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PIETSCH–MAUREY–ROSENTHAL FACTORIZATION OF SUMMING MULTILINEAR OPERATORS

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To the memory of Paweł Domański

Abstract: The main purpose of this paper is the study of a new class of summing multilinear operators acting from the product of Banach lattices with some nontrivial lattice convexity. A mixed Pietsch–Maurey–Rosenthal type factorization theorem for these operators is proved under weaker convexity requirements than the ones that are needed in the Maurey–Rosenthal factorization through products of L^q-spaces. A by-product of our factorization is an extension of multilinear operators defined by a q-concavity type property to a product of special Banach function lattices which inherit some lattice–geometric properties of the domain spaces, as order continuity and p-convexity. Factorization through Fremlin's tensor products is also analyzed. Applications are presented to study a special class of linear operators between Banach function lattices that can be characterized by a strong version of q-concavity. This class contains q-dominated operators, and so the obtained results provide a new factorization theorem for operators from this class.

Keywords: extension, summing multilinear operator, factorization, p-convex, Banach lattice.

1. Introduction

Domination inequalities for multilinear operators are of interest in applications to factorization of various types of operators (see [2, 4, 5, 11]). In the case of operators defined on products of Banach lattices, these dominations are deeply related to Banach lattice geometric notions, as q-convexity or q-concavity. It should be pointed out that domination does not lead in general to a nice factorization in the multilinear case. However, in some situations the relation between domination and factorization works as in the linear case. We recall that the famous Pietsch's factorization theorem is given by a domination result associated to summability properties also in the multilinear case, in which L^p-spaces are involved. We also point out that under the assumption of some variants of convexity properties of

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the involved lattices, the Maurey–Rosenthal multilinear theorem allows to link a q-concavity type domination inequality with a factorization/extension of the multilinear operator.

In this paper we are concerned with the analysis of some new lattice geometric properties that we call p-strong q-concavity (see Section 2). The motivation for this is to prove domination/factorization characterizations for multilinear operators from the product of Banach lattices that satisfy a certain vector norm inequality. Recall that in the case of linear operators acting in Banach lattices, if an operator $T\colon X \to Y$ is q-summing then it is also q-concave. This is the main lattice-type property that is normally used when a summability property for an operator among Banach lattices is considered. Indeed, this implies – using the Maurey–Rosenthal factorization and under the assumption of q-convexity of the domain lattice –, that the operator factors through an L^q-space. For $1 \leqslant p < q$, we define r by $1/r := 1/p - 1/q$. Then we can easily see that

$$\begin{aligned}\sup_{x^*\in B_{X^*}} \Big(\sum_{k=1}^n |\langle x_k, x^*\rangle|^q\Big)^{1/q} &\leqslant \sup_{x^*\in B_{X^*}} \sup_{(\beta_k)\in B_{\ell^r}} \Big(\sum_{k=1}^n |\beta_k\langle x_k, x^*\rangle|^p\Big)^{1/p} \\ &\leqslant \sup_{(\beta_k)\in B_{\ell^r}} \Big\|\Big(\sum_{k=1}^n |\beta_k x_k|^p\Big)^{1/p}\Big\|_X \leqslant \Big\|\Big(\sum_{k=1}^n |x_k|^q\Big)^{1/q}\Big\|_X\end{aligned}$$

for every finite sequence $(x_k)_{k=1}^n$ in the Banach lattice X. A look to the definitions (see Section 2) shows that the implications

$$p\textit{-summing} \Rightarrow p\textit{-strongly } q\textit{-concave} \Rightarrow q\textit{-concave}$$

hold for operators acting in Banach lattices.

The main advantage in using this new lattice property – p-strong q-concavity – is that *the requirement on the q-convexity of the original space can be relaxed* and still obtain a standard factorization theorem. Indeed, Maurey–Rosenthal theorem implies that q-summability of the operator *plus* q-convexity of the domain space allows a strong factorization through an L^q-space. In the preliminary paper [6], it is shown that for $1 \leqslant p < q$, every p-strongly q-concave operator – and so every q-summing operator – acting in a p-convex space factors strongly through a Banach function lattice space of the new class $S^q_{X_p}(\xi)$, that admits an easy description and whose lattice properties are naturally associated to p-strongly q-concave operators. The aim of this paper is to draw the complete picture for this class of lattice dominations/factorizations of operators by analyzing their multilinear variants. By applying them to the linear case we will show new factorization theorems for the classical q-dominated (linear) operators among Banach lattices.

In Section 2 we sketch some background from the theory of general Banach lattices, Fremlin's tensor product of Banach lattices, and also summing operators. We also provide examples which motivates our study.

In Section 3 we study a new class of summing multilinear operators acting from the product of Banach lattices with nontrivial lattice convexity. We prove an extension theorem for these operators acting in products of Banach lattices with some nontrivial convexity. We give a mixed Pietsch–Maurey–Rosenthal type factorization theorem for the multilinear case. We show that a particular class of multilinear operators defined by a q-concavity type property can be extended to a product of Banach lattice satisfying some lattice-geometric properties, as order continuity and p-convexity.

In section 4 we show the relation among summability of multilinear operators from suitable products of Banach function lattices and Fremlin tensor product. Factorization theorems are also proved.

In Section 5 we focus our attention on the factorization of the linear dominated operators associated to a new geometric definition introduced in the paper. Indeed, the lattice–geometric domination inequality appearing in the definition of the p-strongly q-concave operators motivates the definition of the dual notion.

2. Notation and background material

The purpose of this section is to sketch some background from the theory of general Banach lattices and summing operators. We shall also take the opportunity to establish some notation. For a given dual pair $\langle X, Y, (\cdot,\cdot)\rangle$ the evaluation map (x, y) is denoted by $\langle x, y\rangle$ for all $x \in X$, $y \in Y$.

For notations concerning vector lattices we follow [1, 10], and for tensor products of Banach lattices we follow [8, 9]. Let $(E, \leqslant)$ be a vector lattice (called also a Riesz space). If $A \subset E$, then $A^+ := \{x \in A;\ x \geqslant 0\}$. Let us recall, that if A is a subset of a Banach lattice E, then a functional $x^* \in E^*$ satisfying the condition $\langle x, x^*\rangle > 0$ whenever $0 < x \in A$ is called strictly positive on A. It is known that strictly positive functionals on E exist when E has the order continuous norm and a weak unit (see [1, Theorem 12.43] or [10, Proposition 1.b.15]).

We also recall that a Banach lattice possessing order continuous norm and a weak unit is order isomorphic to a Banach function lattice on a finite measure space.

We recall that if (Ω, Σ, μ) is a *σ-finite* measure space $L^0(\mu)$ denotes the space of μ-a.e. equal equivalence classes of functions. A *Banach function lattice* is a Banach space $X \subset L^0(\mu)$ with a norm $\|\cdot\|_X$ such that if $f \in L^0(\mu)$, $g \in X$ and $|f| \leqslant |g|$ μ-a.e. then $f \in X$ and $\|f\|_X \leqslant \|g\|_X$. Every Banach function space is a Banach lattice with the pointwise μ-a.e. order. The Köthe dual X' of a Banach function space X is the subspace of the dual space X^* of the functionals that has an integral representation, that is, $x^* \in X^*$ for which there exists $x' \in L^0(\mu)$ such that

$$\langle x, x^*\rangle = \int_\Omega xx'\,d\mu, \qquad x \in X.$$

In what follows we consider a dual pair $\langle X, X'\rangle$ with the evaluation map $(x, x') \mapsto \langle x, x'\rangle := \int_\Omega xx'\,d\mu$ for all $(x, x') \in X \times X'$.

A Banach function lattice is said to have the *Fatou property* if for every sequence (f_n) in X such that $0 \leqslant f_n \uparrow f$ a.e. and $\sup_n \|f_n\|_X < \infty$, it follows that $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$. This is equivalent to the fact that $X = X''$ with equality of norms.

We use $\mathcal{M}(K)$ to denote the space of regular Borel probability spaces on a compact Hausdorff space. We recall that the weak* topology on the dual E^* of a Banach space E is the topology of pointwise convergence. Then the unit ball is compact, by the Banach-Alaoglu theorem.

The normed space $(\mathbb{R}^n, \|\cdot\|_p)$ is denoted by ℓ_p^n for $1 \leqslant p \leqslant \infty$, where as usual for any $x = (t_1, \dots, t_n) \in \mathbb{R}^n$,

$$\|x\|_p = \Big(\sum_{k=1}^n |t_k|^p\Big)^{1/p},$$

and

$$\|x\|_\infty = \max_{1 \leqslant k \leqslant n} |t_k|.$$

In what follows the unit ball $B_{\ell_p^n}$ is denoted by B_p^n for short.

Given a Banach lattice X, a Banach space Y, and numbers $1 \leqslant p,\, q < \infty$, an operator $T\colon X \to Y$ is said to be *q-concave* if there exists $C_{(q)} > 0$ such that

$$\Big(\sum_{k=1}^n \|Tx_k\|_Y^q\Big)^{1/q} \leqslant C_{(q)} \Big\| \Big(\sum_{k=1}^n |x_k|^q\Big)^{1/q} \Big\|_X$$

for every choice of elements $x_1, ..., x_n$ in X. The infimum of the values $C_{(q)}$ for which the inequality above is satisfied will be denoted by $M_{(q)}(T)$.

A Banach lattice X is said to be *p-convex*, $1 \leqslant p < \infty$, respectively *q-concave*, $1 \leqslant q < \infty$, if there are positive constants $C^{(p)}$ and $C_{(q)}$ such that

$$\Big\| \Big(\sum_{k=1}^n |x_k|^p\Big)^{1/p} \Big\|_X \leqslant C^{(p)} \Big(\sum_{k=1}^n \|x_k\|_X^p\Big)^{1/p},$$

respectively,

$$\Big(\sum_{k=1}^n \|x_k\|_X^q\Big)^{1/q} \leqslant C_{(q)} \Big\| \Big(\sum_{k=1}^n |x_k|^q\Big)^{1/q} \Big\|_X$$

for every finite sequence $(x_k)_{k=1}^n$ in X. The least such $C^{(p)}$ (respectively, $C_{(q)}$)) is denoted by $M^{(p)}(X)$ (respectively, $M_{(q)}(X)$). It is well-known that a p-convex Banach (q-concave) lattice can always be renormed with a lattice norm in such a way that $M^{(p)}(X) = 1$ ($M_{(q)}(X) = 1$). *Henceforth, throughout the paper we will always assume that* $M^{(p)}(X) = 1$. We refer to [10, Ch. 1.d] or [12, Ch.2] for information about the classical geometric concepts of (lattice) p-convexity and q-concavity.

If a Banach lattice X is p-convex with $1 \leqslant p < \infty$, then its *p-concavification* is a Banach lattice X_p (see [10, p. 54] for details). Note that in the case of a Banach function lattice X on (Ω, Σ, μ), X_p is identified with the space of all $f \in L^0(\mu)$ so that $|f|^{1/p} \in X$ and equipped with the norm $\|f\|_{X_p} = \||f|^{1/p}\|_X^p$.

We will use Fremlin tensor products of Banach lattices. Let $X_1, \dots, X_n$ and Y be Archimedean Riesz spaces. An n-linear map

$$B\colon X_1 \times \cdots \times X_n \to Y$$

is called positive if $B(x_1, \dots, x_n) \in Y^+$ whenever $x_k \in X_k^+$, $1 \leqslant k \leqslant n$; it is called a Riesz n-morphism if $B(|x_1|, \dots, |x_n|) = |B(x_1, \dots, x_n)|$ for all $x_k \in X_k$, $1 \leqslant k \leqslant n$.

Following [9] (see also [13]) one can construct an Archimedean Riesz space $X_1 \bar{\otimes} \cdots \bar{\otimes} X_n$ and a Riesz morphism (called the *Fremlin map*) $\bigotimes$.

We recall fundamental properties of this construction;

(a) $X_1 \otimes \cdots \otimes X_n$ is dense in $X_1 \bar{\otimes} \cdots \bar{\otimes} X_n$, i.e., for any $u \in X_1 \bar{\otimes} \cdots \bar{\otimes} X_n$ there exist $x_k \in X_k^+$ $(1 \leqslant k \leqslant n)$ such that for all $\varepsilon > 0$ there is a $v \in X_1 \bar{\otimes} \cdots \bar{\otimes} X_n$ with $|u - v| \leqslant \varepsilon(x_1 \otimes \cdots \otimes x_k)$.

(b) If $u \in X_1 \bar{\otimes} \cdots \bar{\otimes} X_n$, then there exist $x_k \in X_k^+$ $(1 \leqslant k \leqslant n)$ such that $|u| \leqslant x_1 \otimes \cdots \otimes x_n$.

If $X_1, \dots, X_n$ are Banach lattices, then we can define the positive-projective norm $\|\cdot\|_{|\pi|}$ on $X_1 \bar{\otimes} \cdots \bar{\otimes} X_n$ by

$$\|x\|_{|\pi|} = \inf \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} \|x_{i,j}\|_{X_j};\ x_{i,j} \in X_j^+,\ |x| \leqslant \sum_{i=1}^{n} x_{i,1} \otimes \cdots \otimes x_{i,m} \right\}.$$

We define the *Fremlin tensor product* to be the Banach lattice $X_1 \otimes_{|\pi|} \cdots \otimes_{|\pi|} X_m$ given by the completion of $X_1 \bar{\otimes} \cdots \bar{\otimes} X_n$ with respect to $\|\cdot\|_{|\pi|}$.

We note that in the case of Banach function lattices $X_1, \dots, X_m$ on measure spaces $(\Omega_1, \Sigma_1, \mu_1), \dots, (\Omega_n, \Sigma_n, \mu_n)$, respectively, we can define the Riesz space $X_1 \bar{\otimes} \ldots \bar{\otimes} X_n$ generated by

$$\{x_1 \odot \cdots \odot x_n;\ x_j \in X_j,\ 1 \leqslant j \leqslant n\}$$

in $L^0(\mu_1 \times \cdots \times \mu_n)$, where

$$(x_1 \odot \cdots \odot x_n)(\omega_1, \dots, \omega_n) := x_1(\omega_1) \cdots x_n(\omega_n)$$

for all $(x_1, \dots, x_n) \in X_1 \times \cdots \times X_n$ and $(\omega_1, \dots, \omega_n) \in \Omega_1 \times \cdots \times \Omega_n$.

Let us introduce now the notion that motivates the multilinear definition given in this paper. Let $1 \leqslant p \leqslant q < \infty$. Consider a linear operator $T\colon X \to E$ from a Banach lattice X into a Banach space E. We will say that T is *p-strongly q-concave* if there exists $C > 0$ such that

$$\Big(\sum_{k=1}^{n} \|Tx_k\|_E^q \Big)^{1/q} \leqslant C \sup_{(\beta_k) \in B_r^n} \Big\| \Big(\sum_{k=1}^{n} |\beta_k x_k|^p \Big)^{1/p} \Big\|_X$$

for every finite sequence $(x_k)_{k=1}^n$ in X, where $1 < r \leqslant \infty$ is such that $1/r = 1/p - 1/q$.

We present some examples showing the nature of *linear* p-strongly q-concave operators. The reader can find more examples in [6].

Fix $1 \leqslant p < q < \infty$ and let $1/r = 1/p - 1/q$. Clearly that r/p and q/p are conjugate exponents, that is, $1/(r/p) + 1/(q/p) = 1$. Since

$$\sup_{(\beta_k)\in B_r^n} \Big\| \Big(\sum_{k=1}^n |\beta_k x_k|^p \Big)^{1/p} \Big\|_X \leqslant \Big\| \Big(\sum_{k=1}^n |x_k|^q \Big)^{1/q} \Big\|_X,$$

it follows that a p-strongly q-concave operator is always q-concave.

Now observe that if $1 \leqslant p \leqslant q < \infty$ and X is a p-concave Banach lattice, then the identity map $\iota\colon X \to X$ is p-strongly q-concave. To see this we fix a finite sequence $(x_k)_{k=1}^n$ in a p-concave Banach function lattice X. Without loss of generality we may assume that $M_{(p)}(X) = 1$. Let $\alpha_k = \|x_k\|^{q/r}/(\sum_{k=1}^n \|x_k\|^q)^{1/r}$ for each $1 \leqslant k \leqslant n$. Since $q = (pq/r) + p$, $\sum_{k=1}^n \alpha_k^r = 1$ and so

$$\begin{aligned}\Big(\sum_{k=1}^n \|x_k\|_X^q \Big)^{1/q} &= \Big(\sum_{k=1}^n \|x_k\|^{pq/r} \cdot \|x_k\|^p \Big)^{1/q} \leqslant \Big\| \Big(\sum_{k=1}^n \big(|x_k| \|x_k\|^{q/r} \big)^p \Big)^{1/p} \Big\|^{p/q} \\ &\leqslant \sup_{(\beta_k)\in B_r^n} \Big\| \Big(\sum_{k=1}^n |\beta_k\, x_k|^p \Big)^{1/p} \Big\|^{p/q} \Big(\sum_{k=1}^n \|x_k\|^q \Big)^{p/(rq)}.\end{aligned}$$

Hence

$$\Big(\sum_{k=1}^n \|x_k\|_X^q \Big)^{p/q^2} = \Big(\sum_{k=1}^n \|x_k\|_X^q \Big)^{1/q - p/(rq)} \leqslant \sup_{(\beta_k)\in B_r^n} \Big\| \Big(\sum_{k=1}^n |\beta_k\, x_k|^p \Big)^{1/p} \Big\|^{p/q},$$

and this gives the above mentioned statement.

We note that the above observation shows that all L^p-spaces are p-strongly q-concave, thus an operator acting in L^p-space is so. However, there are of course other situations.

We show an example of a p-strongly q-concave operator acting in a p-convex Banach function lattice that is not an L^p space. To see this we need to define special spaces and show some preliminary results.

Let $1 < p < q < \infty$ and let p' be the conjugate number given by $1/p' = 1 - 1/p$. Assume that (Ω, Σ, μ) is a σ-finite measure space such that there is a measurable partition $(A_k)_{k=1}^\infty$ of Ω with $\mu(A_k) = 1$ for each k. Consider the sequence of characteristic functions $(\chi_{A_k})_k$ and define an order continuous Banach function lattice

$$Y := \Big\{ f \in L^0(\mu);\ (f\chi_{A_k})_{k=1}^\infty \in \big(\oplus L^p(\mu|_{A_k}) \big)_{\ell_q} \Big\}$$

equipped with the norm

$$\|f\|_Y := \|(f\chi_{A_k})\|_{(\oplus L^p(\mu|_{A_k}))_{\ell_q}} = \Big(\sum_{k=1}^\infty \Big(\int_{A_k} |f|^p \, d\mu \Big)^{q/p} \Big)^{1/q}, \qquad f \in Y.$$

It is easy to check that

$$\|\chi_{A_k}\|_{(Y_p)'} = 1, \qquad k \in \mathbb{N}.$$

This implies that for each $k \in \mathbb{N}$ we have a functional $x_k^* \in B_{(Y_p)^*}$ given by

$$x_k^*(f) = \int_{A_k} f \, d\mu, \qquad f \in Y_p.$$

Now observe that the linear map T defined by

$$Tf = \Big(\frac{1}{2^{k/q}} \int_{A_k} f \, d\mu \Big)_k, \qquad f \in Y$$

is bounded from Y to ℓ_q.

For the Borel regular measure on $B_{(Y_p)^*}$ given by $\nu = \sum_{k=1}^{\infty} 2^{-k} \, \delta_{x_k^*}$, we denote by $S^q_{Y_p}(\nu)$ the space of all $f \in L^0(\mu)$ such that

$$\|f\|_{p,q;\nu} := \Big(\int_{B^+_{(Y_p)^*}} \big| \langle |f|^p, y^* \rangle \big|^{q/p} \, d\nu(y^*) \Big)^{1/q} < \infty.$$

A direct computation shows that

$$\|f\|_{p,q,\nu} = \Big(\sum_{k=1}^{\infty} \frac{1}{2^k} \Big(\int_{A_k} |f|^p \, d\mu \Big)^{q/p} \Big)^{1/q}, \qquad f \in Y.$$

Therefore, $\| \cdot \|_{p,q,\nu} \leqslant \| \cdot \|_Y$ and the operator T can be extended to the space $S^q_{Y_p}(\nu)$, since

$$\begin{aligned} \|Tf\|_{\ell^q} &= \Big\| \Big(\frac{1}{2^{k/q}} \int_{A_k} f \, d\mu \Big)_k \Big\|_{\ell^q} \\ &\leqslant \Big(\sum_{k=1}^{\infty} \frac{1}{2^k} \Big(\int_{A_k} |f|^p \, d\mu \Big)^{q/p} \mu(A_k)^{q/p'} \Big)^{1/q} = \|f\|_Y. \end{aligned}$$

Now observe that for any $f_1, \ldots, f_n \in Y$ we have

$$\begin{aligned} \sum_{k=1}^{n} \|Tf_k\|^q_{\ell^q} &\leqslant \int_{B^+_{(Y_p)^*}} \Big(\sum_{k=1}^{n} \big| \langle |f_k|^p, y^* \rangle \big|^{q/p} \Big) \, d\nu(y^*) \\ &\leqslant \sup_{y^* \in B^+_{(Y_p)^*}} \sum_{k=1}^{n} \big| \langle |f_k|^p, y^* \rangle \big|^{q/p} \\ &= \sup_{(\beta_k) \in B^n_r} \Big\| \Big(\sum_{k=1}^{n} |\beta_k f_k|^p \Big)^{1/p} \Big\|^q_Y, \end{aligned}$$

where we have used Lemma 3.3 (see Section 3 below).

3. Summing multilinear operators on products of Banach lattices

In what follows we assume that the m-tuples $(p_1, \dots, p_m)$, $(q_1, \dots, q_m)$ and $(r_1, \dots, r_m)$ of real numbers satisfy $1 \leqslant p_j \leqslant q_j$, $1/r_j = 1/p_j - 1/q_j$ for each $1 \leqslant j \leqslant m$. We also define q by $1/q := 1/q_1 + \ldots + 1/q_m$.

A multilinear operator $T\colon X_1 \times \cdots \times X_m \to Y$ —where X_j is a Banach lattice and Y is a Banach space—, is said to be $(p_1, \dots, p_m)$-strongly $(q_1, \dots, q_m)$-concave whenever there exists a constant $C > 0$ such that for any finite sequence $(x_k^j)_{k=1}^n$ in X_j, $1 \leqslant j \leqslant m$, we have that

$$\Big(\sum_{k=1}^n \|T(x_k^1, \dots, x_k^m)\|_Y^q\Big)^{1/q} \leqslant C \prod_{j=1}^m \sup_{(\beta_k^j) \in B_{\ell^{r_j}}} \Big\|\Big(\sum_{k=1}^n |\beta_k^j x_k^j|^{p_j}\Big)^{1/p_j}\Big\|_{X_j}.$$

We will use a lemma which is a general version of Lemma 2 in [6].

Lemma 3.1. *Let $1 \leqslant p < \infty$ and let E be a p-convex Banach lattice. Then*

$$\sup_{(\beta_k) \in B_r^n} \Big\|\Big(\sum_{k=1}^n |\beta_k x_k|^p\Big)^{1/p}\Big\|_E = \sup_{x^* \in B^+_{(E_p)^*}} \Big(\sum_{k=1}^n \langle |x_k|^p, x^*\rangle^{q/p}\Big)^{1/q}$$

for every choice of $(x_k)_{k=1}^n$ in E where $1/r = 1/p - 1/q$.

Proof. Fix a finite set $\{x_1, \dots, x_n\}$ of E, and note that $(\ell^{q/p})^* = \ell^{r/p}$, by $r/p + q/p = 1$. Then we have that

$$\begin{aligned}
\sup_{(\beta_k) \in B_r^n} \Big\|\Big(\sum_{k=1}^n |\beta_k x_k|^p\Big)^{1/p}\Big\|_E^p &= \sup_{(\beta_k) \in B_r^n} \Big\|\sum_{k=1}^n |\beta_k x_k|^p\Big\|_{E_p} \\
&= \sup_{(\beta_k) \in B_r^n} \sup_{x^* \in B^+_{(E_p)^*}} \Big\langle \sum_{k=1}^n |\beta_k x_k|^p,\, x^*\Big\rangle \\
&= \sup_{x^* \in B^+_{(E_p)^*}} \sup_{(\alpha_k) \in B^n_{r/p}} \sum_{k=1}^n |\alpha_k| \langle |x_k|^p, \varphi\rangle \\
&= \sup_{x^* \in B^+_{(E_p)^*}} \Big(\sum_{k=1}^n \Big(\langle |x_k|^p,\, x^*\rangle\Big)^{q/p}\Big)^{p/q}. \qquad \blacksquare
\end{aligned}$$

Now we state our first main theorem.

Theorem 3.2. *Let X_j be p_j-convex Banach lattices and let $1 \leqslant q_j < \infty$ for each $1 \leqslant j \leqslant m$. If $1/q = 1/q_1 + \ldots + 1/q_m$, then the following are equivalent statements about a multilinear operator T from $X_1 \times \cdots \times X_m$ to a Banach space E.*

(i) *T is $(p_1, \dots, p_m)$-strongly $(q_1, \dots, q_m)$-concave.*

(ii) *There is a constant $C>0$ such that for every $(x_1,\dots,x_m)\in X_1\times\cdots\times X_m$*

$$\|T(x_1,\dots,x_m)\|_Y \leqslant C\prod_{j=1}^m\Big(\int_{B^+_{((X_j)_{p_j})^*}}\langle |x_j|^{p_j},\, x_j^*\rangle^{q_j/p_j}\,d\nu_j(x_j^*)\Big)^{1/q_j},$$

where ν_j is a probability Borel measure on the weak compact set $B^+_{((X_j)_p)^*}$ for each $1\leqslant j\leqslant m$.*

Proof. (i) $\Rightarrow$ (ii). Fix finite sequences $(x_i^j)_{i=1}^n$ in X_j for each $1\leqslant j\leqslant m$.

First consider Lemma 3.1 (with $m=1$ and $E=X_j$ for each j) for factors in the product of the left hand side of the inequality that provides the definition of $(p_1,...,p_m)$-strongly $(q_1,...,q_m)$-concave m-linear operator. We obtain that the following inequality is equivalent to the one in this definition

$$\Big(\sum_{i=1}^n\|T(x_i^1,\dots,x_i^m)\|_Y^q\Big)^{1/q}\leqslant C\prod_{j=1}^m\sup_{x_j^*\in B^+_{((X_1)_{p_j})^*}}\Big(\sum_{i=1}^n\Big(\big\langle |x_i^j|^{p_j},\, x_j^*\big\rangle\Big)^{q_j/p_j}\Big)^{1/q_j}.$$

From this on, the proof uses some methods from [3, 11]. We only sketch the main arguments for the convenience of the reader.

Using Young's inequality, we obtain that the inequality above implies

$$\sum_{i=1}^n\|T(x_i^1,\dots,x_i^m)\|_Y^q\leqslant C^q\sum_{j=1}^m\Big(\frac{q}{q_j}\sup_{x_j^*\in B^+_{((X_j)_{p_j})^*}}\sum_{i=1}^n\Big(\big\langle |x_i^j|^{p_j},\, x_j^*\big\rangle\Big)^{q_j/p_j}\Big).$$

We now define a convex set of continuous real functions

$$\psi\colon \mathcal{M}(B^+_{((X_1)_{p_1})^*})\times\cdots\times\mathcal{M}(B^+_{((X_m)_{p_m})^*})\to\mathbb{R},$$

each one associated to each finite sets of finite sequences as the ones at the beginning of the proof, and given by the formula

$$\psi(\eta_1,\dots,\eta_m):=\sum_{i=1}^n\|T(x_i^1,\dots,x_i^m)\|_Y^q - C^q\sum_{j=1}^m\Big(\frac{q}{q_j}\int_{B^+_{((X_j)_{p_j})^*}}\sum_{i=1}^n\Big(\big\langle |x_i^j|^{p_j},\, x_j^*\big\rangle\Big)^{q_j/p_j}\,d\eta_j(x_j^*)\Big).$$

Note that $\mathcal{M}(B^+_{((X_j)_{p_j})^*})$ is a compact set with the product topology defined by means of the weak* topology of the dual of each Banach space $(X_j)_{p_j}$ for each $1\leqslant j\leqslant m$. Recall that these are Banach spaces as a consequence of the requirement that each of them is p_i-convex. The functions are continuous with respect to the product topology and satisfy all the properties needed for applying Ky Fan's Lemma (see, e.g., [7, Lemma 9.10]). This gives the existence of an element

$$(\nu_1,\dots,\nu_m)\in\mathcal{M}(B^+_{((X_1)_{p_1})^*})\times\cdots\times\mathcal{M}(B^+_{((X_m)_{p_m})^*})$$

satisfying

$$\sum_{i=1}^{n} \|T(x_i^1, \ldots, x_i^m)\|_Y^q - C^q \sum_{j=1}^{m} \Big(\frac{q}{q_j} \int_{B^+_{((X_j)_{p_j})^*}} \sum_{i=1}^{n} \Big(\langle |x_i^j|^{p_j}, x_j^* \rangle \Big) \Big)^{q_j/p_j} d\nu_j(x_j^*) \leqslant 0.$$

Now, the multilinearity of T allows to use a direct argument for choosing the right constants for getting from the above estimation for the product domination that is shown in (ii) (see [3, Theorem 1] for the details).

To conclude it is enough to observe that the converse inequality is obvious by using Lemma 3.1. ■

The next result deals with factorization of $(p_1, \ldots, p_m)$-strongly $(q_1, \ldots, q_m)$-concave multilinear operators. Motivated by the above result we define Banach lattices which are connected with the obtained characterization of these operators.

Let $1 \leqslant p \leqslant q < \infty$. For a given p-convex Banach function lattice X and a regular Borel probability measure ν on $B_{(X_p)^*}$ equipped with the weak*-topology we define on X a functional by

$$\|x\|_{p,q,\nu} := \Big(\int_{B^+_{(X_p)^*}} \langle |x|^p, x^* \rangle^{q/p} d\,\nu(x^*) \Big)^{1/q}, \qquad x \in X.$$

We put $F^q_{X_p}(\nu) := (X, \|\cdot\|_{p,q,\nu})$. Clearly $\rho(\cdot) := \|\cdot\|_{p,q,\nu}$ defines a lattice seminorm on X. If N is the null ideal of ρ, i.e., $N = \{x \in X;\ \rho(x) = 0\}$, then X/N is a normed lattice (under the natural order) equipped with the norm

$$\|[x]\| := \rho(x), \qquad [x] \in X/N.$$

The norm completion $\widetilde{F}^q_{X_p}(\nu)$ of X/N with respect to the above lattice norm is a Banach lattice. Note that $\|x\|_{p,q,\nu} \leqslant \|x\|_X$ for all $x \in X$ implies that the map i_X defined by

$$\iota_X(x) = [x], \qquad x \in X$$

is bounded from X to $\widetilde{F}^q_{X_p}(\nu)$.

We have the following useful lemma.

Lemma 3.3. *Let $1 \leqslant p \leqslant q < \infty$ and let X be a p-convex Banach lattice.*

(i) *If there exists a strictly positive functional on X_p, then for every $\nu \in \mathcal{M}(B^+_{(X_p)^*})$ there exists $\widetilde{\nu} \in \mathcal{M}(B^+_{(X_p)^*})$ such that $F_{X_p}(\widetilde{\nu}) = (X, \|\cdot\|_{p,q,\widetilde{\nu}})$ is a normed lattice and*

$$\frac{1}{2}\|x\|_{p,q,\nu} \leqslant \|x\|_{p,q,\widetilde{\nu}} \leqslant \|x\|_X, \qquad x \in X.$$

(ii) *If X is an order continuous Banach function lattice on (Ω, Σ, μ), then for every $\nu \in \mathcal{M}(B^+_{(X_p)^*})$ there exists a probability Borel measure $\xi \in \mathcal{M}(B^+_{(X_p)'})$ such that the completion of $F_{X_p}(\widetilde{\nu})$ —for $\widetilde{\nu}$ as in* (ii)*— is an* *order continuous Banach function lattice* $(S^q_{X_p}(\xi), \|\cdot\|)$ *on* (Ω, Σ, μ) *given by*

$$S^q_{X_p}(\xi) := \Bigg\{ f \in L^0(\mu); \\ \|f\| = \left(\int_{B^+_{(X_p)'}} \left(\int_\Omega |f(\omega)|^p \, h(\omega) \, d\mu(\omega) \right)^{q/p} d\xi(h) \right)^{1/q} < \infty \Bigg\}$$

and satisfying

$$\frac{1}{2}\|x\|_{p,q,\nu} \leqslant \|x\| \leqslant \|x\|_X, \qquad x \in X.$$

Proof. (i). Let $x^* \in (X_p)^*$ be a norm one strictly positive functional on X_p and let $\delta_{x^*} \in \mathcal{M}(B^+_{(X_p)^*})$ be the associated Dirac measure. For a given $\nu \in \mathcal{M}(B^+_{(X_p)^*})$, we define $\widetilde{\nu} := 1/2(\nu + \delta_{x^*}) \in \mathcal{M}(B^+_{(X_p)^*})$. It is obvious that $\widetilde{\nu}$ satisfies the required properties.

(ii). Our hypothesis that X (and so X_p) is order continuous implies that for every $x^* \in (X_p)^*$ there exists a unique $h = h_{x^*} \in (X_p)'$ such that

$$\langle |x|^p, x^* \rangle = \int_\Omega |x|^p h \, d\mu, \qquad x \in X$$

with $\|x^*\|_{(X_p)^*} = \|h\|_{(X_p)'}$, and moreover the map $(X_p)^* \ni x^* \mapsto h_{x^*}$ is an order isometrical isomorphism. We denote the restriction of this map to $B_{(X_p)^*}$ by φ. Clearly, φ is a topological homeomorphism of $B_{(X_p)^*}$ equipped with the weak* topology onto $B_{(X_p)'}$ equipped with the pointwise topology induced by $\sigma(X', X)$.

For a fixed $\nu \in \mathcal{M}(B^+_{(X_p)^*})$, we define $\nu_\varphi \in \mathcal{M}(B^+_{(X_p)'})$ by $\nu_\varphi(A) := \nu(\varphi^{-1}(A))$ for any Borel subset of $B^+_{(X_p)'}$. Then for every $x \in X$, we get that

$$\begin{aligned} \int_{B^+_{(X_p)^*}} \langle |x|^p, x^* \rangle^{q/p} \, d\nu(x^*) &= \int_{\varphi^{-1}(B^+_{(X_p)'})} \langle |x|^p, \varphi^{-1}(h_{x^*}) \rangle^{q/p} \, d\nu(x^*) \\ &= \int_{B^+_{(X_p)'}} \left(\int_\Omega |x|^p h d\mu \right)^{q/p} d\nu_\varphi(h). \end{aligned}$$

Since $(X_p)'$ is a Banach function lattice on (Ω, Σ, μ) there exists $h \in B_{(X_p)'}$ with $h > 0$ on Ω. Then $\xi := 1/2(\nu_\varphi + \delta_h) \in \mathcal{M}(B_{(X_p)'})$, where δ_h is a Dirac measure generated by h.

Combining the above formula with [6, Proposition 1], we conclude that

$$(S^q_{X_p}(\xi), \|\cdot\|)$$

is the desired Banach function lattice. ∎

We are now ready to state the following factorization theorem.

Theorem 3.4. *Let X_j be p_j-convex Banach function lattices and let $1 \leqslant q_j < \infty$ for each $1 \leqslant j \leqslant m$. If $1/q = 1/q_1 + \ldots + 1/q_m$, then the following are equivalent statements about a multilinear operator T from $X_1 \times \cdots \times X_m$ to a Banach space Y.*

(i) *T is $(p_1, \ldots, p_m)$-strongly $(q_1, \ldots, q_m)$-concave.*

(ii) *There are probability Borel measures ν_j in $\mathcal{M}(B_{((X_j)_{p_j})^*})$ for each $1 \leqslant j \leqslant m$, and a multilinear operator S such that T factors as*

$$\begin{array}{ccc} X_1 \times \cdots \times X_m & \xrightarrow{\quad T \quad} & Y \\ \Big\downarrow{\scriptstyle \iota_1 \times \cdots \times \iota_m} & \nearrow_{S} & \\ \widetilde{F}^{q_1}_{X_{p_1}}(\nu_1) \times \cdots \times \widetilde{F}^{q_m}_{X_{p_m}}(\nu_m) & & \end{array}$$

where $\iota_j = \iota_{X_j}$ for each $1 \leqslant j \leqslant m$.

Proof. (i) $\Rightarrow$ (ii). From Theorem 3.2, it follows that there is a constant $C > 0$ such that for every $(x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m$,

$$\|T(x^1, \ldots, x^m)\|_Y \leqslant C \prod_{j=1}^{m} \Big(\int_{B^+_{((X_j)_{p_j})^*}} \langle |x_j|^{p_j}, x_j^* \rangle^{q_j/p_j} \, d\nu_j(x_j^*) \Big)^{1/q_j},$$

where ν_j is a probability Borel measure on the weak*-compact set $B^+_{((X_j)_p)^*}$ for each $1 \leqslant j \leqslant m$. This implies that

$$\|T(x_1, \ldots, x_m)\|_Y \leqslant C\rho_1(x_1) \cdots \rho_m(x_m)$$

holds for all $(x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m$ with $\rho_j(\cdot) = \|\cdot\|_{P_j, q_j, \nu_j}$ for each $1 \leqslant j \leqslant m$. In particular, this implies that the formula

$$T_0([x_1], \ldots, [x_m]) := T(x_1, \ldots, x_m), \qquad (x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m$$

defines a bounded multilinear operator from $X_1/N_1 \times \cdots \times X_m/N_m$ to Y, where $N_j = \{x \in X_j;\, \rho_j(x) = 0\}$ for each $1 \leqslant j \leqslant m$. Denote by S the unique multilinear continuous extension of T_0 to $\widetilde{F}^{q_1}_{X_{p_1}} \times \cdots \times \widetilde{F}^{q_m}_{X_{p_m}}$. Clearly we have that $\iota_1 \times \cdots \times \iota_m$ given by

$$(\iota_1 \times \cdots \times \iota_m)(x_1, \ldots, x_m) := ([x_1], \ldots, [x_m])$$

for all $(x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m$ is a bounded linear operator from $X_1 \times \cdots \times X_m$ to $\widetilde{F}^{q_1}_{X_{p_1}}(\nu_1) \times \cdots \times \widetilde{F}^{q_m}_{X_{p_m}}(\nu_m)$ and so we have the required factorization

$$T = S \circ (\iota_1 \times \cdots \times \iota_m).$$

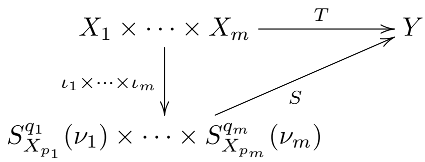
The implication (ii) $\Rightarrow$ (i) is obvious. ∎

Combing the above corollary with Lemma 3.3 we obtain the following result for the case of order continuous Banach function lattices.

Corollary 3.5. *Let X_j be order continuous p_j-convex Banach function lattices on measure spaces $(\Omega_j, \Sigma_j, \mu_j)$ and let $1 \leqslant q_j < \infty$, $1 \leqslant j \leqslant m$. If $1/q = 1/q_1 + \ldots + 1/q_m$, then the following are equivalent statements about an m-linear operator T from $X_1 \times \cdots \times X_m$ to a Banach space Y.*

(i) *T is $(p_1, \ldots, p_m)$-strongly $(q_1, \ldots, q_m)$-concave.*

(ii) *There are probability measures ν_j in $\mathcal{M}(B_{((X_j)_{p_j})'})$ for each $1 \leqslant j \leqslant m$, and a multilinear operator S such that T factors through the product of Banach function lattices $S^{q_j}_{X_{p_j}}(\nu_j)$ on the corresponding measure spaces $(\Omega_j, \Sigma_j, \mu_j)$ as*



where $\iota_j \colon X_j \to S^{q_j}_{X_{p_j}}(\nu_j)$ are continuous inclusions for $1 \leqslant j \leqslant m$.

4. Domination and the Fremlin tensor product

In this section we show the relation between summability of multilinear operators from suitable products of Banach function lattices and Fremlin tensor products. This will provide a class of multilinear operators which is different from the one analyzed in the previous section. The main difference is that the factorization is in the present case defined by a multilinear operator with values in a tensor product structure and a linear map, in an opposite way as what happens with the class of $(p_1, \ldots, p_m)$-strongly $(q_1, \ldots, q_m)$-concave operators.

Theorem 4.1. *Let $T \colon X_1 \times \cdots \times X_m \to Y$ be a Banach space valued multilinear operator, where X_j, $1 \leqslant j \leqslant m$, are Banach lattices. Suppose that $X_1 \otimes_{|\pi|} \cdots \otimes_{|\pi|} X_m$ is embedded in the p-convex Banach lattice E. Then the following statements are equivalent.*

(i) *There is a constant $C > 0$ such that for each $1 \leqslant j \leqslant m$ and every choice of $(x_i^j)_{i=1}^n$ in X_j,*

$$\Big(\sum_{i=1}^n \|T(x_i^1, \ldots, x_i^m)\|_Y^q\Big)^{1/q} \leqslant C \sup_{(\beta_i) \in B_r^n} \Big\|\Big(\sum_{i=1}^n \big|\beta_i\big(x_i^1 \otimes \cdots \otimes x_i^m\big)\big|^p\Big)^{1/p}\Big\|_E.$$

(ii) *There is a constant $C > 0$ such that for every $(x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m$,*

$$\|T(x_1, \ldots, x_m)\|_Y \leqslant C\Big(\int_{B^+_{(E_p)^*}} \big\langle |x_1 \otimes \cdots \otimes x_m|^p,\, x^*\big\rangle^{q/p}\, d\nu\Big)^{1/q},$$

where ν is a probability Borel measure on the weak compact set $B^+_{(E_p)^*}$.*

Proof. The argument follows the lines of the one given for Theorem 3.2. The p-convexity of E implies that E_p is a Banach lattice. By the inclusion of the Fremlin tensor product $X_1 \otimes_{|\pi|} \cdots \otimes_{|\pi|} X_m \hookrightarrow E$ we have that all the tensors $x_1 \otimes \cdots \otimes x_m$ are in E, and so $|x_1 \otimes \cdots \otimes x_m|^p$ define a continuous function in $C(B_{(E_p)^*})$, where $B_{(E_p)^*}$ is equipped with the induced topology by the weak* topology of $(E_p)^*$. From this point on, the proof using Ky Fan's Lemma is similar to the one of Theorem 3.2, using Lemma 3.1 for defining the right set of functions $\phi\colon \mathcal{M}(B_{(E_p)^*}) \to \mathbb{R}$, where only functions as $|x_1 \otimes \cdots \otimes x_m|^p$ are considered.

The converse implication is easily obtained by a direct calculation. ∎

In the case of Banach function lattices on measure spaces we obtain the following results on factorization of multilinear operators.

Corollary 4.2. *Let $T\colon X_1 \times \cdots \times X_m \to Y$ be a Banach space valued multilinear operator, where X_j are Banach function lattices on $(\Omega_j, \Sigma_j, \mu_j)$, $1 \leqslant j \leqslant m$. Suppose that $X_1 \otimes_{|\pi|} \cdots \otimes_{|\pi|} X_m$ is continuously embedded in E, where $E = E(\mu_1 \times \cdots \times \mu_m)$ is a p-convex Banach function lattice on the product measure space. Then the following statements are equivalent.*

(i) *There is a constant $C > 0$ such that for each $1 \leqslant j \leqslant m$ and for every choice of sequences $(x_i^j)_{i=1}^n$ in X_j,*

$$\Big(\sum_{i=1}^n \|T(x_i^1, \ldots, x_i^m)\|_Y^q \Big)^{1/q} \leqslant C \sup_{(\beta_i) \in B_r^n} \Big\| \Big(\sum_{i=1}^n \big|\beta_i \big(x_i^1 \odot \cdots \odot x_i^m\big)\big|^p \Big)^{1/p} \Big\|_E.$$

(ii) *There is a constant $C > 0$ such that for every $(x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m$,*

$$\|T(x_1, \ldots, x_m)\|_E \leqslant C \Big(\int_{B^+_{(E_p)^*}} \big\langle |x_1 \odot \cdots \odot x_m|^p, \, x^* \big\rangle^{q/p} \, d\nu \Big)^{1/q},$$

where ν is a probability Borel measure on the weak compact set $B^+_{(E_p)^*}$.*

Using the same proof but changing single tensors $x_1 \otimes \cdots \otimes x_m$ by finite combinations of these products, we obtain the corresponding factorization theorem.

Corollary 4.3. *Under the assumptions of Theorem 4.1 on the spaces $X_1, \ldots, X_m$, E and the multilinear operator $T\colon X_1 \times \cdots \times X_m \to Y$, the following statements are equivalent.*

(i) *There is a constant $C > 0$ such that for each $1 \leqslant j \leqslant m$ and for every choice of matrices $(x_{i,k}^j)_{i=1,k=1}^{N,M}$ in X_j, $(\lambda_{i,k})_{i=1,k=1}^{N,M}$ in $\mathbb{R}$,*

$$\Big(\sum_{i=1}^N \Big\| \sum_{k=1}^M \lambda_{i,k} T(x_{i,k}^1, \ldots, x_{i,k}^m) \Big\|_Y^q \Big)^{1/q}$$
$$\leqslant C \sup_{(\beta_i) \in B_r^n} \Big\| \Big(\sum_{i=1}^N \Big| \beta_i \Big(\sum_{k=1}^M \lambda_{i,k} \big(x_{i,k}^1 \otimes \cdots \otimes x_{i,k}^m\big) \Big) \Big|^p \Big)^{1/p} \Big\|_E.$$

(ii) *The operator T admits the following factorization*

$$\begin{array}{ccc} X_1 \times \cdots \times X_m & \xrightarrow{\;T\;} & E \\ \otimes \big\downarrow & \nearrow_{S} & \\ \widetilde{F}^q_{E_p}(\nu) & & \end{array}$$

where ν is a probability Borel measure on the weak compact set $B^+_{(E_p)^*}$ and $\bigotimes$ is the Fremlin map.*

5. Factorization of p-strongly q-dominated operators.

In this section we prove a factorization theorem for a special class of linear operators between Banach lattices. We start with the following definition. Let $1 \leqslant q < \infty$ and $1/q + 1/q' = 1$, $1 \leqslant p_1 \leqslant q_1 = q$, $1 \leqslant p_2 \leqslant q_2 = q'$ and $1/r_1 = 1/p_1 - 1/q_1$, $1/r_2 = 1/p_2 - 1/q_2$. An operator $T\colon X \to Y$ between Banach lattices is said to be (p_1, p_2)-*strongly* (q_1, q_2)-*concave* whenever

$$\Big|\sum_{k=1}^{n} \langle T x_k, y_k^* \rangle\Big| \leqslant C \sup_{(\alpha_k) \in B^n_{r_1}} \Big\| \Big(\sum_{k=1}^{n} |\alpha_k x_k|^{p_1} \Big)^{1/p_1} \Big\|_X \times \sup_{(\beta_k) \in B^n_{r_2}} \Big\| \Big(\sum_{k=1}^{n} |\beta_k y_k^*|^{p_2} \Big)^{1/p_2} \Big\|_{Y^*},$$

for every choice of sequences $(x_k)_{k=1}^n$ in X and $(y_k^*)_{k=1}^n$ in Y^*.

We note that general examples of (p_1, p_2)-strongly (q_1, q_2)-concave operators are given by the classical q-dominated operators. Indeed, an operator $T\colon X \to Y$ is said to be q-dominated $(1 \leqslant q < \infty)$ if

$$\Big|\sum_{k=1}^{n} \langle T(x_k), y_k^* \rangle\Big| \leqslant C \sup_{x^* \in B_{X^*}} \Big(\sum_{k=1}^{n} |\langle x_k, x^* \rangle|^q \Big)^{1/q} \sup_{y^{**} \in B_{Y^{**}}} \Big(\sum_{k=1}^{n} |\langle y_k^*, y^{**} \rangle|^{q'} \Big)^{1/q'}$$

for every choice of $(x_k)_{k=1}^n$ in X and $(y_k^*)_{k=1}^n$ in Y^*.

Since $1/q = 1 - 1/q'$, $1/q' = 1 - 1/q$ and

$$\sup_{x^* \in B_{X^*}} \Big(\sum_{k=1}^{n} |\langle x_k, x^* \rangle|^q \Big)^{1/q} \cdot \sup_{y^{**} \in B_{Y^{**}}} \Big(\sum_{k=1}^{n} |\langle y_k^*, y^{**} \rangle|^{q'} \Big)^{1/q'} \leqslant \sup_{(\alpha_k) \in B^n_{q'}} \Big\| \sum_{i=1}^{n} |\alpha_k x_k| \Big\|_X \cdot \sup_{(\beta_k) \in B^n_{q}} \Big\| \sum_{k=1}^{n} |\beta_k y_k^*| \Big\|_{Y^*},$$

we conclude that $T\colon X \to Y$ is $(1,1)$-strongly (q, q')-concave operator.

Before showing the results, let us observe the following fact. Suppose that $T\colon X \to Y$ is an operator between Banach lattices such that X is p_1-convex and Y is p_2'-concave with $1/p_2 + 1/p_2' = 1$. Then it follows from Theorem 3.2 that T is (p_1, p_2)-strongly (q_1, q_2)-concave operator if and only if there exist $C > 0$ and probability measures $\nu_1 \in \mathcal{M}(B_{(X_{p_1})^*})$ and $\nu_2 \in \mathcal{M}(B_{((Y^*)_{p_2})^*})$ such that for every $(x, y^*) \in X \times Y^*$,

$$|\langle Tx, y^*\rangle| \leqslant C\Big(\int_{B^+_{(X_{p_1})^*}} \big\langle |x|^{p_1},\, x^*\big\rangle^{q_1/p_1}\, d\nu_1(x^*)\Big)^{1/q_1} \times \Big(\int_{B^+_{((Y^*)_{p_2})^*}} \big\langle |y^*|^{p_2},\, y^{**}\big\rangle^{q_2/p_2}\, d\nu_2(y^{**})\Big)^{1/q_2}.$$

We need the following lattice formula (see [14, Proposition 12.6]).

Proposition 5.1. *Let E be a Banach lattice, $x_k \in E$ $(1 \leqslant k \leqslant n)$, and $1 \leqslant p \leqslant \infty$. Then*

$$\Big\|\Big(\sum_{k=1}^n |x_k|^p\Big)^{1/p}\Big\|_E = \sup\Big\{\sum_{k=1}^n \langle x_k, x_k^*\rangle;\ x_k^* \in E^*,\ \Big\|\Big(\sum_{k=1}^n |x_k^*|^{p'}\Big)^{1/p'}\Big\|_{E^*} \leqslant 1\Big\}.$$

An application of the above proposition is the following corollary.

Corollary 5.2. *Let $1 \leqslant p_1 < q$, $1 \leqslant p_2 < q'$, and let r_1 and r_2 be given by $1/r_1 = 1/p_1 - 1/q$ and $1/r_2 = 1/p_2 - 1/q'$. Assume that $T\colon X \to Y$ is an operator between Banach lattices such that X is p_1-convex and Y be p_2'-concave. If there exists a constant $C > 0$ such that for every sequence $(x_k)_{k=1}^n$,*

$$\inf_{(\alpha_k)\in B^n_{r_2}} \Big\|\Big(\sum_{k=1}^n \Big|\frac{T(x_k)}{\alpha_k}\Big|^{p_2'}\Big)^{1/p_2'}\Big\|_Y \leqslant C \sup_{(\beta_k)\in B^n_{r_1}} \Big\|\Big(\sum_{k=1}^n |\beta_k x_k|^{p_1}\Big)^{1/p_1}\Big\|_X,$$

then T is (p_1, p_2)-strongly (q, q')-concave.

Proof. From Proposition 5.1, it follows that it is enough to show that

$$\sup\Big\{\Big|\sum_{k=1}^n \langle T(x_k), y_k^*\rangle\Big|;\ y_k^* \in Y^*,\ \sup_{(\gamma_k)\in B^n_{r_2}} \Big\|\Big(\sum_{i=1}^n |\gamma_k y_k^*|^{p_2}\Big)^{1/p_2}\Big\|_{Y^*} \leqslant 1\Big\} \leqslant C \sup_{(\beta_k)\in B^n_{r_1}} \Big\|\Big(\sum_{k=1}^n |\beta_k x_k|^{p_1}\Big)^{1/p_1}\Big\|_X$$

for every choice of finite sequences $(x_k)_{k=1}^n$ in X and $(y_k^*)_{k=1}^n$ in Y^*.

Assume that $p_2 > 1$; the proof for $p_2 = 1$ is the same with the obvious changes in the computations. Fix now (α_k) in $B^n_{r_2}$. We have

$$\sup\Big\{\Big|\sum_{k=1}^n \langle T(x_k), y_k^*\rangle\Big|;\ y_k^* \in Y^*,\ \sup_{(\gamma_k)\in B^n_{r_2}} \Big\|\Big(\sum_{k=1}^n |\gamma_k y_k^*|^{p_2}\Big)^{1/p_2}\Big\|_{Y^*} \leqslant 1\Big\}$$
$$\leqslant \sup\Big\{\Big|\sum_{k=1}^n \langle T(x_k), y_k^*\rangle\Big|;\ y_k^* \in Y^*,\ \Big\|\Big(\sum_{k=1}^n |\alpha_k y_k^*|^{p_2}\Big)^{1/p_2}\Big\|_{Y^*} \leqslant 1\Big\}.$$

Thus, we get that

$$\sup\Big\{\Big|\sum_{k=1}^n \langle T(x_k), y_k^*\rangle\Big|;\ \sup_{(\gamma_k)\in B^n_{r_2}} \Big\|\Big(\sum_{k=1}^n |\gamma_k y_k^*|^{p_2}\Big)^{1/p_2}\Big\|_{Y^*} \leqslant 1\Big\}$$
$$\leqslant \inf_{(\alpha_k)\in B^n_{r_2}} \Big\|\Big(\sum_{k=1}^n \Big|\frac{T(x_k)}{\alpha_k}\Big|^{p_2'}\Big)^{1/p_2'}\Big\|_{Y}.$$

Combining with Proposition 5.1, the proof is completed. ■

Note that Corollary 5.2 shows that some classical operators are (p_1, p_2)-strongly (q_1, q_2)-concave. Consider the following example. Let $([0,1], \mathcal{B}, \mu)$ be Lebesgue measure space, and let $(A_k)_{k=1}^\infty$ be the decreasing sequence of the intervals $A_k := [0, 1/2^{k-1}]$ for each $k \in \mathbb{N}$. Consider the *integral evaluation operator* $T\colon L^1[0,1] \to \ell^\infty$ given by

$$T(x) := \Big(\int_{A_k} x\, d\mu\Big)_k, \qquad x \in L^1[0,1].$$

We claim that T satisfies the assumptions of Corollary 5.2 with $q = 2 = q'$, $p_1 = p_2 = 1$ and $r_1 = r_2 = 2$. To see this fix a finite set $\{x_1, ..., x_n\}$ in $L^1[0,1]$ and define the following constants,

$$\alpha_{0,i} := \frac{\int_{[0,1]} |x_i|\, d\mu}{\big(\sum_{i=1}^n \big(\int_{[0,1]} |x_i|\, d\mu\big)^2\big)^{1/2}}, \qquad 1 \leqslant i \leqslant n.$$

Note that $\big(\sum_{i=1}^n \alpha_{0,i}^2\big)^{1/2} = 1$. Then

$$\inf_{(\alpha_i)\in B_2^n} \Big\|\sup_{1\leqslant i\leqslant n} \Big|\frac{Tx_i}{\alpha_i}\Big|\Big\|_{\ell^\infty} \leqslant \Big\|\sup_{1\leqslant i\leqslant n} \Big|\frac{(\int_{A_k} x_i\, d\mu)_k}{\alpha_{i,0}}\Big|\Big\|_{\ell^\infty}$$
$$\leqslant \Big\|\sup_{1\leqslant i\leqslant n} \Big|\Big(\frac{\int_{A_k} x_i\, d\mu}{\int_{[0,1]} |x_i|\, d\mu}\Big)_k\Big|\Big(\sum_{i=1}^n \Big(\int_{[0,1]} |x_i|\, d\mu\Big)^2\Big)^{1/2}\Big\|_{\ell^\infty}$$
$$\leqslant \Big(\sum_{i=1}^n \Big(\int_{[0,1]} |x_i|\, d\mu\Big)^2\Big)^{1/2}$$
$$\leqslant \sup_{(\beta_i)\in B_2^n} \sum_{i=1}^n |\beta_i| \int_{[0,1]} |x_i| d\mu = \sup_{(\beta_i)\in B_2^n} \Big\|\sum_{i=1}^n |\beta_i x_i|\Big\|_{L^1[0,1]}.$$

Thus, Corollary 5.2 applies and so T is $(1,1)$-strongly $(2,2)$-concave.

Theorem 5.3. *Let $1 \leqslant p_1, p_2, q_1, q_2$ be real numbers such that $p_1 \leqslant q_1$ and $p_2 \leqslant q_2 = q_1'$, where $1/q_1 + 1/q_1' = 1$. Let (Ω, Σ, μ) be a finite measure space. Let X be an order continuous p_1-convex Banach function space and Y a p_2'-concave order continuous Banach function lattice with the Fatou property, where $1/p_2 + 1/p_2' = 1$. Assume that Y' is also order continuous. The following statements about an operator $T\colon X \to Y$ are equivalent.*

(i) *T is (p_1, p_2)-strongly (q_1, q_2)-concave.*

(ii) *There is a constant $C > 0$ such that for every choice of $(x_k)_{k=1}^n$ in X and $(y_k^*)_{k=1}^n$ in Y^*,*

$$\sup\Big\{\Big|\sum_{k=1}^n \langle T(x_k), y_k^*\rangle\Big| : \sup_{(\alpha_k)\in B^n_{r_2}} \Big\|\Big(\sum_{k=1}^n |\alpha_k y_k^*|^{p_2}\Big)^{1/p_2}\Big\|_{Y^*} \leqslant 1\Big\}$$
$$\leqslant C \sup_{(\beta_k)\in B^n_{r_1}} \Big\|\Big(\sum_{k=1}^n |\beta_k x_k|^{p_1}\Big)^{1/p_1}\Big\|_X.$$

(iii) *T admits the factorization*

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ {\scriptstyle \iota_X}\big\downarrow & & \big\uparrow{\scriptstyle (\iota'_Y)'} \\ S^{q_1}_{X_{p_1}}(\nu_1) & \xrightarrow[T_0]{} & (S^{q_2}_{Y'_{p_2}}(\nu_2))' \end{array}.$$

Proof. The equivalence between (i) and (ii) is just given by Corollary 5.2. Let us show the equivalence of (i) and (iii). Applying Corollary 3.4, we conclude that (i) implies that the bounded bilinear form on $X \times Y'$ given by

$$(x, y') \mapsto \langle T(x), y'\rangle, \qquad (x, y') \in X \times Y',$$

admits a bilinear continuous extension from the product $S^{q_1}_{X_{p_1}}(\nu_1) \times S^{q_2'}_{(Y')_{p_2'}}(\nu_2)$ of Banach function lattices for some probability Borel measure spaces, i.e., there exists a continuous bilinear form $S\colon E \times F \to \mathbb{R}$ such that

$$S(i_X(x), i_{Y'}(y')) = \langle T(x), y'\rangle, \qquad (x, y') \in X \times Y',$$

where $E := S^{q_1}_{X_{p_1}}(\nu_1)$, $F := S^{q_2'}_{(Y')_{p_2'}}(\nu_2)$ and

$$\iota_X\colon X \to E, \qquad \iota_{Y'}\colon Y' \to F$$

are continuous inclusions.

The required factorization follows then by using standard arguments. At first we observe that for any fixed $f \in E$ the formula $\langle T_0(f), \cdot\rangle := S(f, \cdot)$ defines a continuous functional on F with

$$\sup_{g\in F} |\langle T_0(f), g\rangle| = \sup_{g\in F} |S(f, g)| \leqslant \|S\| \|f\|_E\,.$$

This clearly implies that $T_0\colon E \to F$ is a bounded linear operator with $\|T_0\| \leqslant \|S\|$.

Since Y is p_2'-concave, Y' is p_2-convex. Our assumption on Y' yields that $(Y')_{p_2}$ is also order continuous. Consequently, we have that the Köthe adjoint of the inclusion $\iota_{Y'}: Y' \to F$ appearing in the factorization given by Corollary 3.4 for the bilinear map can be considered,

$$(\iota_{Y'})': F' \to (Y')'.$$

Combining the Köthe duality with $Y'' = Y$ (by the Fatou property), we obtain the required factorization shown in (iii). The converse is obvious. ■

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