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A NOTE ON THE CONGRUENCES WITH SUMS OF POWERS OF BINOMIAL COEFFICIENTS

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Abstract: Let $p \geqslant 7$ be a prime, $l \geqslant 0$ be an integer and k, m be two positive integers, we obtain the following congruences,

$$\sum_{s=lp}^{(l+1)p-1} \binom{kp-1}{s}^m \equiv \begin{cases} \binom{k-1}{l}^m 2^{km(p-1)} \pmod{p^3}, & \text{if } 2 \nmid m, \\ \binom{k-1}{l}^m \binom{kmp-2}{p-1} \pmod{p^4}, & \text{if } 2 \mid m; \end{cases}$$

and

$$\sum_{s=lp}^{(l+1)p-1} (-1)^s \binom{kp-1}{s}^m \equiv \begin{cases} (-1)^l \binom{k-1}{l}^m 2^{km(p-1)} \pmod{p^3}, & \text{if } 2 \mid m, \\ (-1)^l \binom{k-1}{l}^m \binom{kmp-2}{p-1} \pmod{p^4}, & \text{if } 2 \nmid m. \end{cases}$$

Let p and q are distinct odd primes and k be a positive integer, we have

$$\binom{kpq-1}{(pq-1)/2} \equiv \binom{kp-1}{(p-1)/2}\binom{kq-1}{(q-1)/2} \pmod{pq}.$$

Keywords: binomial coefficients, prime powers, congruences.

1. Introduction

The well-known identities

$$\sum_{s=0}^{n} \binom{n}{s} = 2^n, \qquad \sum_{s=0}^{n} \binom{n}{s}^2 = \binom{2n}{n}$$

and

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} = 0, \qquad \sum_{s=0}^{2n} (-1)^s \binom{2n}{s}^2 = (-1)^n \binom{2n}{n}$$

are true for any positive integer n.

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In 1879, Lucas [3] proved that

$$\binom{p-1}{s} \equiv (-1)^s \pmod{p}$$

for any s, $0 \leqslant s \leqslant p-1$, with p prime. In 1895, Morley [4] showed that for any prime $p \geqslant 5$,

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{(p-1)/2} \equiv 4^{p-1} \pmod{p^3}.$$

In 2002, Cai and Granville [1] showed several arithmetic properties on the residues of binomial coefficients and their products modulo primes powers, e.g.,

$$\binom{pq-1}{(pq-1)/2} \equiv \binom{p-1}{(p-1)/2}\binom{q-1}{(q-1)/2} \pmod{pq},$$

for any distinct odd primes p and q. They also proved that if $p \geqslant 5$ is a prime and m is an integer, then

$$\sum_{s=0}^{p-1} \binom{p-1}{s}^m \equiv \begin{cases} 2^{m(p-1)} \pmod{p^3}, & \text{if } 2 \nmid m, \\ \binom{mp-2}{p-1} \pmod{p^4}, & \text{if } 2 \mid m; \end{cases}$$

and

$$\sum_{s=0}^{p-1} (-1)^s \binom{p-1}{s}^m \equiv \begin{cases} 2^{m(p-1)} \pmod{p^3}, & \text{if } 2 \mid m, \\ \binom{mp-2}{p-1} \pmod{p^4}, & \text{if } 2 \nmid m. \end{cases}$$

Let

$$C(m,k,l) = \sum_{s=lp}^{(l+1)p-1} \binom{kp-1}{s}^m, \qquad C'(m,k,l) = \sum_{s=lp}^{(l+1)p-1} (-1)^s \binom{kp-1}{s}^m.$$

In this paper, we obtain the following theorems.

Theorem 1. *Let $p \geqslant 7$ be a prime, $l \geqslant 0$ be an integer and k, m be two positive integers. Then*

$$C(m,k,l) \equiv \begin{cases} \binom{k-1}{l}^m 2^{km(p-1)} \pmod{p^3}, & \text{if } 2 \nmid m, \\ \binom{k-1}{l}^m \binom{kmp-2}{p-1} \pmod{p^4}, & \text{if } 2 \mid m; \end{cases}$$

and

$$C'(m,k,l) \equiv \begin{cases} (-1)^l \binom{k-1}{l}^m 2^{km(p-1)} \pmod{p^3}, & \text{if } 2 \mid m, \\ (-1)^l \binom{k-1}{l}^m \binom{kmp-2}{p-1} \pmod{p^4}, & \text{if } 2 \nmid m. \end{cases}$$

When $l=0$, we have

Remark 1. Let $p \geqslant 7$ be a prime and $k,\ m$ be integers. Then

$$C(m,k,0)=\sum_{s=0}^{p-1}\binom{kp-1}{s}^m \equiv \begin{cases} 2^{km(p-1)} \pmod{p^3}, & \text{if } 2\nmid m,\\ \binom{kmp-2}{p-1} \pmod{p^4}, & \text{if } 2\mid m;\end{cases}$$

and

$$C'(m,k,0)=\sum_{s=0}^{p-1}(-1)^s\binom{kp-1}{s}^m \equiv \begin{cases} 2^{km(p-1)} \pmod{p^3}, & \text{if } 2\mid m,\\ \binom{kmp-2}{p-1} \pmod{p^4}, & \text{if } 2\nmid m.\end{cases}$$

By Theorem 1 and Remark 1, it is obviously that

Remark 2. Let $p \geqslant 7$ be a prime, $l \geqslant 0$ be an integer and $k,\ m$ be two positive integers. Then

$$C(m,k,l) \equiv \begin{cases} \binom{k-1}{l}^m C(m,k,0) \pmod{p^3}, & \text{if } 2\nmid m,\\ \binom{k-1}{l}^m C(m,k,0) \pmod{p^4}, & \text{if } 2\mid m;\end{cases}$$

and

$$C'(m,k,l) \equiv \begin{cases} (-1)^l\binom{k-1}{l}^m C'(m,k,0) \pmod{p^3}, & \text{if } 2\mid m,\\ (-1)^l\binom{k-1}{l}^m C'(m,k,0) \pmod{p^4}, & \text{if } 2\nmid m.\end{cases}$$

When $m=1$, since $\binom{k-1}{l}=\binom{k-2}{l}+\binom{k-2}{l-1}$, by Theorem 1, we have

Corollary 1. *Let $p \geqslant 7$ be a prime and k be a positive integer. Then*

$$C(1,k,l) \equiv (C(1,k-1,l)+C(1,k-1,l-1))2^{p-1} \pmod{p^3}.$$

Theorem 2. *Let p and q are distinct odd primes and k be a positive integer, we have*

$$\binom{kpq-1}{(pq-1)/2} \equiv \binom{kp-1}{(p-1)/2}\binom{kq-1}{(q-1)/2} \pmod{pq}.$$

2. Preliminaries

In order to prove Theorem 1, we need the following lemmas.

Lemma 1 ([2, 5]). *Let $p \geqslant 5$ be a prime and $k \leqslant p-4$ be a positive integer. Then*

$$\sum_{i=1}^{p-1}\frac{1}{i^k} \equiv \frac{k}{k+1}pB_{p-1-k} \pmod{p^2}.$$

$$\sum_{i=1}^{\frac{p-1}{2}}\frac{1}{i^k} \equiv \begin{cases} (2^k-2)\left(2\frac{B_{p-k}}{p-k}-\frac{B_{2p-1-k}}{2p-1-k}\right) \pmod{p^2}, \text{if } k>1 \text{ is odd},\\ \frac{k(2^{k+1}-1)}{2(k+1)}pB_{p-1-k} \pmod{p^2}, \text{if } k \text{ is even},\\ -2q_p(2)+pq_p^2(2) \pmod{p^2}, \text{if } k=1,\end{cases}$$

where B_n is the n-th Bernoulli number and $q_p(n)=(n^{p-1}-1)/p$ is the Fermat quotient.

Lemma 2 ([7]). *Let $p \geqslant 5$ be a prime and $u > v > 0$ be integers. Then*

$$\binom{up}{vp} \equiv \binom{u}{v} \pmod{p^3}.$$

Lemma 3 ([8]). *Let $p \geqslant 5$ be a prime and n, α be two positive integers. Then*

$$\sum_{1 \leqslant l_1 < l_2 < \cdots < l_n \leqslant p-1} \frac{1}{l_1^\alpha l_2^\alpha \cdots l_n^\alpha} \equiv \begin{cases} 0 \pmod{p^2}, & \text{if } 2 \nmid n\alpha, \\ 0 \pmod{p}, & \text{if } 2 | n\alpha. \end{cases}$$

Lemma 4. *Let $p \geqslant 3$ be a prime and r be an integer, we have*

$$\sum_{s=1}^{p-1} (-1)^s \sum_{i=1}^{s} \frac{1}{i^r} = \frac{1}{2^r} \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i^r}.$$

Proof. Exchange the order of sums, we have

$$\sum_{s=1}^{p-1} (-1)^s \sum_{i=1}^{s} \frac{1}{i^r} = \sum_{i=1}^{p-1} \frac{1}{i^r} \sum_{s=i}^{p-1} (-1)^s.$$

If i is odd, then $\sum\limits_{s=i}^{p-1} (-1)^s = 0$. When i is even, then $\sum\limits_{s=i}^{p-1} (-1)^s = 1$, let $i \to 2i$, we obtain

$$\sum_{i=1}^{p-1} \frac{1}{i^r} \sum_{s=i}^{p-1} (-1)^s = \frac{1}{2^r} \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i^r}.$$

Therefore,

$$\sum_{s=1}^{p-1} (-1)^s \sum_{i=1}^{s} \frac{1}{i^r} = \frac{1}{2^r} \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i^r}.$$

We complete the proof of Lemma 4. ■

Lemma 5. *Let $p \geqslant 7$ be a prime, we have*

$$\sum_{s=1}^{p-1} (-1)^s \left(\sum_{i=1}^{s} \frac{1}{i} \right)^2 \equiv q_p^2(2) \pmod{p}.$$

Proof. By harmonic shuffle relation, we have

$$\begin{aligned}\sum_{s=1}^{p-1}(-1)^s\left(\sum_{i=1}^{s}\frac{1}{i}\right)^2 &= \sum_{s=1}^{p-1}(-1)^s\left(2\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij}+\sum_{i=1}^{s}\frac{1}{i^2}\right)\\ &= 2\sum_{s=1}^{p-1}(-1)^s\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij}+\sum_{s=1}^{p-1}(-1)^s\sum_{i=1}^{s}\frac{1}{i^2}\\ &= 2\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{ij}\sum_{s=j}^{p-1}(-1)^s+\sum_{s=1}^{p-1}(-1)^s\sum_{i=1}^{s}\frac{1}{i^2}\\ &= \sum_{1\leqslant i<j\leqslant p-1}\frac{1+(-1)^j}{ij}+\sum_{s=1}^{p-1}(-1)^s\sum_{i=1}^{s}\frac{1}{i^2}\\ &= \sum_{1\leqslant i<j\leqslant p-1}\frac{1}{ij}+\sum_{1\leqslant i<j\leqslant p-1}\frac{(-1)^j}{ij}+\sum_{s=1}^{p-1}(-1)^s\sum_{i=1}^{s}\frac{1}{i^2}.\end{aligned}$$

By Lemma 3, we have

$$\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{ij}\equiv 0 \pmod p.$$

By (101) in [6], we obtain

$$\sum_{1\leqslant i<j\leqslant p-1}\frac{(-1)^j}{ij}\equiv q_p^2(2) \pmod p.$$

By Lemma 1 and Lemma 4, we get

$$\sum_{s=1}^{p-1}(-1)^s\sum_{i=1}^{s}\frac{1}{i^2}\equiv 0 \pmod p.$$

Therefore, we complete the proof of Lemma 5. ■

Lemma 6. *Let $p\geqslant 3$ be a prime and r be an integer, we have*

$$\sum_{s=1}^{p-1}\sum_{i=1}^{s}\frac{1}{i^r}=\sum_{1\leqslant i\leqslant s\leqslant p-1}\frac{1}{i^r}\equiv\begin{cases}-p+1 \pmod{p^3}, & \text{if } r=1,\\ 0 \pmod{p^2}, & \text{if } r \text{ is even},\\ 0 \pmod p, & \text{if } r>1 \text{ is odd}.\end{cases}$$

Proof. Exchange the order of sums, we have

$$\sum_{s=1}^{p-1}\sum_{i=1}^{s}\frac{1}{i^r}=\sum_{i=1}^{p-1}\frac{1}{i^r}\sum_{s=i}^{p-1}1=\sum_{i=1}^{p-1}\frac{p-i}{i^r}=p\sum_{i=1}^{p-1}\frac{1}{i^r}-\sum_{i=1}^{p-1}\frac{1}{i^{r-1}}.$$

When $r=1$, by Lemma 1, we have

$$\sum_{s=1}^{p-1}\sum_{i=1}^{s}\frac{1}{i}=p\sum_{i=1}^{p-1}\frac{1}{i}-\sum_{i=1}^{p-1}1\equiv -p+1 \pmod{p^3}.$$

When r is even, then $r-1$ is odd, by Lemma 1, we have

$$\sum_{s=1}^{p-1}\sum_{i=1}^{s}\frac{1}{i^r}\equiv 0 \pmod{p^2}.$$

When $r>1$ is odd, then $r-1$ is even, by Lemma 1, we have

$$\sum_{s=1}^{p-1}\sum_{i=1}^{s}\frac{1}{i^r}\equiv 0 \pmod{p}.$$

We complete the proof of Lemma 6. ■

Lemma 7. *Let $p\geqslant 5$ be a prime and r be an integer, we have*

$$\sum_{s=1}^{p-1}\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij}=\sum_{1\leqslant i<j\leqslant s\leqslant p-1}\frac{1}{ij}\equiv p-1 \pmod{p^2}, \tag{1}$$

$$\sum_{s=1}^{p-1}\sum_{1\leqslant i<j<t\leqslant s}\frac{1}{ijt}\equiv 1 \pmod{p},$$

$$\sum_{s=1}^{p-1}\sum_{1\leqslant i<j\leqslant s}\frac{1}{i^2j}\equiv 0 \pmod{p}$$

and

$$\sum_{s=1}^{p-1}\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij^2}\equiv 0 \pmod{p}.$$

Proof. Exchange the order of sums, we have

$$\begin{aligned}\sum_{s=1}^{p-1}\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij}&=\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{ij}\sum_{s=j}^{p-1}1=\sum_{1\leqslant i<j\leqslant p-1}\frac{p-j}{ij}\\&=p\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{ij}-\sum_{1\leqslant i\leqslant j\leqslant p-1}\frac{1}{i}+\sum_{i=1}^{p-1}\frac{1}{i}.\end{aligned}$$

By Lemma 1, Lemma 3 and Lemma 6, we have

$$\sum_{s=1}^{p-1}\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij}\equiv p-1 \pmod{p^2}.$$

Exchange the order of sums, we have

$$\sum_{s=1}^{p-1}\sum_{1\leqslant i<j<t\leqslant s}\frac{1}{ijt}=\sum_{1\leqslant i<j<t\leqslant p-1}\frac{1}{ijt}\sum_{s=t}^{p-1}1=\sum_{1\leqslant i<j<t\leqslant p-1}\frac{p-t}{ijt}$$
$$=p\sum_{1\leqslant i<j<t\leqslant p-1}\frac{1}{ijt}-\sum_{1\leqslant i<j\leqslant t\leqslant p-1}\frac{1}{ij}+\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{ij}.$$

By Lemma 3 and (1), we have

$$\sum_{s=1}^{p-1}\sum_{1\leqslant i<j<t\leqslant s}\frac{1}{ijt}\equiv -p+1\equiv 1 \pmod p.$$

Similarly, we deduce that

$$\sum_{s=1}^{p-1}\sum_{1\leqslant i<j\leqslant s}\frac{1}{i^2 j}\equiv 0 \pmod p,\quad \sum_{s=1}^{p-1}\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij^2}\equiv 0 \pmod p.$$

We complete the proof of Lemma 7. ■

3. Proofs of the theorems

Proof of Theorem 1. Let $s\to lp+s$, then

$$C(m,k,l)=\sum_{s=0}^{p-1}\binom{kp-1}{lp+s}^m=\sum_{s=0}^{p-1}\prod_{i=1}^{lp+s}\left(\frac{kp-i}{i}\right)^m$$
$$=\prod_{i=1}^{lp}\left(\frac{kp-i}{i}\right)^m\left[1+\sum_{s=1}^{p-1}\prod_{i=lp+1}^{lp+s}\left(\frac{kp-i}{i}\right)^m\right]. \tag{2}$$

In (2), the first product

$$\prod_{i=1}^{lp}\left(\frac{kp-i}{i}\right)^m=\binom{kp-1}{lp}^m=\left\{\frac{k-l}{k}\binom{kp}{lp}\right\}^m,$$

by Lemma 2,

$$\prod_{i=1}^{lp}\left(\frac{kp-i}{i}\right)^m\equiv\left\{\frac{k-l}{k}\binom{k}{l}\right\}^m\equiv\binom{k-1}{l}^m \pmod{p^3}. \tag{3}$$

Let $i\to lp+i$ in (2), then

$$\sum_{s=1}^{p-1}\prod_{i=lp+1}^{lp+s}\left(\frac{kp-i}{i}\right)^m=\sum_{s=1}^{p-1}\prod_{i=1}^{s}\left(\frac{kp-lp-i}{lp+i}\right)^m=\sum_{s=1}^{p-1}(-1)^{sm}\prod_{i=1}^{s}\left(\frac{i+lp-kp}{i(1+\frac{lp}{i})}\right)^m.$$

Since

$$\frac{1}{1+\frac{lp}{i}} \equiv 1-\frac{lp}{i}+\frac{(lp)^2}{i^2}-\frac{(lp)^3}{i^3} \pmod{p^4},$$

we have

$$\begin{aligned}
&\sum_{s=1}^{p-1}\prod_{i=lp+1}^{lp+s}\left(\frac{kp-i}{i}\right)^m \equiv \sum_{s=1}^{p-1}(-1)^{sm}\prod_{i=1}^{s}\left(1-\frac{kp}{i}+\frac{klp^2}{i^2}-\frac{kl^2p^3}{i^3}\right)^m \qquad (4)\\
&\equiv \sum_{s=1}^{p-1}(-1)^{sm}\Bigg\{1-kp\sum_{i=1}^{s}\frac{1}{i}+p^2\left(k^2\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij}+kl\sum_{i=1}^{s}\frac{1}{i^2}\right)\\
&\quad -p^3\left[k^3\sum_{1\leqslant i<j<t\leqslant s}\frac{1}{ijt}+k^2l\sum_{1\leqslant i<j\leqslant s}\left(\frac{1}{i^2j}+\frac{1}{ij^2}\right)+kl^2\sum_{i=1}^{s}\frac{1}{i^3}\right]\Bigg\}^m \pmod{p^4}.
\end{aligned}$$

By harmonic shuffle relation

$$\begin{aligned}
\left(\sum_{i=1}^{s}\frac{1}{i}\right)^2 &= \sum_{1\leqslant i<j\leqslant s}\frac{2}{ij}+\sum_{i=1}^{s}\frac{1}{i^2},\\
\left(\sum_{i=1}^{s}\frac{1}{i}\right)^3 &= \sum_{1\leqslant i<j<t\leqslant s}\frac{6}{ijt}+3\sum_{1\leqslant i<j\leqslant s}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right)+\sum_{i=1}^{s}\frac{1}{i^3},\\
\sum_{i=1}^{s}\frac{1}{i}\sum_{i=1}^{s}\frac{1}{i^2} &= \sum_{1\leqslant i<j\leqslant s}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right)+\sum_{i=1}^{s}\frac{1}{i^3},\\
\sum_{i=1}^{s}\frac{1}{i}\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij} &= \sum_{1\leqslant i<j<t\leqslant s}\frac{3}{ijt}+\sum_{1\leqslant i<j\leqslant s}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right),
\end{aligned}$$

and applying the following multinomial theorem to (4)

$$(x_1+x_2+\cdots+x_s)^n = \sum_{\substack{n_1+n_2+\cdots+n_s=n\\ n_1,n_2,\cdots,n_s\geqslant 0}}\binom{n}{n_1,n_2,\cdots,n_s}x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s},$$

where $\binom{n}{n_1,n_2,\cdots,n_s}$ is the generalized binomial coefficients, and $\binom{n}{n_1,n_2,\cdots,n_s} = \frac{n!}{n_1!n_2!\cdots n_s!}$.

It is not hard to see that

$$
\begin{aligned}
&\sum_{s=1}^{p-1}\prod_{i=lp+1}^{lp+s}\left(\frac{kp-i}{i}\right)^m \\
&\qquad\equiv \sum_{s=1}^{p-1}(-1)^{sm}\Bigg\{1-kmp\sum_{i=1}^{s}\frac{1}{i}+m^2k^2p^2\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij} \\
&\qquad\quad+\left(\binom{m}{2}k^2+mkl\right)p^2\sum_{i=1}^{s}\frac{1}{i^2}-m^3k^3p^3\sum_{1\leqslant i<j<t\leqslant s}\frac{1}{ijt} \\
&\qquad\quad-\left(\binom{m}{2}mk^3+m^2k^2l\right)p^3\sum_{1\leqslant i<j\leqslant s}\left(\frac{1}{i^2j}+\frac{1}{ij^2}\right) \\
&\qquad\quad-\left(\binom{m}{3}k^3+m(m-1)k^2l+mkl^2\right)p^3\sum_{i=1}^{s}\frac{1}{i^3}\Bigg\} \pmod{p^4}. \quad (5)
\end{aligned}
$$

When m is odd, $(-1)^{sm}=(-1)^s$, from (5), we have

$$
\begin{aligned}
\sum_{s=1}^{p-1}\prod_{i=lp+1}^{lp+s}\left(\frac{kp-i}{i}\right)^m &\equiv \sum_{s=1}^{p-1}(-1)^s\{1-kmp\sum_{i=1}^{s}\frac{1}{i}+\frac{m^2k^2}{2}p^2\left(\sum_{i=1}^{s}\frac{1}{i}\right)^2 \\
&\quad+\left(-\frac{m^2k^2}{2}+mkl\right)p^2\sum_{i=1}^{s}\frac{1}{i^2}\} \pmod{p^3}.
\end{aligned}
$$

By Lemma 1, Lemma 4 and Lemma 5, we obtain

$$
\begin{aligned}
\sum_{s=1}^{p-1}\prod_{i=lp+1}^{lp+s}\left(\frac{kp-i}{i}\right)^m &\equiv -\frac{kmp}{2}\left(-2q_p(2)+pq_p^2(2)\right)+\frac{m^2k^2}{2}p^2q_p^2(2) \\
&\equiv kmpq_p(2)+\binom{km}{2}p^2q_p^2(2) \pmod{p^3}. \quad (6)
\end{aligned}
$$

Combining (2), (3) and (6), we have

$$
\begin{aligned}
C(m,k,l) &\equiv \binom{k-1}{l}^m\left(1+kmpq_p(2)+\binom{km}{2}p^2q_p^2(2)\right) \\
&\equiv \binom{k-1}{l}^m(1+pq_p(2))^{km} \\
&\equiv \binom{k-1}{l}^m 2^{km(p-1)} \pmod{p^3}.
\end{aligned}
$$

When m is even, $(-1)^{sm}=1$, from (5), we have

$$\begin{aligned}\sum_{s=1}^{p-1}&\prod_{i=lp+1}^{lp+s}\left(\frac{kp-i}{i}\right)^m\\ &\equiv\sum_{s=1}^{p-1}\Bigg\{1-kmp\sum_{i=1}^{s}\frac{1}{i}+m^2k^2p^2\sum_{1\leqslant i<j\leqslant s}\frac{1}{ij}\\ &\quad+\left(\binom{m}{2}k^2+mkl\right)p^2\sum_{i=1}^{s}\frac{1}{i^2}-m^3k^3p^3\sum_{1\leqslant i<j<t\leqslant s}\frac{1}{ijt}\\ &\quad-\left(\binom{m}{2}mk^3+m^2k^2l\right)p^3\sum_{1\leqslant i<j\leqslant s}\left(\frac{1}{i^2j}+\frac{1}{ij^2}\right)\\ &\quad-\left(\binom{m}{3}k^3+m(m-1)k^2l+mkl^2\right)p^3\sum_{i=1}^{s}\frac{1}{i^3}\Bigg\}\pmod{p^4}.\end{aligned}$$

By Lemma 6 and Lemma 7, we deduce that

$$\begin{aligned}\sum_{s=1}^{p-1}&\prod_{i=lp+1}^{lp+s}\left(\frac{kp-i}{i}\right)^m\\ &\equiv p-1+kmp(p-1)+k^2m^2p^2(p-1)-k^3m^3p^3\pmod{p^4}.\end{aligned}\tag{7}$$

Combining (2), (3), (7) and Lemma 2, we have

$$\begin{aligned}C(m,k,l)&\equiv\binom{k-1}{l}^m(p+kmp(p-1)+k^2m^2p^2(p-1)-k^3m^3p^3)\\ &\equiv\binom{k-1}{l}^m(1-km)p(1+kmp+k^2m^2p^2)\\ &\equiv\binom{k-1}{l}^m\frac{(1-km)p}{1-kmp}\\ &\equiv\binom{k-1}{l}^m\frac{(1-km)p}{1-kmp}\frac{1}{km}\binom{kmp}{p}\\ &\equiv\binom{k-1}{l}^m\binom{kmp-2}{p-1}\pmod{p^4}.\end{aligned}$$

As for the proof of $C'(m,k,l)$, only add $(-1)^{lp+s}$ in (2) and change $(-1)^{sm}$ into $(-1)^{s(m+1)}$ in (4) and (5).

This completes the proof of Theorem 1. ■

Proof of Theorem 2. For non-negative integers m and n and a prime p, Lucas's congruence relation holds,

$$\binom{m}{n}\equiv\prod_{i=0}^{k}\binom{m_i}{n_i}\pmod{p},$$

where $m = m_k p^k + m_{k-1}p^{k-1} + \cdots + m_1 p + m_0$ and $n = n_k p^k + n_{k-1}p^{k-1} + \cdots + n_1 p + n_0$ are the base p expansions of m and n respectively. This uses the convention that $\binom{m}{n} = 0$ if $m < n$.

By Lucas's congruence, we have

$$\binom{kpq-1}{(pq-1)/2} \equiv \binom{kp-1}{(p-1)/2}\binom{q-1}{(q-1)/2} \pmod{q}$$

and

$$\binom{kq-1}{(q-1)/2} \equiv \binom{k-1}{0}\binom{q-1}{(q-1)/2} \equiv \binom{q-1}{(q-1)/2} \pmod{q}.$$

Hence, we can obtain

$$\binom{kpq-1}{(pq-1)/2} \equiv \binom{kp-1}{(p-1)/2}\binom{kq-1}{(q-1)/2} \pmod{q}.$$

Similarly for prime p, we have

$$\binom{kpq-1}{(pq-1)/2} \equiv \binom{kp-1}{(p-1)/2}\binom{kq-1}{(q-1)/2} \pmod{p}.$$

Since p and q are distinct primes, we have

$$\binom{kpq-1}{(pq-1)/2} \equiv \binom{kp-1}{(p-1)/2}\binom{kq-1}{(q-1)/2} \pmod{pq}.$$

This completes the proof of Theorem 2. ■

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