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MEAN SQUARE ASYMPTOTIC STABILITY IN NONLINEAR STOCHASTIC NEUTRAL VOLTERRA-LEVIN EQUATIONS WITH POISSON JUMPS AND VARIABLE DELAYS

Mimia Benhadri, Halim Zeghdoudi

Abstract: In this paper, we use the contraction mapping principle to obtain mean square asymptotic stability results of a nonlinear stochastic neutral Volterra-Levin equation with Poisson jumps and variable delays. An asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some previous results due to Burton [5], Becker and Burton [4] and Jin and Luo [10], Ardjouni and Djoudi [1]. Finally, an illustrative example is given.

Keywords: fixed points theory, Poisson jumps, asymptotically stable in mean square, neutral stochastic differential equations, variable delays.

1. Introduction

In recent years, the stability of stochastic differential equations has been studied by using Lyapunov functions, which has led to a lot of better results, see for example, [18–20] and so on. Unfortunately, a number of difficulties have been encountered in the study of stability by means of Lyapunov's direct method. Luckily, Luo [14] and Burton et al. [2,3, 5–7] have successfully solved these problems by the application of fixed point theory. Since the method is in its initial stages, we are sure that the investigators will obtain much better results than by using the method of Lyapunov functions which is old and has been previously made in the literature.

Very recently, many scholars have begun to deal with the stability of stochastic delay differential equations by using fixed point theory (see, for example, [1, 6, 14–26]). More precisely, Appleby [3] and Burton [6] (see pp. 315–328) considered

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the almost sure stability for some classical equations by splitting the stochastic differential equation into two equations, one is a fixed stochastic problem and the other is a deterministic stability problem with forcing function. Luo [14] studied, the mean square asymptotic stability for a class of linear scalar neutral stochastic differential equations by means of fixed point theory. Furthermore, Luo [15], Luo and Taniguchi [16] used fixed point theory to study the exponential stability of mild solutions of stochastic partial differential equations with bounded delays and with infinite delays. Wu et al. [21] applied fixed point theory to study the of a class nonlinear neutral stochastic differential equations with variable time delays. Dezhi Liu, Guiyuan Yang, Wei Zhang [9], studied, the mean square asymptotic stability of a nonlinear neutral stochastic differential equation with Poisson jumps.

As we know, Volterra first investigated the unperturbed Volterra–Levin equation in [12] and later Levin used this equation to model a certain biological problem in [13]. Recently, a few researchers began to study the stability of stochastic Volterra–Levin equations. Without the aid of the Lyapunov's method, the stability of stochastic Volterra–Levin equations were discussed in [3], [6], and [16]. More explicitly, by using the fixed point theory, Appleby and Burton studied the almost sure stability of stochastic Volterra–Levin equations under several specific conditions in [3] and [6]. In [16], Luo applied the same way to discuss the mean square exponential stability under a set of weaker conditions. However, to the best of the authors' knowledge, till now, stochastic Volterra–Levin equations with Poisson jumps have not been studied owing to the difficulty of mathematics.

The Poisson jumps have become very popular in recent years (see [9, 11]), because it is extensively used to model many of the phenomena arising in areas such as economics, finance, physics, biology, medicine and other sciences. For example, if a system jumps from a "normal state" to a "bad state", the strength of the system is random. It is natural and necessary to include a jump term in the stochastic differential equations. Therefore, in this paper, the first attempt is made towards investigating the stability of stochastic delay differential equations with Poisson jumps by using fixed point theory.

This paper is organized as follows: Section 1 is devoted to introduce and give basic preliminaries on stochastic neutral Volterra-Levin equation with variable delay. In Section 2 we give the main result about mean square asymptotic stability and its proof. In Section 3, some well-known results are generalized and an example is given to illustrate our theory.

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P\right)$ be a probability space with a filtration satisfying the usual conditions, i.e., the filtration is continuous on the right and $\mathcal{F}_0$ contains all P-zero sets. Let $\{W(t), t \geqslant 0\}$ be a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $\{N(dt, du), t \in \mathbb{R}^+, u \in \mathbb{R}\}$ is a centred Poisson random measure with parameter $\pi(du)\,dt$.

Now, we consider the following stochastic neutral Volterra-Levin equation with variable delays and Poisson jumps:

$$d\left[x(t)-Q\left(t,x\left(t-\tau_1\left(t\right)\right)\right)\right]=-\left(\int_{t-\tau_1(t)}^{t}a(t,s)x(s)ds\right)dt$$
$$+G\left(t,x\left(t\right),x(t-\tau_2\left(t\right))\right)dW\left(t\right) \tag{1.1}$$
$$+\int_{-\infty}^{+\infty}h\left(t,x\left(t\right),x(t-\tau_3\left(t\right)),u\right)\widetilde{N}\left(dt,du\right),\quad t\geqslant 0,$$

with the initial condition

$$x(t)=\psi\left(t\right)\qquad \text{for}\ \ t\in\left[m\left(0\right),0\right],$$

where $\psi\in C\left(\left[m\left(0\right),0\right],\mathbb{R}\right)$ and

$$m\left(0\right)=\min\left\{\inf\left(s-\tau_j\left(s\right),s\geqslant 0\right),j=1,2,3\right\}.$$

Where $Q:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}, G:\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, h:\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$ are continuous, $x:\left[m\left(0\right),\infty\right[\times\Omega\to\mathbb{R}$, and $a\in C\left(\mathbb{R}^+\times\left[m\left(0\right),\infty\right[,\mathbb{R}\right)$, and $\tau_j\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ satisfy $t-\tau_j\left(t\right)\to\infty$ as $t\to\infty$ for $j=1,2,3$. $\widetilde{N}\left(dt,du\right)=N\left(dt,du\right)-\pi\left(du\right)dt$ is a compensated Poisson random measure which is independent of $\left\{W\left(t\right)\right\}$. Suppose $\int_{-\infty}^{+\infty}\pi\left(du\right)<\infty$ and the following conditions are satisfied:

(i) There exists a positive constant $K_1>0$ such that for all $x,y\in\mathbb{R}$,

$$\left|Q\left(t,x\right)-Q\left(t,y\right)\right|\leqslant K_1\left|x-y\right|. \tag{1.2}$$

We also assume that

$$Q\left(t,0\right)=0. \tag{1.3}$$

(ii) The global Lipschitz condition: there exists a positive constant $K_2>0$ such that

$$\left|G\left(t,x_1,y_1\right)-G\left(t,x_2,y_2\right)\right|^2\vee\int_{-\infty}^{+\infty}\left|h\left(t,x_1,y_1,u\right)-h\left(t,x_2,y_2,u\right)\right|^2\pi\left(du\right)$$
$$\leqslant K_2\left(\left|x_1-x_2\right|^2+\left|y_1-y_2\right|^2\right), \tag{1.4}$$

where $x_1,y_1,x_2,y_2\in\mathbb{R}$. We also assume that

$$G\left(t,0,0\right)=h\left(t,0,0,u\right)=0. \tag{1.5}$$

Definition 1.1. The zero solution of (1.1) is called mean square stable if for any $\varepsilon>0$, there exists a $\delta\left(\varepsilon\right)>0$ such that the inequality $\left\|\psi\right\|^2<\delta$ guarantees $E\left(\sup_{s\in[0,t]}\left|x\left(s\right)\right|^2\right)<\varepsilon$ for all $t>0$. If, in addition, we have $\lim_{t\to\infty}E\left(\sup_{s\in[0,t]}\left|x\left(s\right)\right|^2\right)=0$ for all initial condition, then the zero solution of (1.1) is called asymptotically mean square stable.

Special cases of equation (1.1) have been investigated by many authors. For example, Burton in [5], Becker and Burton in [4], Jin and Luo in [10] studied the equation

$$x'(t) = -\int_{t-\tau_1(t)}^{t} a(t,s)x(s)ds, \tag{1.6}$$

and have respectively proved the following theorems.

Theorem 1.1 (Burton [5]). *Suppose that $\tau_1(t) = r$ and there exists a constant $\alpha < 1$ such that*

$$2\int_{t-r}^{t} |A(t,s)|\, ds \leqslant \alpha \qquad \text{for all } \ t \geqslant 0,$$

and

$$\int_0^t A(s,s)ds \to \infty \qquad \text{as } \ t \to \infty,$$

where

$$A(t,s) = \int_{t-s}^{r} a(u+s,s)du \qquad \text{with } \ A(t,t) = \int_0^r a(u+t,t)du.$$

Then the zero solution of (1.6) is asymptotically stable.

Theorem 1.2 (Becker and Burton [4]). *Suppose that τ_1 is differentiable, $t-\tau_1(t)$ is strictly increasing, and there exist constants $k \geqslant 0$, $\alpha \in (0,1)$ such that for $t \geqslant 0$,*

$$-\int_0^t G(s,s)ds \leqslant k, \tag{1.7}$$

$$\int_{t-\tau_1(t)}^{t} |G(t,s)|\, ds + \int_0^t e^{-\int_s^t G(u,u)du} |G(s,s)| \left(\int_{s-\tau_1(s)}^{s} |G(s,u)|\, du \right) ds \leqslant \alpha, \tag{1.8}$$

with

$$G(t,s) = \int_t^{f(s)} a(u,s)du, \qquad G(t,t) = \int_t^{f(t)} a(u,t)du.$$

where f is the inverse function of $t-\tau_1(t)$. Then for each continuous initial function $\psi : [m(0),0] \to \mathbb{R}$, there is a unique continuous function $x : [m(0),\infty) \to \mathbb{R}$ with $x(t) = \psi(t)$ on $[m(0),0]$ that satisfies (1.6) on $[0,\infty)$. Moreover, x is bounded on $[m(0),\infty)$. Furthermore, the zero solution of (1.6) is stable at $t = 0$. If, in addition,

$$\int_0^t G(s,s)ds \to \infty \qquad \text{as } \ t \to \infty, \tag{1.9}$$

then $x(t) \to 0$ as $t \to \infty$.

Theorem 1.3 (Jin and Luo [10]). *Let τ_1 be differentiable. Suppose that there exist constants $k \geqslant 0, \alpha \in (0,1)$ and a function $h \in C(\mathbb{R}^+, \mathbb{R})$ such that for $t \geqslant 0$,*

$$-\int_0^t h(s)ds \leqslant k, \tag{1.10}$$

and

$$\begin{aligned}\int_{t-\tau_1(t)}^t &|h(s) + B(t,s)|\,ds \\ &+ \int_0^t e^{-\int_s^t h(u)du} \left|\left[h(s-\tau_1(s)) + B(s, s-\tau_1(s))\right](1-\tau_1'(s))\right| ds \\ &+ \int_0^t e^{-\int_s^t h(u)du} |h(s)| \left(\int_{s-\tau_1(s)}^s |h(u) + B(s,u)|\,du\right) ds \leqslant \alpha,\end{aligned} \tag{1.11}$$

where

$$B(t,s) = \int_t^s a(u,s)du \qquad \text{with} \;\; B(t, t-\tau_1(t)) = \int_t^{t-\tau_1(t)} a(u, t-\tau_1(t))du. \tag{1.12}$$

Then for each continuous initial function $\psi : [m(0), 0] \to \mathbb{R}$, there is a unique continuous function $x : [m(0), \infty) \to \mathbb{R}$ with $x(t) = \psi(t)$ on $[m(0), 0]$ that satisfies (1.6) on $[0, \infty)$. Moreover, x is bounded on $[m(0), \infty)$. Furthermore, the zero solution of (1.6) is stable at $t = 0$. If , in addition,

$$\int_0^t h(s)ds \to \infty \qquad \text{as} \;\; t \to \infty, \tag{1.13}$$

then $x(t) \to 0$ as $t \to \infty$.

In a recent work, we have studied the nonlinear neutral integro-differential equation with variable delay

$$x'(t) = -\int_{t-\tau_1(t)}^t a(t,s)x(s)ds + Q\left(t, x(t-\tau_1(t))\right), \tag{1.14}$$

and have obtained the following result.

Theorem 1.4 (Ardjouni and Djoudi [1]). *Suppose (1.2) and (1.3) hold. Let τ_1 be differentiable, and suppose that there exist continuous function $h : [m(0), \infty) \to \mathbb{R}$ and a constant and a constant $\alpha \in (0,1)$ such that for $t \geqslant 0$*

$$\liminf_{t\to\infty} \int_0^t h(s)ds > -\infty, \tag{1.15}$$

and

$$\begin{aligned} K_1 &+ \int_{t-\tau_1(t)}^{t} |h(s) + B(t,s)|\, ds \\ &+ \int_0^t e^{-\int_s^t h(u)du} \left[|(h(s-\tau_1(s)) + B(s,s-\tau_1(s)))\,(1-\tau_1'(s))| + K_1 \,|h\,(s)| \right] ds \\ &+ \int_0^t e^{-\int_s^t h(u)du} \,|h\,(s)| \left(\int_{s-\tau_1(s)}^{s} |h(u) + B(s,u)|\, du \right) ds \leqslant \alpha, \end{aligned} \tag{1.16}$$

where B *are given by* (1.12). *Then the zero solution of* (1.14) *is asymtotically stable if and only if*

$$\int_0^t h\,(t) \to \infty \qquad as \quad t \to \infty. \tag{1.17}$$

Remark 1.1. The result of Becker and Burton obtained in Theorem 1.2 requires that $t - \tau_1\,(t)$ is strictly increasing. However, in Theorem 1.3, this condition is clearly removed. Also, the conditions of stability in Theorem 1.3 are less restrictive than Theorem 1.2. That is, Theorem 1.3 improves Theorems 1.1 and 1.2. Moreover, if we let $Q\,(t,x) = 0$ in (1.14) then the equation reduces to (1.6). Consequently, Theorem 1.4 is a generalization of both theorems 1.1, 1.2 and 1.3.

Our objective here is to improve Theorem 1.3 and extend it to investigate a wide class of stochastic neutral Volterra-Levin equation with Poisson jumps and variable delays presented in (1.1). To do this we define a suitable continuous function H (see Theorem 2.1 below) and find conditions for H, with no need of further assumptions on the inverse of the delay $t - \tau_1\,(t)$, so that for a given continuous initial function a mapping P for (1.1) is constructed in such a way to map a complete metric space S in itself and in which P possesses a fixed point. This procedure will enable us to establish and prove an mean square asymptotically stable for the zero solution of (1.1) with a necessary and sufficient condition and with less restrictive conditions. The results obtained in this paper improve and generalize the main results in [1, 4, 5,10]. We provide an example to illustrate our claim.

2. Main results

For each $\psi \in C([m\,(0)\,,0]\,,\mathbb{R})$, a solution of (1.1) through $(0,\psi)$ is a continuous function $x : [m\,(0)\,,\sigma) \to \mathbb{R}$ for some positive constant $\sigma > 0$ such that x satisfies (1.1) on $[0,\sigma)$ and $x = \psi$ on $[m\,(0)\,,0]$. We denote such a solution by $x(t) = x(t,0,\psi)$. We define $\|\psi\| = \max\{|\psi\,(t)| : m\,(0) \leqslant t \leqslant 0\}$. Our aim here is to generalize Theorem 1.3 to equation (1.1) by giving a necessary and sufficient condition for an mean square asymptotically stable of the zero solution. It is known that studying the stability of an equation using a fixed point technic involves the construction of a suitable fixed point mapping. This can be an arduous task. So, to construct our mapping, we begin by transforming (1.1) to a more tractable, but equivalent, equation, which we then invert to obtain an equivalent integral equation from which we derive a fixed point mapping. After that, we define a suitable complete space, depending on the initial condition, so that the mapping is a contraction. Using Banach's contraction mapping principle, we obtain a solution for this mapping, and hence a solution for (1.1), which is mean square asymptotically stable.

First, we have to transform (1.1) into an equivalent equation that possesses the same basic structure and properties to which we apply the variation of parameters to define a fixed point mapping.

Lemma 2.1. *Equation* (1.1) *is equivalent to*

$$\begin{aligned} d\left[x(t) - Q\left(t, x\left(t - \tau_1\left(t\right)\right)\right)\right] = {} & B(t, t - \tau_1\left(t\right))(1 - \tau_1'\left(t\right))x(t - \tau_1\left(t\right)) \\ & + \frac{d}{dt}\int_{t-\tau_1(t)}^{t} B(t,s)x(s)ds \\ & + G\left(t, x\left(t\right), x(t - \tau_2\left(t\right))\right) dW\left(t\right) \\ & + \int_{-\infty}^{+\infty} h\left(t, x\left(t\right), x(t - \tau_3\left(t\right)), u\right) \widetilde{N}\left(dt, du\right), \ t \geqslant 0, \end{aligned} \tag{2.1}$$

where

$$B(t,s) = \int_t^s a(u,s)du \qquad \text{and} \qquad B(t, t-\tau_1\left(t\right)) = \int_t^{t-\tau_1(t)} a(u, t-\tau_1\left(t\right))du.$$

Proof. Differentiating the integral term in (2.1) , we obtain

$$\begin{aligned} \frac{d}{dt}\int_{t-\tau_1(t)}^{t} B(t,s)x(s)ds = {} & B(t,t)x(t) - B(t, t-\tau_1\left(t\right))(1-\tau_1'\left(t\right))x(t-\tau_1\left(t\right)) \\ & + \int_{t-\tau_1(t)}^{t} \frac{\partial}{\partial t}B(t,s)x(s)ds. \end{aligned}$$

Substituting this into (2.1) , it follows that (2.1) is equivalent to (1.1) provided B satisfies the following conditions

$$B(t,t) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial t}B(t,s) = -a(t,s). \tag{2.2}$$

This equality implies

$$B(t,s) = -\int_0^t a(u,s)du + \phi\left(s\right), \tag{2.3}$$

for some function ϕ, and $B(t,s)$ must satisfy

$$B(t,t) = -\int_0^t a(u,t)du + \phi\left(t\right) = 0.$$

Consequently,

$$\phi\left(t\right)=\int_{0}^{t}a(u,t)du.$$

Substituting this into (2.3) , we obtain

$$B(t,s)=-\int_{0}^{t}a(u,s)du+\int_{0}^{s}a(u,s)du=\int_{t}^{s}a(u,s)du.$$

This definition of B satisfies (2.2) . Consequently, (1.1) is equivalent to (2.1). ■

Theorem 2.1. *Suppose that τ_1 is differentiable, and there exist continuous functions $H:[m(0),\infty[\rightarrow\mathbb{R}$ and a constant $\alpha\in(0,1)$ such that for $t\geqslant 0$*

$$\lim_{t\rightarrow\infty}\inf\int_{0}^{t}H(s)ds>-\infty, \tag{2.4}$$

$$\begin{aligned}K_1&+\int_{s-\tau_1(s)}^{s}\left|H(z)+B(s,z)\right|dz\\&+\int_{0}^{s}e^{-\int_z^s H(u)du}\left[\left|\left(H(z-\tau_1(z))+B(z,z-\tau_1(z))\left(1-\tau_1'(z)\right)\right)\right|+K_1\left|H\left(z\right)\right|\right]dz\\&+\int_{0}^{s}e^{-\int_z^s H(u)du}\left|H\left(z\right)\right|\left(\int_{z-\tau_1(z)}^{z}\left|H(u)+B(z,u)\right|du\right)dz\\&+2\left(2K_2\int_{0}^{s}e^{-2\int_z^s H(u)du}dz\right)^{\frac{1}{2}}\leqslant\alpha,\end{aligned} \tag{2.5}$$

and for $s\in[0,t]$ and a positive constant $\alpha<1$, the following inequality holds: where

$$B(t,s)=\int_{t}^{s}a(u,s)du\qquad\textit{with}\ \ B(t,t-\tau_1\left(t\right))=\int_{t}^{t-\tau_1(t)}a(u,t-\tau_1\left(t\right))du.$$

Then the zero solution of (1.1) is mean square asymtotically stable if and only if

$$\int_{0}^{t}H\left(s\right)ds\rightarrow\infty\qquad as\ \ t\rightarrow\infty. \tag{2.6}$$

Proof. First, we suppose that (2.6) holds. We set

$$K=\sup_{t\geqslant 0}\left\{e^{-\int_0^t H(s)ds}\right\}. \tag{2.7}$$

Denote by S the Banach space of all $\mathcal{F}$-adapted processes $x(t,\omega) : [m(0), \infty[\times \Omega \to \mathbb{R}$ which are almost surely continuous in t with norm

$$S = \left\{ \begin{array}{r} x \in C\left([m(0), \infty[\times \Omega : \mathbb{R}\right) : \|x\|_{[0,t]} = E\left(\sup_{s\in[0,t]} |x(s)|^2\right)^{\frac{1}{2}} \to 0 \\ \text{as } t \to \infty,\ x(s,\omega) = \psi(s) \text{ for } s \in [m(0), 0] \end{array} \right\}. \tag{2.8}$$

Rewrite (1.1) in the following equivalent form

$$\begin{aligned} d\left[x(t) - Q\left(t, x\left(t - \tau_1(t)\right)\right)\right] = {} & B(t, t-\tau_1(t))(1-\tau_1'(t))x(t-\tau_1(t)) \\ & + \frac{d}{dt}\int_{t-\tau_1(t)}^{t} B(t,s)x(s)ds \\ & + G\left(t, x(t), x(t-\tau_2(t))\right) dW(t) \\ & + \int_{-\infty}^{+\infty} h\left(t, x(t), x(t-\tau_3(t)), u\right) \widetilde{N}(dt, du),\ t \geqslant 0 \end{aligned} \tag{2.9}$$

Multiplying both sides of (2.9) by $\exp\left(\int_0^t H(u)du\right)$ and integrating with respect to s from 0 to t, we obtain

$$\begin{aligned} x(t) = {} & \left(\psi(0) - Q(0, \psi(-\tau_1(0)))\right) e^{-\int_0^t H(u)du} + Q(t, x(t-\tau_1(t)) \\ & + \int_0^t e^{-\int_s^t H(u)du} d\left(\int_{s-\tau_1(s)}^{s} [H(u) + B(s,u)]\, ds\right)) x(u)du \\ & + \int_0^t e^{-\int_s^t H(u)du} \left[H(s-\tau_1(s)) + B(s, s-\tau_1(s))\right](1-\tau_1'(s))\, x(s-\tau_1(s))ds \\ & - \int_0^t e^{-\int_s^t H(u)du} H(s)\, Q\left(s, x(s-\tau_1(s))\right) ds \\ & + \int_0^t e^{-\int_s^t H(u)du} G\left(s, x(s), x(s-\tau_2(s))\right) dW(s) \\ & + \int_0^t \int_{-\infty}^{+\infty} e^{-\int_s^t H(u)du} h\left(s, x(s), x(s-\tau_3(s)), u\right) \widetilde{N}(ds, du). \end{aligned}$$

Performing an integration by parts, we obtain

$$\begin{aligned}
x(t) &= \left(\psi(0) - Q(0,\psi(-\tau_1(0)) - \int_{-\tau_1(0)}^{0} [H(s)+B(0,s)]\,\psi(s)\,ds\right) e^{-\int_0^t H(u)du} \\
&\quad + Q(t, x(t-\tau_1(t)) + \int_{t-\tau_1(t)}^{t} [H(s)+B(t,s)]\,x(s)\,ds \\
&\quad + \int_0^t e^{-\int_s^t H(u)du} [H(s-\tau_1(s)) + B(s,s-\tau_1(s))]\,(1-\tau_1'(s))\,x(s-\tau_1(s))ds \\
&\quad - \int_0^t e^{-\int_s^t H(u)du} H(s)\,Q(s, x(s-\tau_1(s)))\,ds \\
&\quad - \int_0^t e^{-\int_s^t H(u)du} H(s) \left(\int_{s-\tau_1(s)}^{s} [H(u)+B(s,u)]\,x(u)\,du\right) ds \\
&\quad + \int_0^t e^{-\int_s^t H(u)du} G(s, x(s), x(s-\tau_2(s)))\,dW(s) \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} e^{-\int_s^t H(u)du} h(s, x(s), x(s-\tau_3(s)), u)\,\widetilde{N}(ds,du).
\end{aligned}$$

Now use this equality to define an operator $P : S \to S$ by $(Px)(t) = \psi(t)$ if $t \in [m(0), 0]$ and for $t \geqslant 0$, we let

$$\begin{aligned}
(Px)(t) &= \left(\psi(0) - Q(0,\psi(-\tau_1(0)) - \int_{-\tau_1(0)}^{0} [H(s)+B(0,s)]\,\psi(s)\,ds\right) e^{-\int_0^t H(u)du} \\
&\quad + Q(t, x(t-\tau_1(t)) + \int_{t-\tau_1(t)}^{t} [H(s)+B(t,s)]\,x(s)\,ds \\
&\quad + \int_0^t e^{-\int_s^t H(u)du} [H(s-\tau_1(s)) + B(s,s-\tau_1(s))](1-\tau_1'(s))x(s-\tau_1(s))ds \\
&\quad - \int_0^t e^{-\int_s^t H(u)du} H(s)\,Q(s, x(s-\tau_1(s)))\,ds \\
&\quad - \int_0^t e^{-\int_s^t H(u)du} H(s) \left(\int_{s-\tau_1(s)}^{s} [H(u)+B(s,u)]\,x(u)\,du\right) ds \\
&\quad + \int_0^t e^{-\int_s^t H(u)du} G(s, x(s), x(s-\tau_2(s)))\,dW(s) \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} e^{-\int_s^t H(u)du} h(s, x(s), x(s-\tau_3(s)), u)\,\widetilde{N}(ds,du) \\
&= \sum_{i=1}^{3} I_i(t), \qquad (2.10)
\end{aligned}$$

where

$$I_1(t) = \left(\psi(0) - Q(0, \psi(-\tau_1(0)) - \int_{-\tau_1(0)}^{0} [H(s) + B(0,s)]\,\psi(s)\,ds\right) e^{-\int_0^t H(u)du}$$
$$+ Q(t, x(t - \tau_1(t)) + \int_{t-\tau_1(t)}^{t} [H(s) + B(t,s)]\,x(s)\,ds$$
$$+ \int_0^t e^{-\int_s^t H(u)du} [H(s-\tau_1(s)) + B(s, s-\tau_1(s))]\,(1-\tau_1'(s))\,x(s-\tau_1(s))ds$$
$$- \int_0^t e^{-\int_s^t H(u)du} H(s)\,Q(s, x(s-\tau_1(s)))\,ds$$
$$- \int_0^t e^{-\int_s^t H(u)du} H(s) \left(\int_{s-\tau_1(s)}^{s} [H(u) + B(s,u)]\,x(u)\,du\right) ds,$$

$$I_2(t) = \int_0^t e^{-\int_s^t H(u)du} G(s, x(s), x(s-\tau_2(s)))\,dW(s),$$

$$I_3(t) = \int_0^t \int_{-\infty}^{+\infty} e^{-\int_s^t H(u)du} h(s, x(s), x(s-\tau_3(s)), u)\,\widetilde{N}(ds, du).$$

In order to the conclusion, we give four steps as follows.

First step: we must prove the mean square continuity of P on $[0, \infty[$. Let $x \in S, t_1 > 0$, and $|r|$ be sufficiently small, then

$$E|P(x)(t_1 + r) - P(x)(t_1)|^2 \leqslant 3\sum_{i=1}^{3} E|I_i(t_1 + r) - I_i(t_1)|^2.$$

It is easy to obtain that

$$E|I_1(t_1 + r) - I_1(t_1)|^2 \to 0 \qquad \text{as } \ r \to 0.$$

Furthermore,

$$E|I_2(t_1 + r) - I_2(t_1)|^2$$
$$\leqslant 2E\left|\int_0^{t_1} \left(e^{-\int_{t_1}^{t_1+r} H(u)du} - 1\right) e^{-\int_0^{t_1} H(u)du} G(s, x(s), x(s-\tau_2(s)))dW(s)\right|^2$$
$$+ 2E\left|\int_{t_1}^{t_1+r} e^{-\int_s^{t_1+r} H(u)du} G(s, x(s), x(s-\tau_2(s)))dW(s)\right|^2$$
$$\leqslant 2E\int_0^{t_1} \left(e^{-\int_{t_1}^{t_1+r} H(u)du} - 1\right)^2 e^{-2\int_0^{t_1} H(u)du} |G(s, x(s), x(s-\tau_2(s)))|^2\,ds$$
$$+ 2E\int_{t_1}^{t_1+r} e^{-2\int_s^{t_1+r} H(u)du} |G(s, x(s), x(s-\tau_2(s)))|^2\,ds \to 0,$$

as $r \to \infty$, and

$$
\begin{aligned}
&E\left|I_3\left(t_1+r\right)-I_3\left(t_1\right)\right|^2 \\
&\quad\leqslant 2E\left|\int_0^{t_1}\int_{-\infty}^{+\infty}\left(e^{-\int_{t_1}^{t_1+r}H(u)du}-1\right)e^{-\int_0^{t_1}H(u)du}\right. \\
&\qquad\left.\times h(s,x\left(s\right),x(s-\tau_3\left(s\right)),u)\widetilde{N}\left(ds,du\right)\right|^2 \\
&\qquad+2E\left|\int_{t_1}^{t_1+r}\int_{-\infty}^{+\infty}e^{-\int_s^{t_1+r}H(u)du}h(s,x\left(s\right),x(s-\tau_3\left(s\right)))\widetilde{N}\left(ds,du\right)\right|^2 \\
&\quad\leqslant 2E\int_0^{t_1}\int_{-\infty}^{+\infty}\left(e^{-\int_{t_1}^{t_1+r}H(u)du}-1\right)^2e^{-2\int_0^{t_1}H(u)du} \\
&\qquad\times\left|h(s,x\left(s\right),x(s-\tau_3\left(s\right)),u)\right|^2\pi\left(du\right)ds \\
&\qquad+2E\int_{t_1}^{t_1+r}\int_{-\infty}^{+\infty}e^{-2\int_s^{t_1+r}H(u)du}\left|h(s,x\left(s\right),x(s-\tau_3\left(s\right)),u)\right|^2\pi\left(du\right)ds\to 0,
\end{aligned}
$$

as $r \to 0$. Therefore, P is mean square continuous on $[0, \infty[$.

Second step: we prove that $P(S) \subset S$, that is to say, for any $\varepsilon > 0$, there exists $T_1 > 0$, such that $s \geqslant T_1$, then $E\left|x(s)\right|^2 < \varepsilon$ and $E\left|x(t-\tau_i(t))\right|^2 < \varepsilon, i = 1, 2, 3$ and $x(t) \to 0$ as $t \to \infty$. Hence

$$
\begin{aligned}
E\sup_{s\in[0,t]}\left|I_2\left(s\right)\right|^2 &\leqslant E\int_0^t e^{-2\int_s^t H(u)du}\left|G(s,x\left(s\right),x(s-\tau_2\left(s\right)\right|^2ds \\
&\leqslant E\int_0^{T_1}e^{-2\int_s^t H(u)du}K_2\left(\left|x\left(s\right)\right|^2+\left|x(s-\tau_2\left(s\right)\right|^2\right)ds \\
&\quad+E\int_{T_1}^t e^{-2\int_s^t H(u)du}K_2\left(\left|x\left(s\right)\right|^2+\left|x(s-\tau_2\left(s\right)\right|^2\right)ds \\
&\leqslant 2E\left(\sup_{\sigma\geqslant m(0)}\left|x(\sigma)\right|^2\right)K_2\int_0^{T_1}e^{-2\int_s^t H(u)du}ds \\
&\quad+2\varepsilon K_2\int_{T_1}^t e^{-2\int_s^t H(u)du}ds. \qquad (2.11)
\end{aligned}
$$

By condition (2.5) and (2.6) , there exists a $T_2 > T_1$ such that $t \geqslant T_2$, we obtain

$$2E\left(\sup_{\sigma\geqslant m(0)} |x(\sigma)|^2\right) K_2 \int_0^{T_1} e^{-2\int_s^t H(u)du} ds < (1-\alpha)\,\varepsilon.$$

Now we can get $E\left(\sup_{s\in[0,t]} |I_2\,(s)|^2\right) < (1-\alpha)\,\varepsilon + \alpha\varepsilon = \varepsilon$. Similarly,

$$\begin{aligned}
&E\left(\sup_{s\in[0,t]} |I_3\,(s)|^2\right) \\
&\qquad \leqslant E\int_0^{T_1}\int_{-\infty}^{+\infty} e^{-2\int_s^t H(u)du} \left|h(s,x\,(s)\,,x(s-\tau_3\,(s)\right|^2 \pi\,(du)\,ds \\
&\qquad\quad + E\int_{T_1}^{t}\int_{-\infty}^{+\infty} e^{-2\int_s^t H(u)du} \left|h(s,x\,(s)\,,x(s-\tau_3\,(s)\right|^2 \pi\,(du)\,ds \\
&\qquad \leqslant 2E\left(\sup_{\sigma\geqslant m(0)} |x(\sigma)|^2\right) K_2 \int_0^{T_1} e^{-2\int_s^t H(u)du} ds \\
&\qquad\quad + 2\varepsilon K_2 \int_{T_1}^{t} e^{-2\int_s^t H(u)du} ds \\
&\qquad < (1-\alpha)\,\varepsilon + \alpha\varepsilon = \varepsilon,
\end{aligned} \tag{2.12}$$

and it is very easy to get $E\left(\sup_{s\in[0,t]} |I_1\,(s)|^2\right) \to 0$ as $t\to\infty$, thus $P\,(S)\subset S$.

Third step: now, we will show that P is a contractive mapping. From the condition (2.5) , there exists some constant $L > 0$, such that

$$\begin{aligned}
&\left(1+\frac{1}{L}\right)\left(K_1 + \int_{t-\tau_1(t)}^{t} |H(s)+B(t,s)|\,ds\right. \\
&+\int_0^t e^{-\int_s^t H(u)du}\left\{\left|\left[H(s-\tau_1(s)) + B(s,s-\tau_1(s))\right](1-\tau_1'(s))\right| + K_1\,|H\,(s)|\right\} ds \\
&\left.+\int_0^t e^{-\int_s^t H(u)du}\,|H\,(s)|\left(\int_{s-\tau_1(s)}^{s} |H(u)+B(s,u)|\,du\right) ds\right)^2 \\
&+\left(1+\frac{1}{L}+\frac{1+L}{1-L}\right)\left(2K_2\int_0^t e^{-2\int_s^t H(u)du} ds\right) \leqslant \alpha^2 < 1.
\end{aligned} \tag{2.13}$$

For any $x, y \in S$, it follows from (2.10), conditions (2.5) and (2.6), an

$$
\begin{aligned}
& E \sup_{s\in[0,t]} \left|(Px)(s) - (Py)(s)\right|^2 \\
& \leqslant E \sup_{s\in[0,t]} \Bigg| Q(s, x(s-\tau_1(s)) - Q(s, y(s-\tau_1(s)) \\
& \quad + \int_{s-\tau_1(s)}^{s} [H(z) + B(s,z)] (x(z) - y(z)) \, dz \\
& \quad + \int_0^s e^{-\int_z^s H(u)du} [H(z-\tau_1(z)) + B(z, z-\tau_1(z))] (1-\tau_1'(z)) \\
& \quad \times (x(z-\tau_1(z)) - y(z-\tau_1(z))) \, dz \\
& \quad - \int_0^s e^{-\int_z^s H(u)du} H(z) (Q(s, x(s-\tau_1(s)) - Q(s, y(s-\tau_1(s)))) \, dz \\
& \quad - \int_0^s e^{-\int_z^s H(u)du} H(z) \left(\int_{z-\tau_1(z)}^{z} [H(u) + B(z,u)] (x(u) - y(u)) \, du \right) dz \\
& \quad + \int_0^s e^{-\int_z^s H(u)du} [G(z, x(z), x(z-\tau_2(z))) - G(z, y(z), y(z-\tau_2(z)))] \, dW(z) \\
& \quad + \int_0^s \int_{-\infty}^{+\infty} e^{-\int_z^s H(u)du} \\
& \quad \times [h(z, x(z), x(z-\tau_3(z)), u) - h(z, y(z), y(z-\tau_3(z)), u)] \widetilde{N}(dz, du) \Bigg|^2 \\
& \leqslant E \sup_{s\in[0,t]} |x(s) - y(s)|^2 \sup_{s\in[0,t]} \Bigg\{ \left(1 + \frac{1}{L}\right) \Bigg(K_1 + \int_{s-\tau_1(s)}^{s} [H(z) + B(s,z)] \, dz \\
& \quad + \int_0^s e^{-\int_z^s H(u)du} [H(z-\tau_1(z)) + B(z, z-\tau_1(z))] (1-\tau_1'(z)) + K_1 |H(z)| \, dz \\
& \quad + \int_0^s e^{-\int_z^s H(u)du} H(z) \left(\int_{z-\tau_1(z)}^{z} |H(u) + B(z,u)| \, du \right) dz \Bigg)^2 \\
& \quad + \left(1 + \frac{1}{L}\right) \left(2K_2 \int_0^s e^{-2\int_z^s H(u)du} dz \right) + \left(\frac{1+L}{1-L} \right) \left(2K_2 \int_0^s e^{-2\int_z^s H(u)du} dz \right) \Bigg\} \\
& \leqslant \alpha^2 E \sup_{s\in[0,t]} |x(s) - y(s)|^2 .
\end{aligned}
$$

So P is contraction mapping.

Now we can observe that P has unique fixed points $x(t)$ in S by the contraction mapping principle, it is a solution of (1.1) with $x(s) = \psi(s)$ on $s \in [m(0), 0]$ and $E|x(t)|^2 \to 0$ as $t \to \infty$.

Fourth step: we will prove that the zero solution of (1.1) is mean square asymptotic stability. Let $\varepsilon > 0$ be given and choose $\delta > 0$ $(\delta < \varepsilon)$ satisfying the following condition:

$$(1+L)\,\delta\left(1+K_1+\int_{-\tau_1(0)}^{0}|H(s)+B(0,s)|\,ds\right)^2 e^{-2\int_0^{t^*}H(u)du} + \left(1+\frac{1}{L}\right)\alpha^2\varepsilon<\varepsilon, \quad (2.14)$$

where L is defined in (2.14). If $x\,(t)=x\,(t,0,\psi)$ is a solution of (1.1) with $\|\psi\|^2<\delta$, then $x\,(t)=(Px)\,(t)$ as defined in (2.10). We claim that $E\,|x\,(t)|^2<\varepsilon$ for all $t\geqslant 0$. Notice that $E\,|x\,(t)|^2=\|\psi\|^2<\varepsilon$ for $t\in[m\,(0)\,,0]$. If there exists $t^*>0$ such that $E\,|x\,(t^*)|^2=\varepsilon$ and $E\,|x\,(t)|^2<\varepsilon$ for $t\in[m\,(0)\,,t^*)$, then (2.10) and (2.13) imply that

$$\begin{aligned}
&E\,|x\,(t^*)|^2\\
&\leqslant(1+L)\,\|\psi\|^2\left(1+K_1+\int_{-\tau_1(0)}^{0}|H(s)+B(0,s)|\,ds\right)^2 e^{-2\int_0^{t^*}H(u)du}\\
&\quad+\varepsilon\left(1+\frac{1}{L}\right)\left\{\left(1+\frac{1}{L}\right)\left(K_1+\int_{t^*-\tau_1(t^*)}^{t^*}|H(s)+B(t^*,s)|\,ds\right.\right.\\
&\quad+\int_0^{t^*}e^{-\int_s^{t^*}H(u)du}\,|[H(s-\tau_1(s))+B(s,s-\tau_1(s))]\,(1-\tau_1'(s))|+K_1\,|H(s)|\,ds\\
&\quad\left.+\int_0^{t^*}e^{-\int_s^{t^*}H(u)du}\,|H\,(s)|\left(\int_{s-\tau_1(s)}^{s}|H(u)+B(s,u)|\,du\right)ds\right)^2\\
&\quad\left.+\varepsilon\left(1+\frac{1}{L}+\frac{1+L}{1-L}\right)\left(\int_0^{t^*}2K_2e^{-2\int_s^{t^*}H(u)du}ds\right)\right\}\\
&\leqslant(1+L)\,\delta\left(1+K_1+\int_{-\tau_1(0)}^{0}|H(s)+B(0,s)|\,ds\right)^2 e^{-2\int_0^{t^*}H(u)du}\\
&\quad+\left(1+\frac{1}{L}\right)\alpha^2\varepsilon<\varepsilon, \qquad (2.15)
\end{aligned}$$

which contradicts the definition of t^*. Thus the zero solution of (1.1) is mean square asymtotically stable if (2.6) holds.

Conversely, we suppose that (2.6) fails. From (2.4), there exists a sequence $\{t_n\}$ with $t_n\to\infty$ as $n\to\infty$ such that $\lim_{n\to\infty}\int_0^{t_n}H(u)du=l$ for some $l\in\mathbb{R}$. We may also choose a positive constant J satisfying

$$-J\leqslant\int_0^{t_n}H(u)du\leqslant+J,$$

for all $n \geqslant 1$. To simplify the expression, we define

$$\begin{aligned}\omega(s) &:= \left|\left[H(s-\tau_1(s))+B(s,s-\tau_1(s))\right](1-\tau_1'(s))\right| + K_1\left|H(s)\right| \\ &\quad + \left|H(s)\right|\int_{s-\tau_1(s)}^{s}\left|H(u)+B(s,u)\right|du \end{aligned}\tag{2.16}$$

for all $s \geqslant 0$. From (2.5), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} H(u)du}\omega(s)\,ds \leqslant \alpha, \tag{2.17}$$

which implies that

$$\int_0^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds \leqslant \alpha e^{\int_0^{t_n} H(u)du} \leqslant e^J. \tag{2.18}$$

Therefore, the sequence $\left\{\int_0^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds\right\}$ has a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty}\int_0^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds = \gamma, \tag{2.19}$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds \leqslant \frac{\delta_0}{8K}, \tag{2.20}$$

for all $n \geqslant m$, where $\delta_0 > 0$ satisfies $8\delta_0 K^2 e^{2J} + \alpha^2 \leqslant 1$.

Now we consider the solution $x(t) = x(t, t_m, \psi)$ of (1.1) with $\|\psi(t_m)\|^2 = \delta_0$ and $\|\psi(s)\|^2 < \delta_0$ for $s < t_m$. By the similar method in (2.15), we have $E|x(t)|^2 \leqslant 1$ for $t \geqslant t_m$. We may choose ψ so that

$$\begin{aligned}G(t_m) &:= \psi(t_m) - Q(t_m, \psi(t_m - \tau_1(t_m))) \\ &\quad - \int_{t_m-\tau_1(t_m)}^{t_m}\left[H(s)+B(t_m,s)\right]\psi(s)\,ds \geqslant \frac{\delta_0}{2}.\end{aligned}\tag{2.21}$$

In follows from $(2.10), (2.20)$ and (2.21) with $x(t) = (Px)(t)$ that for $n \geqslant m$,

$$\begin{aligned}&E\left|x(t_n) - Q(t_n, \psi(t_n - \tau_1(t_n))) - \int_{t_n-\tau_1(t_n)}^{t_n}\left[H(s)+B(t_n,s)\right]\psi(s)\,ds\right|^2 \\ &\geqslant G^2(t_m)\,e^{-2\int_{t_m}^{t_n} H(u)du} - 2G(t_m)\,e^{-\int_{t_m}^{t_n} H(u)du}\int_{t_m}^{t_n} e^{-\int_s^{t_n} H(u)du}\omega(s)\,ds \\ &\geqslant \frac{\delta_0}{2}e^{-2\int_{t_m}^{t_n} H(u)du}\left(\frac{\delta_0}{2} - 2K\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds\right) \geqslant \frac{\delta_0^2}{8}e^{-2J} > 0.\end{aligned}\tag{2.22}$$

If the zero solution of (1.1) is mean square asymptotically stable, then $E\left|x\left(t\right)\right|^{2}=E\left|x\left(t,t_{m},\psi\right)\right|^{2}\rightarrow 0$ as $t\rightarrow\infty$. Since $t_{n}-\tau_{j}(t_{n})\rightarrow\infty$ as $n\rightarrow\infty$, for $j=1,2,3$ and conditions (2.4) and (2.5) hold,

$$E\left|x\left(t_{n}\right)-Q(t_{n},x\left(t_{n}-\tau_{1}\left(t_{n}\right)\right))-\int_{t_{n}-\tau_{1}\left(t_{n}\right)}^{t_{n}}\left[H(s)+B(t_{n},s)\right]x\left(s\right)ds\right|^{2}\rightarrow 0,$$

as $t\rightarrow\infty$, which contradicts (2.22). Hence condition (2.6) is necessary in order that (1.1) has a solution $x\left(t,0,\psi\right)\rightarrow 0$ as $t\rightarrow\infty$. The proof is complete. ■

3. Remarks and an example

In this section, we give some remarks and an example to illustrate the applications of Theorem 2.1.

Remark 3.1. It follows from the first part of the proof of Theorem 2.1 that the zero solution of (1.1) is stable under (2.4) and (2.5). Moreover, Theorem 2.1 still holds if (2.5) is satisfied for $t\geqslant t_{\sigma}$ for some $t_{\sigma}\in\mathbb{R}$.

Remark 3.2. When $h\left(t,x\left(t\right),x(t-\tau_{3}\left(t\right)),u\right)=0$ and $G\left(t,x\left(t\right),x(t-\tau_{2}\left(t\right))\right)=0$, equation (1.1) reduce to

$$\frac{d}{dt}x(t)=-\left(\int_{t-\tau_{1}(t)}^{t}a(t,s)x(s)ds\right)+\frac{d}{dt}Q\left(t,x\left(t-\tau_{1}\left(t\right)\right)\right),\qquad t\geqslant 0,$$

which is recently studied in Ardjouni and Djoudi [1].

Example 3.1. Consider the following stochastic Volterra-Levin equation with variable delays:

$$\begin{aligned}d\left[x(t)-a_{1}x\left(t-\tau_{1}\left(t\right)\right)\right]=-a_{2}\left(\int_{t-\tau_{1}(t)}^{t}x(s)ds\right)dt+a_{3}x(t-\tau_{2}\left(t\right))dW\left(t\right)\\+a_{4}x(t-\tau_{3}\left(t\right))d\widetilde{N}\left(t\right),\qquad t\geqslant 0.\end{aligned}\tag{3.1}$$

Its initial condition is equipped with $x(0)=c$ (constant) and a_{i}, $i=1,2,3,4$, are positive constants. $\tau_{1}\left(t\right),\tau_{2}\left(t\right)$ and $\tau_{3}\left(t\right)$ are variable delays. Set $\tau_{2}\left(t\right)=1$ and $H\left(t\right)=a_{2}$, under the condition (2.5) of Theorem 2.1,

$$K_{1}+\int_{s-\tau_{1}(s)}^{s}\left|H(z)+B(s,z)\right|dz\leqslant a_{1}+a_{2}\int_{s-1}^{s}\left|1+z-s\right|dz\leqslant a_{1}+\frac{1}{2}a_{2},$$

$$\begin{aligned}\int_{0}^{s}e^{-\int_{z}^{s}H(u)du}\left[\left|\left(H(z-\tau_{1}(z))+B(z,z-\tau_{1}(z))\left(1-\tau_{1}'(z)\right)\right)\right|+K_{1}\left|H\left(z\right)\right|\right]dz\\\leqslant a_{1}a_{2}e^{-a_{2}s}\int_{0}^{s}e^{a_{2}z}dz\leqslant a_{1},\end{aligned}$$

$$\int_0^s e^{-\int_z^s H(u)du} |H(z)| \left(\int_{z-\tau_1(z)}^z |H(u) + B(z,u)| \, du \right) dz$$
$$\leqslant a_2^2 e^{-a_2 s} \int_0^s e^{a_2 z} \left(\int_{z-1}^z |1 + u - z| \, du \right) dz$$
$$\leqslant \frac{1}{2} a_2^2 e^{-a_2 s} \int_0^s e^{a_2 z} dz \leqslant \frac{1}{2} a_2,$$

$$2 \left(2K_2 \int_0^s e^{-2\int_z^s H(u)du} dz \right)^{\frac{1}{2}} \leqslant 2 \left(2 \max(a_3, a_4) \int_0^s e^{-2a_2(s-z)} dz \right)^{\frac{1}{2}}$$
$$\leqslant 2 \left[\max(a_3, a_4) \right]^{\frac{1}{2}} \left(\frac{1}{a_2} \right)^{\frac{1}{2}}.$$

Let $a_1 = 0.07, a_2 = 0.08, a_3 = a_4 = 0.01$ then by (2.5), it is easy to see that all the conditions of Theorem 2.1 hold for

$$\alpha = 2a_1 + a_2 + 2 \left[\max(a_3, a_4) \right]^{\frac{1}{2}} \left(\frac{1}{a_2} \right)^{\frac{1}{2}} \simeq 0.92 < 1.$$

Thus, Theorem 2.1 implies that the zero solution of (4.1) is asymptotically stable. ■

Conclusion. This work studies the problem of mean square asymptotic stability of a nonlinear stochastic neutral Volterra-Levin equation with Poisson jumps and variable delays. As the main tool, it used the contraction mapping principle to obtain asymptotic stability results As the main result, the paper establishes an asymptotic stability theorem with a necessary and sufficient condition. This improves and extends some previous results due to Burton [5], Becker and Burton [4] and Jin and Luo [10], Ardjouni and Djoudi [1]. Actually, the methods used in proofs are an appropriate modification of those in [2], [14] and other cited references. Finally, an illustrative example is given.

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Addresses: Mimia Benhadri: Frères Mentouri Constantine University, Department of Mathematics, Algeria;
Halim Zeghdoudi: LaPS Laboratory, Badji-Mokhtar University, Box 12, Annaba, 23000, Algeria.

E-mail: mbenhadri@yahoo.com, halim.zeghdoudi@univ-annaba.dz

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