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A SHORT REMARK ON CONSECUTIVE COINCIDENCES OF A CERTAIN MULTIPLICATIVE FUNCTION

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Abstract: We study integral solutions n of the equation $A(n+k) = A(n)$, where A is a certain multiplicative function related to Jordan's totient function.

Keywords: multiplicative function, Jordan's totient function.

1. Introduction

If $f : \mathbf{N} \to \mathbf{N}$ is a multiplicative number-theoretic function taking positive integral values, such as Euler's totient function, the divisor function or functions closely related, a quite intriguing problem is to study those n for which $f(n) = f(n+1)$ or more generally $f(n) = f(n+k)$ where $k \in \mathbf{N}$ is fixed. If e.g. $f(n) = \phi(n)$ is Euler's totient function, there has been quite a number of interesting results on this subject, cf. [4] and the literature given there. Though in general it seems quite difficult to find explicit solutions of $\phi(n) = \phi(n+k)$, one conjectures that there are always infinitely many [5]. So far this conjecture has not been proved in a single case. Similar assertions concern the divisor function $\sigma(n)$ and in this connection analogous questions about perfect numbers or amicable numbers, cf. e.g. [3].

On the other hand, one may ask for simple and explicit examples of multiplicative functions $f : \mathbf{N} \to \mathbf{N}$, closely related to the classical ones, such that there are only finitely many n with $f(n) = f(n+k)$. In this short note we will construct such a function in a simple way. In fact, using Jordan's totient functions (which are simple generalizations of Euler's phi-function, the definition will be recalled in sect. 2), we will define a multiplicative function $A : \mathbf{N} \to \mathbf{N}$ in a similar way as the classical ones are defined, however with the main differences that the Euler factor at the prime 3 has been slightly modified and also the product of the Euler

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p-factors for all primes p converges to a non-zero value. We shall prove that for each given odd $k \in \mathbf{N}$, there are only finitely many n with $A(n) = A(n+k)$. Once the proper definition of A has been found, the proof is quite simple and only relies on the unequal parity of n and $n+k$ and some elementary estimates.

2. Statement of result

We define $A : \mathbf{N} \to \mathbf{N}$ by

$$A(n) := n^2 \prod_{p|n, p\neq 3} \Big(1 + \frac{1}{p^2}\Big). \tag{1}$$

Then A clearly is multiplicative.

Recall the definition of the ℓ-th Jordan totient function

$$J_\ell(n) := n^\ell \prod_{p|n} \Big(1 - \frac{1}{p^\ell}\Big) \qquad (n \in \mathbf{N})$$

[2, p. 46]. As is easy to see $J_\ell(n)$ counts the number of ℓ-tuples of positive integers all less or equal to n that form a coprime $(\ell+1)$-tuple together with n. Clearly $J_1(n) = \phi(n)$.

Regarding (1) we note that

$$n^2 \prod_{p|n} \Big(1 + \frac{1}{p^2}\Big) = \frac{J_4(n)}{J_2(n)}.$$

We remind the reader that $J_\ell(n)$ is a useful and interesting number-theoretic function which for example (among other things) is demonstrated by the classical identity

$$\# Sp_m(\mathbf{Z}/n\mathbf{Z}) = n^{m^2} \prod_{\ell=1}^{m} J_{2\ell}(n)$$

(cf. [1]). Here as usual $Sp_m \subset GL_{2m}$ denote the symplectic group of degree $2m$.

Theorem. *Let k be a fixed odd positive integer. Then the equation $A(n) = A(n+k)$ has only finitely many solutions $n \in \mathbf{N}$.*

Remark. It would be interesting to investigate solutions of the equations $J_\ell(n) = J_\ell(n+k)$, for fixed ℓ and k, in a similar way as was done in the case $\ell = 1$ for the Euler phi-function.

3. Proof of Theorem

We rewrite the equality $A(n) = A(n+k)$ as

$$\frac{A(n)}{n^2} = \Big(1 + \frac{k}{n}\Big)^2 \frac{A(n+k)}{(n+k)^2}. \tag{2}$$

Let us first suppose that n is even. Then $n+k$ is odd and from (1) and (2) we obtain

$$\Big(1+\frac{1}{2^2}\Big)\prod_{\substack{p\mid n\\ p\neq 3\\ p \text{ odd}}}\Big(1+\frac{1}{p^2}\Big)=\Big(1+\frac{k}{n}\Big)^2\prod_{\substack{p\mid n+k\\ p\neq 3\\ p \text{ odd}}}\Big(1+\frac{1}{p^2}\Big). \tag{3}$$

The left-hand side in (3) is bounded from below by $1+\frac{1}{2^2}$. Hence we find that

$$1+\frac{1}{2^2}<\Big(1+\frac{k}{n}\Big)^2\prod_{\substack{p\mid n+k\\ p\neq 3\\ p \text{ odd}}}\Big(1+\frac{1}{p^2}\Big)<\Big(1+\frac{k}{n}\Big)^2\prod_{\substack{p \text{ odd}\\ p\neq 3}}\Big(1+\frac{1}{p^2}\Big)$$

and hence

$$\Big(1+\frac{1}{2^2}\Big)^2<\Big(1+\frac{k}{n}\Big)^2\prod_{p\neq 3}\Big(1+\frac{1}{p^2}\Big)=\Big(1+\frac{k}{n}\Big)^2\cdot\frac{1}{1+\frac{1}{3^2}}\prod_{p}\Big(1+\frac{1}{p^2}\Big).$$

We have

$$\prod_{p}\Big(1+\frac{1}{p^2}\Big)=\frac{\prod_p(1-\frac{1}{p^4})}{\prod_p(1-\frac{1}{p^2})}=\frac{\zeta(2)}{\zeta(4)}.$$

Inserting the values

$$\zeta(2)=\frac{\pi^2}{6},\qquad \zeta(4)=\frac{\pi^4}{90}$$

we therefore finally get

$$c<\Big(1+\frac{k}{n}\Big)^2 \tag{4}$$

with

$$c:=\Big(1+\frac{1}{2^2}\Big)^2\Big(1+\frac{1}{3^2}\Big)\cdot\frac{\pi^2}{15}=\frac{\pi^2}{8.64}.$$

Since $c>1$ we get a contradiction from (4) when n is large.

If n is odd, then $n+k$ is even, and we can proceed as before with the roles of n and $n+k$ interchanged. This proves our assertion.

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