of the equation $P_{6n+1} = x^3+y^3+z^3$. ■

Remark 2.2. Let us note that the expression for P_{6n+1} from the proof of Theorem 2.1, can be deduced from the polynomial identity

$$64t^3(2t^6-1) = (4t^3-21)^3 + (4t^3+21)^3 - (22t)^3$$

by multiplying both sides by $\frac{1}{64}t^3$, and then taking $t = 2^n$. Moreover, the first expression for P_{6n+5} follows from the identity

$$t^3(t^6-2) = (t^3+t^2+1)^3 + (t^3-t^2-1)^3 - (t(t^2+2))^3$$

by multiplying both sides by $\frac{1}{8}t^3$, and then taking $t = 2^{n+1}$.

Corollary 2.3. *For each even perfect number* N*, the number of representations of* N *as a sum of three cubes of integers is* $\geqslant 2$.

Proof. From Theorem 2.1, we know that for each odd prime $p > 3$, the number $N = P_p$ has at least two representations as a sum of three cubes of integers.

For $p = 2, 3$ we have

$$P_2 = 2^3 - 1^3 - 1^3 = 65^3 - 43^3 - 58^3, \qquad P_3 = 3^3 + 1^3 = 14^3 + 13^3 - 17^3,$$

and get the result. ■

We firmly believe that equation (1) has a solution in integers for each $n \in \mathbb{N}_+$ (see Conjecture 3.3). Unfortunately, we were unable to prove such statement. Instead, we offer the following

Theorem 2.4. *For each $n \in \mathbb{N}_+$, the number P_n can be represented as a sum of four cubes of integers.*

Proof. Let us note the classical identity

$$t^3 - 2(t-1)^3 + (t-2)^3 = 6(t-1),$$

and observe that $P_{2n} \equiv 0 \pmod{6}$. Thus, by taking

$$t = \frac{1}{3}(2^{2(2n-1)} - 2^{2(n-1)} + 3)$$

we get the representation of the number P_{2n} as a sum of four cubes.

In order to represents P_{2n+1}, we note the identity

$$(3t-12)^3 - (3t-13)^3 - t^3 + (t-9)^3 = 2(9t-130).$$

Using simple induction, we easily get the congruence $P_{2n+1} \equiv 10 \pmod{18}$ for $n \in \mathbb{N}_+$. Thus, by taking

$$t = \frac{1}{9}(2^{4n} - 2^{2n-1} + 130)$$

we get the representation of the number $P_{2n+1}, n \in \mathbb{N}$, as a sum of four cubes. Our theorem is proved. ■

3. Numerical results, questions and conjectures

In order to gain more precise insight into the problem we performed a search for solutions of the equation (1) in integers. Because we are mainly interested in solutions in non-negative integers we use the following procedure. First of all, let us recall that for $a, b \in \mathbb{Z}$ we have $a^3 + b^3 \equiv 0, 1, 2, 7, 8 \pmod{9}$. Moreover, we observed that the sequence $(P_n \pmod{9})_{n\in\mathbb{N}_+}$ is periodic of the (pure) period 6. More precisely:

$$(P_n \pmod{9})_{n\in\mathbb{N}_+} = \overline{(1, 6, 1, 3, 1, 0)}.$$

For given n and each $x \in \{0, \ldots, \lfloor P_n^{1/3} \rfloor\}$ satisfying $(P_n - x^3) \pmod 9 \in \{0, 1, 2, 7, 8\}$, we computed the set

$$D_n(x) = \{d \in \mathbb{N}_+ : \ P_n - x^3 \equiv 0 \pmod d\},$$

i.e., the set of all positive divisors of the number $P_n - x^3$. The congruence condition is useful in some cases because it reduces the number of computations which need to be performed. Indeed, if $n \equiv 2, 4 \pmod 6$ then $P_n \equiv 6, 3 \pmod 9$ respectively, and we need to have $x \equiv 2 \pmod 3$ ($x \equiv 1 \pmod 3$). Unfortunately, in remaining cases we need to compute all values of x in order to find non-negative solutions. Next, for each $d \in D_n(x)$ such that $d \leqslant (P_n - x^3)/d$, we solved the system of equations

$$d = y + z, \qquad \frac{P_n - x^3}{d} = y^2 - yz + z^2$$

for y, z and get

$$y = \frac{1}{6}\left(3d \pm \sqrt{3\left(\frac{4(P_n - x^3)}{d} - d^2\right)}\right),$$
$$z = \frac{1}{6}\left(3d \mp \sqrt{3\left(\frac{4(P_n - x^3)}{d} - d^2\right)}\right).$$

In consequence, if the numbers y, z computed in this way were integers we got a solution of the equation (1). The number of possible cases which need to be considered is bounded by

$$\sum_{i=1}^{\lfloor P_n^{1/3} \rfloor} \sigma_0(P_n - i^3),$$

where $\sigma_0(n)$ is the number of positive divisors of n.

The described procedure was implemented in Magma computational package [8], and allows us to get all solutions in positive integers of equation (1) with $n \leqslant 40$. The results of our computations are presented in Table 1 below. We also added the value of $g := \gcd(x, y, z)$.

Table 1. All solutions of the Diophantine equation $P_n = x^3 + y^3 + z^3$ in non-negative integers x, y, z and $n \leqslant 40$.

n	(x,y,z)	g	n	(x,y,z)	g
3	$(0,1,3)$	1	31	$(1024,1014784,1080320)$	2^{10}
5	$(4,6,6)$	2		$(53824,684032,1256896)$	2^6
7	$(4,4,20)$	2^2		$(90112,464896,1301504)$	2^{10}
9	$(10,23,49)$	1		$(342016,581120,1274368)$	2^9
11	$(18,94,108)$	2		$(435712,977920,1088000)$	2^9
	$(28,73,119)$	1		$(452624,712312,1227976)$	2^3
13	$(16,176,304)$	2^4		$(642957,702144,1192051)$	1
15	$(87,273,802)$	1		$(649984,956288,1049728)$	2^7
	$(280,488,736)$	2^3	35	$(103936,1058816,8382976)$	2^9
17	$(720,1336,1800)$	2^3		$(825724,2369072,8322436)$	2^2
18	$(144,1224,3192)$	$3\cdot 2^3$		$(1159576,5742485,7364203)$	1
	$(168,1368,3168)$	$3\cdot 2^3$		$(1545844,5658327,7401321)$	1
	$(276,1808,3052)$	2^2		$(2128896,5711872,7332864)$	2^{10}
	$(968,976,3192)$	2^3		$(2565760,2610912,8220960)$	2^5
	$(1284,2076,2856)$	$3\cdot 2^2$		$(4021568,5381152,7175392)$	2^5
	$(1368,1904,2920)$	2^3	36	$(870912,8406528,12088320)$	$3\cdot 2^9$
19	$(64,3520,4544)$	2^6		$(3364928,7935616,12216768)$	2^6
	$(1216,1856,5056)$	2^6		$(3663896,6521760,12671464)$	2^3
	$(1968,3516,4420)$	2^2	37	$(4096,16510976,17035264)$	2^{12}
21	$(976,9088,11312)$	2^4		$(65536,7086080,20869120)$	2^{12}
22	$(13084,14728,14980)$	2^2		$(1409488,9313840,20514944)$	2^4
23	$(10096,19648,29840)$	2^4		$(1690048,2408352,21123936)$	2^5
	$(10398,17175,30721)$	1		$(1940480,12226048,19669504)$	2^9
	$(19776,20992,26304)$	2^6		$(7889536,14446400,18109120)$	2^6
25	$(16,27680,81520)$	2^4		$(2701980,13899489,18889183)$	1
	$(256,61184,69376)$	2^8		$(5169168,15293424,17894080)$	2^4
	$(6208,37888,79808)$	2^6		$(5875248,13984848,18669088)$	2^4
	$(21034,58773,70515)$	1		$(10327879,11144196,19091961)$	1
26	$(3542,93428,112826)$	2	38	$(72704,24487424,28477952)$	2^9
27	$(39808,89600,201856)$	2^7	39	$(3083584,32842240,48722624)$	2^6
	$(83110,154196,168298)$	2		$(14437236,38893888,44692620)$	2^2
28	$(88576,156160,315904)$	2^9		$(26259968,34426624,45177088)$	2^8
29	$(37120,54272,524032)$	2^8		$(29613312,30112512,46079488)$	2^8
	$(292540,340128,430404)$	2^2	40	$(23894752,58850848,72873280)$	2^5
30	$(98816,297216,818944)$	2^8			
	$(120576,440992,787808)$	2^5			

For given n, the time needed to compute solutions with our method was from seconds (for $n \leqslant 25$) to four days in case of $n = 40$. All computations were performed on a typical laptop with generation i7 processor and 16 GB of RAM. Moreover, it should be noted that our procedure also computes (some) solutions satisfying $yz < 0$, which is a consequence of the construction. In consequence, for each $n \in \{2, \ldots, 40\} \setminus \{2, 8, 20\}$, our procedure produces a solution of the equation (1) with $yz<0$, i.e., exactly one among the numbers y,z is negative. In Table 2 below, we present integer solution of the equation (1) without non-negative solutions and with smallest value of $\min\{|x|,|y|,|z|\}$.

Table 2. Certain integer solutions of the Diophantine equation $P_n=x^3+y^3+z^3$ for $n\leqslant 40$ and without non-negative solutions.

n	(x,y,z)	g	n	(x,y,z)	g
4	$(-2,4,4)$	2	24	$(-21716,19656,52340)$	2^2
10	$(-8,64,64)$	2^3	32	$(-5219392,1549376,5285888)$	2^6
12	$(-54,136,182)$	2	33	$(-312056,1171940,3280828)$	2^2
14	$(-430,446,500)$	2	34	$(-2048,4194304,4194304)$	2^{11}
16	$(-32,1024,1024)$	2^5			

Moreover, in Table 3 we present the number of integer solutions which were found by our procedure.

Table 3. The number of integer solutions of the Diophantine equation $P_n=x^3+y^3+z^3$, $n\leqslant 40$, founded by the described procedure.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
	0	1	1	3	2	2	0	3	2	8	2	6	1
n	15	16	17	18	19	20	21	22	23	24	25	26	27
	4	1	8	38	17	0	7	3	18	4	18	4	16
n	28	29	30	31	32	33	34	35	36	37	38	39	40
	4	12	11	17	1	4	6	54	14	75	3	10	3

The search of solutions for $n=2,8,20$ was performed in a similar way, but without the assumption of positivity of P_n-x^3 and with the replacement of P_n-x^3 by $|P_n-x^3|$. In this way, for $n=2$, we found the solutions of the equation (1) presented in the proof of Corollary 2.3. Moreover, we get the equalities

$$P_8=32^3-4^3-4^3=404^3-124^3-400^3,$$
$$P_{20}=8192^3-64^3-64^3=9404^3-472^3-6556^3,$$

which fill the gap in Table 3.

Remark 3.1. Let us also note that the non-negative solutions of the equation (1) for given n often satisfy the condition $\gcd(x,y,z)=2^k$ for certain, not to small, value of k. Having in mind this property, we performed numerical search of positive solutions for certain values of $n>40$. The method employed was the same as in the case $n\leqslant 40$, but instead to work for given n, with P_n we worked with the (smaller) number $M_{k,n}=2^{a_n}2^{3k}(2^n-1)$, where $k\in\{1,2,3,4,5\}$ and $a_n\equiv n-1 \pmod 3$. Each representation of $M_{k,n}$ after multiplication by 2^{3m}, where $m=(n-1-a_n-3k)/3$, leads to the representation of P_n as a sum of three cubes.

Using this approach we found the following representations

$$\begin{aligned}
P_{41} &= (2^{12}\cdot 441)^3 + (2^{12}\cdot 22063)^3 + (2^{12}\cdot 29022)^3,\\
P_{42} &= (2^{9}\cdot 183840)^3 + (2^{9}\cdot 301469)^3 + (2^{9}\cdot 337507)^3,\\
P_{43} &= (2^{14})^3 + (2^{14}\cdot 16255)^3 + (2^{14}\cdot 16511)^3,\\
P_{45} &= (2^{12}\cdot 18326)^3 + (2^{12}\cdot 144043)^3 + (2^{12}\cdot 181837)^3,\\
P_{47} &= (2^{14}\cdot 5835)^3 + (2^{14}\cdot 41149)^3 + (2^{14}\cdot 129702)^3,\\
P_{48} &= (2^{14}\cdot 8479)^3 + (2^{14}\cdot 160641)^3 + (2^{14}\cdot 169400)^3,\\
P_{49} &= (2^{16})^3 + (2^{16}\cdot 65279)^3 + (2^{16}\cdot 65791)^3,\\
P_{51} &= (2^{15}\cdot 91838)^3 + (2^{15}\cdot 252707)^3 + (2^{15}\cdot 380629)^3,\\
P_{60} &= (2^{19}\cdot 522158)^3 + (2^{19}\cdot 877167)^3 + (2^{19}\cdot 1559725)^3.
\end{aligned}$$

Let us observe that for $n \in \{44, 46, 50, 52, 53, 55, 56, 58, 59\}$ we have integer solutions coming from the parametrization given in Theorem 2.1. Moreover, noting the representations

$$\begin{aligned}
P_{54} &= (-2^{16}\cdot 557852)^3 + (2^{16}\cdot 302)^3 + (2^{16}\cdot 908586)^3,\\
P_{57} &= (-2^{16}\cdot 2647337)^3 + (2^{16}\cdot 2070161)^3 + (2^{16}\cdot 3597922)^3
\end{aligned}$$

we get

Corollary 3.2. *For each $n \in \{1, \ldots, 60\}$ the Diophantine equation* (1) *has a solution in integers.*

Our numerical search and Theorem 2.1 suggest the following

Conjecture 3.3. *For each $n \in \mathbb{N}_+$ the Diophantine equation* (1) *has a solution in integers.*

From our table we note that the equation (1) has no solutions in non-negative integers x, y, z for

$$n = 2, 4, 6, 8, 10, 12, 14, 16, 20, 24, 32, 33.$$

This numerical observation lead us to the following

Conjecture 3.4. *For each $\epsilon \in \{0, 1\}$, there are infinitely many $n \equiv \epsilon \pmod 2$ such that the equation* (1) *has no solutions in non-negative integers x, y, z.*

Moreover, according to our numerical search, one can also ask whether the conjecture proposed by Farhi is not too optimistic. Indeed, in his proof of the existence of representations of a perfect number P_p as a sum of five non-negative cubes, with $p \geqslant 3$, he used only the fact that $p \equiv \pm 1 \pmod 6$ and the well-known polynomial identity

$$2t^6 - 1 = (t^2 + t - 1)^3 + (t^2 - t - 1)^3 + 1,$$

i.e., no special property of perfect numbers was used. We also observed that the smallest odd $n \in \mathbb{N}_{\geqslant 3}$, such that the equation (1) has no solutions in positive integers is 33. Due to our limited experimental data ($n \leqslant 40$ in our search), there is no strong reason to believe that for all perfect numbers P_p, the equation $P_p = x^3 + y^3 + z^3$ has a solution in non-negative integers. On the other hand, the first possible candidate for the counterexample to the conjecture is $p = 89$. The corresponding perfect number P_{89} has 54 digits, and the question about the existence of positive integer solutions of the equation $P_{89} = x^3 + y^3 + z^3$ is rather difficult.

It is also interesting to note the equalities

$$P_3 = 1^3 + 3^3, \qquad P_7 = 28^3 - 24^3, \qquad P_9 = 60^3 - 44^3,$$

which give all solutions of the equation $P_n = x^3 + y^3, n \leqslant 140$, in integers. This observation lead us to the following

Question 3.5. *Is the set of integer solutions (in variables n, x, y) of the Diophantine equation $P_n = x^3 + y^3$ finite?*

We expect that the answer is positive.

Remark 3.6. One can also ask about representation of the number P_n as a sum of three squares. In this case we can easily get the answer. Indeed, Gauss proved that the equation $N = x^2 + y^2 + z^2$ has a solution in integers if and only if N is not of the form $4^m(8a + 7)$ for some $a, m \in \mathbb{N}$. In consequence the equation $P_n = x^2 + y^2 + z^2$ has a solution in integers if and only if $n \equiv 0 \pmod 2$.

It would be also interesting to know whether the Diophantine equation

$$P_n = x^2 + y^2 + z^4$$

has infinitely many solutions in integers (x, y, z, n), i.e., we treat the above equation in variables $x, y, z \in \mathbb{Z}$ and $n \in \mathbb{N}$. We expect that this is the case, and numerical computations suggest the existence of solutions with z being power of 2.

References

[1] M. Beck, E. Pine, W. Tarrant, K.Y. Jensen, *New integer representations as the sum of three cubes*, Math. Comp. **76**(259) (2007), 1683–1690.

[2] A. Bremner, *On sums of three cubes*, Canadian Mathematical Society Conference Proceedings **15** (1995), 87–91.

[3] L.E. Dickson, *History of the theory of numbers*, Vol I, Washington: Carnegie Institute of Washington. 1919–1923; reprint ed., Chelsea Publ. Co., New York (1952).

[4] N.D. Elkies, *Rational points near curves and small nonzero $|x^3 - y^2|$ via lattice reduction*, In Algorithmic number theory, pages 33–63, Springer, 2000.

[5] A.S. Elsenhans, J. Jahnel, *New sums of three cubes*, Math. Comp. **78**(266) (2009), 1227–1230.

[6] B. Farhi, *On the representation of an even perfect number as the sum of five cubes*, Funct. Approx. Comment. Math. **57**(2) (2017), 277–278.
[7] S.G. Huisman, *Newer sums of three cubes*, avaliable at: https://arxiv.org/pdf/1604.07746.pdf.
[8] W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.
[9] L.J. Mordell, *On sums of three cubes*, Journal of the LMS **17**(3) (1942), 139–144.

Address: Maciej Ulas: Jagiellonian University, Faculty of Mathematics and Computer Science, Institute of Mathematics, Łojasiewicza 6, 30–348 Kraków, Poland.

E-mail: maciej.ulas@uj.edu.pl

Received: 30 August 2017; **revised:** 31 January 2018