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BOUNDING THE LEAST PRIME IDEAL IN THE CHEBOTAREV DENSITY THEOREM

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Abstract: Let K be a number field and suppose L/K is a finite Galois extension. We establish a bound for the least prime ideal occurring in the Chebotarev Density Theorem. Namely, for every conjugacy class C of $\mathrm{Gal}(L/K)$, there exists a prime ideal $\mathfrak{p}$ of K unramified in L, for which its Artin symbol $\left[\frac{L/K}{\mathfrak{p}}\right] = C$, for which its norm $\mathrm{N}^K_{\mathbb{Q}}\mathfrak{p}$ is a rational prime, and which satisfies

$$\mathrm{N}^K_{\mathbb{Q}}\mathfrak{p} \ll d_L^{40},$$

where $d_L = |\mathrm{disc}(L/\mathbb{Q})|$. All implicit constants are effective and absolute.

Keywords: least prime ideal, Chebotarev Density Theorem, Dedekind zeta function, Deuring-Heilbronn phenomenon.

1. Introduction

Let K be a number field and L/K be a finite Galois extension. For an unramified prime ideal $\mathfrak{p}$ of K, let $\left[\frac{L/K}{\mathfrak{p}}\right]$ be its associated Artin symbol, which is a conjugacy class of $G := \mathrm{Gal}(L/K)$. For a given conjugacy class C of G and $X \geqslant 2$, define

$$\pi_C(X) := \#\Big\{\mathfrak{p} \text{ prime ideals of } K \text{ of degree } 1 : \mathrm{N}^K_{\mathbb{Q}}\mathfrak{p} \leqslant X, \qquad \mathfrak{p} \text{ unramified in } L, \left[\frac{L/K}{\mathfrak{p}}\right] = C\Big\},$$

where $\mathrm{N}^K_{\mathbb{Q}}$ is the absolute norm of K. The Chebotarev Density Theorem [Hei67, Tsc26] states

$$\pi_C(X) \sim \frac{|C|}{|G|}\mathrm{Li}(X),$$

where $\mathrm{Li}(X) = \int_2^X (\log t)^{-1} dt$, so infinitely many such prime ideals exist. One may then ask: when does such a prime ideal $\mathfrak{p}$ of least norm occur?

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Assuming the Generalized Riemann Hypothesis, Lagarias and Odlyzko [LO77] proved that

$$\mathrm{N}_{\mathbb{Q}}^{K}\mathfrak{p} \ll (\log d_L)^2(\log\log d_L)^4,$$

where $d_L = |\mathrm{disc}(K/\mathbb{Q})|$ is the absolute discriminant of L. They also sketched a proof showing the $(\log\log d_L)^4$ factor could be removed entirely. Additionally assuming the Artin Conjecture, V.K. Murty [KM00] showed that

$$\mathrm{N}_{\mathbb{Q}}^{K}\mathfrak{p} \ll \frac{n_K^2}{|C|}\big(\log d_L + [L:K]\log[L:K]\big)^2,$$

where $n_K = [K:\mathbb{Q}]$ is the degree of $K/\mathbb{Q}$.

Unconditionally, Lagarias, Mongtomery and Odlyzko [LMO79] proved that

$$\mathrm{N}_{\mathbb{Q}}^{K}\mathfrak{p} \ll d_L^B \tag{1.1}$$

for some effectively computable absolute constant $B > 0$. In [KN12], Kadiri and Ng made reference to some explicit value of B but the author has been unable to locate that preprint[1]. The purpose of this paper is to show that $B = 40$ is admissible in (1.1).

Theorem 1.1. *Let K be a number field and suppose L/K is a finite Galois extension. For every conjugacy class C of* $\mathrm{Gal}(L/K)$*, there exists a prime ideal* $\mathfrak{p}$ *of K unramified in L, for which its Artin symbol $\left[\frac{L/K}{\mathfrak{p}}\right] = C$, for which its norm $\mathrm{N}_{\mathbb{Q}}^{K}\mathfrak{p}$ is a rational prime, and which satisfies*

$$\mathrm{N}_{\mathbb{Q}}^{K}\mathfrak{p} \ll d_L^{40},$$

where $d_L = |\mathrm{disc}(L/\mathbb{Q})|$. The implied constant is effective and absolute.

Remark.

(i) In several cases, one can reduce the exponent $B = 40$ by straightforward modifications. For example, one can take

$$B = \begin{cases} 36.5 & \text{if } L \text{ has a tower of normal extensions with base } \mathbb{Q}, \\ 24.1 & \text{if } n_L = o(\log d_L), \\ 7.5 & \text{if } \zeta_L(s) \text{ does not have a real zero } \beta_1 = 1 - \frac{\lambda_1}{\log d_L} \\ & \text{satisfying } \lambda_1 = o(1), \end{cases}$$

where $\zeta_L(s)$ is the Dedekind zeta function of L. See the remark at the end of section 5 for details.

(ii) With a slight addition to our arguments, one can deduce a quantitative lower bound for $\pi_C(X)$. See [Zam17, Theorem 1.3.1] for details.

[1]Note added: A preprint of this paper was posted on the arXiv in August 2015 (`arXiv/1508.00287`). Subsequently, in January 2016, the author was informed by Kadiri and Ng of their unpublished work [KN] wherein they prove Theorem 1.1 in the case $K = \mathbb{Q}$.

The proof of Theorem 1.1 is motivated by the original arguments of [LMO79] which are naturally connected with Linnik's celebrated result [Lin44a] on the least rational prime in an arithmetic progression. As such, we take advantage of powerful techniques found in Heath-Brown's work [HB95] on Linnik's constant. We also require explicit estimates related to the zeros of the Dedekind zeta function of L, denoted $\zeta_L(s)$. Recall

$$\zeta_L(s) = \sum_{\mathfrak{N}} (\mathrm{N}_{\mathbb{Q}}^{L}\mathfrak{N})^{-s} \tag{1.2}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}\{s\} > 1$ and where the sum is over integral ideals $\mathfrak{N}$ of L. One key ingredient in our proof is an explicit zero-free region due to Kadiri [Kad12]. She showed that $\zeta_L(s)$ has at most one zero in the rectangle

$$\mathrm{Re}\{s\} > 1 - \frac{0.0784}{\log d_L}, \qquad |\Im\{s\}| \leqslant 1.$$

Furthermore, if such a zero β_1 exists, it is real and simple, and we refer to it as *exceptional*. To handle this exceptional zero $\beta_1 = 1 - \frac{\lambda_1}{\log d_L}$, as Linnik [Lin44b] did for Dirichlet L-functions, we use explicit versions of Deuring-Heilbronn phenomenon for the Dedekind zeta function. We employ such a result due to Kadiri and Ng [KN12] when $\lambda_1 \gg 1$. To cover the remaining case when $\lambda_1 = o(1)$, which we refer to as a *Siegel zero*, we establish another variant of Deuring-Heilbronn phenomenon.

Theorem 1.2. *Suppose $\zeta_L(s)$ has a real zero β_1 and let $\rho' = \beta' + i\gamma'$ be another zero of $\zeta_L(s)$ satisfying*

$$\tfrac{1}{2} \leqslant \beta' < 1 \qquad \text{and} \qquad |\gamma'| \leqslant 1. \tag{1.3}$$

Then, for d_L sufficiently large,

$$\beta' \leqslant 1 - \frac{\log\left(\dfrac{c}{(1-\beta_1)\log d_L}\right)}{35.8 \log d_L},$$

where $c > 0$ is an absolute effective constant.

Remarks.

(i) Kadiri and Ng [KN12] alternatively show that if

$$1 - \frac{\log\log d_L}{13.84 \log d_L} \leqslant \beta' < 1, \qquad |\gamma'| \leqslant 1, \tag{1.4}$$

and d_L is sufficiently large then

$$\beta' \leqslant 1 - \frac{\log\left(\dfrac{1}{(1-\beta_1)\log d_L}\right)}{1.53 \log d_L}.$$

While the repulsion constant 1.53 is much better than 35.8 given by Theorem 1.2, the permitted range of β' in (1.3) is much larger than that of (1.4) therefore allowing Theorem 1.2 to deal with Siegel zeros.

(ii) If $n_L = o(\log d_L)$ then 35.8 can be replaced by 24.01. By a classical theorem of Minkowski, recall $n_L = O(\log d_L)$ so such an assumption is often reasonable.

Theorem 1.2 gives a quantitative bound for [LMO79, Theorem 5.1] and its proof is motivated by this non-explicit version. It involves a careful application of a modified Turán power sum inequality along with several explicit estimates concerning sums over zeros of $\zeta_L(s)$. Using similar arguments, we may establish a quantitative Deuring-Heilbronn phenomenon for only the real zeros of $\zeta_L(s)$ which is stronger than Theorem 1.2.

Theorem 1.3. *Suppose $\zeta_L(s)$ has a real zero β_1 and let β' be another real zero of $\zeta_L(s)$ satisfying $0 < \beta' < 1$. Then, for d_L sufficiently large,*

$$\beta' \leqslant 1 - \frac{\log\left(\dfrac{c}{(1-\beta_1)\log d_L}\right)}{16.6\log d_L},$$

where $c > 0$ is an absolute effective constant.

Remark. Similar to remark (ii) following Theorem 1.2, if $n_L = o(\log d_L)$ then 16.6 can be replaced by 12.01.

Applying the above theorem to the zero $\beta' = 1-\beta_1$ of $\zeta_L(s)$ immediately yields the following corollary which will play a role in our proof of Theorem 1.1.

Corollary 1.4. *Suppose $\zeta_L(s)$ has a real zero β_1. Then, for d_L sufficiently large,*

$$1 - \beta_1 \gg d_L^{-16.6},$$

where the implied constant is absolute and effective.

Remarks. Corollary 1.4 makes explicit [LMO79, Corollary 5.2] and so, as remarked therein, Stark [Sta74] gives a better bound for $1-\beta_1$ when L has a tower of normal extensions with base $\mathbb{Q}$.

Finally, we describe the organization of the paper. Section 2 provides the necessary preliminaries including background on the Dedekind zeta function, a power sum inequality, and some technical estimates. Section 3 contains work on the Deuring-Heilbronn phenomenon proving Theorems 1.2 and 1.3. Section 4 prepares for the proof of Theorem 1.1 and Section 5 contains the concluding arguments divided into the relevant cases.

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2. Preliminaries

2.1. Dedekind zeta function

The background material discussed here on the Dedekind zeta function can be found in [LO77, Hei67]. Consider a number field $L/\mathbb{Q}$ of degree $n_L = [L : \mathbb{Q}]$ with absolute discriminant $d_L = |\mathrm{disc}(L/\mathbb{Q})|$ and ring of integers $\mathcal{O}_L$. The *Dedekind zeta function of* L, denoted $\zeta_L(s)$, can be given as a Dirichlet series by (1.2) or as an Euler product by

$$\zeta_L(s) = \prod_{\mathfrak{P}} \Big(1 - (\mathrm{N}^L_{\mathbb{Q}}\mathfrak{P})^{-s}\Big)^{-1}$$

for $\mathrm{Re}\{s\} > 1$, where the product is over prime ideals $\mathfrak{P}$ of L. The *completed Dedekind zeta function* $\xi_L(s)$ is given by

$$\xi_L(s) = s(s-1)d_L^{s/2}\gamma_L(s)\zeta_L(s), \tag{2.1}$$

where γ_L is the *gamma factor of* L defined by

$$\gamma_L(s) = \Big[\pi^{-\frac{s}{2}}\Gamma\big(\frac{s}{2}\big)\Big]^{r_1+r_2} \cdot \Big[\pi^{-\frac{s+1}{2}}\Gamma\big(\frac{s+1}{2}\big)\Big]^{r_2}. \tag{2.2}$$

Here $r_1 = r_1(L)$ and $2r_2 = 2r_2(L)$ are the number of real and complex embeddings of L respectively. It is well-known that $\xi_L(s)$ is entire and satisfies the functional equation

$$\xi_L(s) = \xi_L(1-s). \tag{2.3}$$

We refer to its zeros as the *non-trivial zeros* ρ of $\zeta_L(s)$, which are known to lie in the strip $0 < \mathrm{Re}\{s\} < 1$. The *trivial zeros* ω of $\zeta_L(s)$ occur at certain non-positive integers arising from poles of the gamma factor of L; namely,

$$\operatorname*{ord}_{s=\omega} \zeta_L(s) = \begin{cases} r_1 + r_2 - 1 & \text{if } \omega = 0, \\ r_2 & \text{if } \omega = -1, -3, -5, \dots, \\ r_1 + r_2 & \text{if } \omega = -2, -4, -6, \dots. \end{cases} \tag{2.4}$$

Using the functional equation and a Hadamard product for $\xi_L(s)$, one can deduce an explicit formula for the logarithmic derivative of $\zeta_L(s)$ given by the lemma below.

Lemma 2.1. *For any number field* L *and* $s \in \mathbb{C}$,

$$-\mathrm{Re}\Big\{\frac{\zeta_L'}{\zeta_L}(s)\Big\} = \frac{1}{2}\log d_L + \mathrm{Re}\Big\{\frac{1}{s-1} - \sum_{\rho}\frac{1}{s-\rho} + \frac{1}{s} + \frac{\gamma_L'}{\gamma_L}(s)\Big\},$$

where the sum is over all the non-trivial zeros ρ *of* $\zeta_L(s)$.

Proof. See [LO77, Lemma 5.1] for example. ■

2.2. Power sum inequality

We record a power sum inequality and its proof from [LMO79, Theorem 4.2] specialized to our intended application.

Lemma 2.2 ([LMO79, Lemma 4.1]). *Define*

$$P(r,\theta) := \sum_{j=1}^{J} \Big(1 - \frac{j}{J+1}\Big) r^j \cos(j\theta).$$

Then

(i) $P(r,\theta) \geqslant -\frac{1}{2}$ *for* $0 \leqslant r \leqslant 1$ *and all* θ.
(ii) $P(1,0) = J/2$.
(iii) $|P(r,\theta)| \leqslant \frac{3}{2}r$ *for* $0 \leqslant r \leqslant 1/3$.

Theorem 2.3. *Let* $\epsilon > 0$ *and a sequence of complex numbers* $\{z_n\}_n$ *be given. Let* $s_m = \sum_{n=1}^{\infty} z_n^m$ *and suppose that* $|z_n| \leqslant |z_1|$ *for all* $n \geqslant 1$. *Define*

$$M := \frac{1}{|z_1|} \sum_n |z_n|. \tag{2.5}$$

Then there exists m_0 *with* $1 \leqslant m_0 \leqslant (12+\epsilon)M$ *such that*

$$\mathrm{Re}\{s_{m_0}\} \geqslant \frac{\epsilon}{48+5\epsilon} |z_1|^{m_0}.$$

Proof. This is a simplified version of [LMO79, Theorem 4.2]; our focus was to reduce their constant 24 to $12+\epsilon$ by some minor modifications. We reiterate the proof here for clarity. Rescaling we may suppose $|z_1| = 1$. Write $z_n = r_n \exp(i\theta_n)$ so $r_n \in [0,1]$. Then

$$\begin{aligned} S_J &:= \sum_{j=1}^{J} \Big(1 - \frac{j}{J+1}\Big) \mathrm{Re}\{s_j\}(1 + \cos j\theta_1) \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{J} \Big(1 - \frac{j}{J+1}\Big) (\cos j\theta_n)(1+\cos j\theta_1) r_n^j \\ &= \sum_{n=1}^{\infty} \big\{ P(r_n,\theta_n) + \tfrac{1}{2}P(r_n, \theta_n - \theta_1) + \tfrac{1}{2} P(r_n, \theta_n + \theta_1) \big\}. \end{aligned}$$

Using Lemma 2.2, we estimate the contribution of each term. For $n = 1$, we obtain a contribution $\geqslant \Big(\frac{J+1}{4} - r_1\Big)$. Terms $n > 1$ satisfying $r_n \geqslant 1/3$ contribute $\geqslant -1 \geqslant -3r_n$. Each of the remaining terms satisfying $r_n < 1/3$ are bounded using Lemma 2.2(iii) and so contribute $\geqslant -3r_n$. Choosing $J = \lfloor (12+\epsilon)M \rfloor$, we deduce

$$S_J \geqslant \frac{J+1}{4} - 3M \geqslant \frac{\epsilon M}{4} \tag{2.6}$$

as $J+1 \geqslant (12+\epsilon)M$. Now, suppose for a contradiction that $\mathrm{Re}\{s_j\} < \frac{\epsilon}{48+5\epsilon}$ for all $1 \leqslant j \leqslant J$. Then, as $(1-\frac{j}{J+1})(1+\cos j\theta_1)$ is non-negative for all $1 \leqslant j \leqslant J$,

$$S_J \leqslant \frac{\epsilon}{48+5\epsilon} \sum_{j=1}^{J} \Big(1 - \frac{j}{J+1}\Big)(1+\cos j\theta_1) < \frac{\epsilon}{48+5\epsilon} \cdot 2P(1,0) = \frac{\epsilon J}{48+5\epsilon}.$$

Comparing with (2.6) and noting $J \leqslant (12+\epsilon)M$, we obtain a contradiction. ■

2.3. Technical estimates

For the application of the power sum inequality, we will require some precise numerical estimates.

Lemma 2.4. *For $\alpha > 0$ and $t \geqslant 0$,*

$$\mathrm{Re}\Big\{\frac{\gamma_L'}{\gamma_L}(\alpha+1) + \frac{\gamma_L'}{\gamma_L}(\alpha+1 \pm it)\Big\} = G_1(\alpha;t)\cdot r_1 + G_2(\alpha;t)\cdot 2r_2,$$

where

$$\begin{aligned} G_1(\alpha;t) &:= \frac{\Delta(\alpha+1,0)+\Delta(\alpha+1,t)}{2} - \log\pi, \\ G_2(\alpha;t) &:= \frac{\Delta(\alpha+1,0)+\Delta(\alpha+2,0)+\Delta(\alpha+1,t)+\Delta(\alpha+2,t)}{4} - \log\pi, \end{aligned} \tag{2.7}$$

and $\Delta(x,y) = \mathrm{Re}\{\frac{\Gamma'}{\Gamma}(\frac{x+iy}{2})\}$.

Remark. For fixed $\alpha > 0$ and $j = 1$ or 2, observe that $G_j(\alpha;t)$ is increasing as a function of $t \geqslant 0$ by [AK14, Lemma 2].

Proof. Denote $\sigma = \alpha+1$. As $\Delta(x,y) = \Delta(x,-y)$, we may assume $t \geqslant 0$. From (2.2), it follows that

$$\begin{aligned} \mathrm{Re}\Big\{\frac{\gamma_L'}{\gamma_L}(\sigma+it)\Big\} &= \frac{1}{2}\Big[(r_1+r_2)\Delta(\sigma,t) + r_2\Delta(\sigma+1,t) - (r_1+2r_2)\log\pi\Big] \\ &= \frac{1}{2}\Big[r_1(\Delta(\sigma,t)-\log\pi) + 2r_2\cdot\big(\tfrac{\Delta(\sigma,t)+\Delta(\sigma+1,t)}{2} - \log\pi\big)\Big]. \end{aligned}$$

Using the same identity for $t = 0$ gives the desired result. ■

Lemma 2.5. *For $\alpha \geqslant 1$ and $t \in \mathbb{R}$,*

$$\begin{aligned} &\sum_{\omega} \Big(\frac{1}{|\alpha+1-\omega|^2} + \frac{1}{|\alpha+1+it-\omega|^2}\Big) \\ &\qquad \leqslant \frac{1}{\alpha}\log d_L + \Big(\frac{G_1(\alpha;|t|)}{\alpha} + 2W_1(\alpha)\Big)\cdot r_1 + \Big(\frac{G_2(\alpha;|t|)}{\alpha} + W_2(\alpha)\Big)\cdot 2r_2 \\ &\qquad\quad + \frac{2}{\alpha^2} + \frac{2}{\alpha+\alpha^2}, \end{aligned}$$

where the sum is over all zeros ω *of* $\zeta_L(s)$ *including trivial ones, the functions* $G_j(\alpha; |t|)$ *are defined by* (2.7),

$$W_1(\alpha) = \sum_{k=0}^{\infty} \frac{1}{(\alpha+1+2k)^2}, \qquad \textit{and} \qquad W_2(\alpha) = \sum_{k=0}^{\infty} \frac{1}{(\alpha+1+k)^2}.$$

Proof. We estimate the trivial and non-trivial zeros separately. From (2.4), notice

$$\sum_{\omega \text{ trivial}} \frac{1}{|\alpha+1-\omega|^2} \leqslant r_1 \sum_{k=0}^{\infty} \frac{1}{(\alpha+1+2k)^2} + r_2 \sum_{k=0}^{\infty} \frac{1}{(\alpha+1+k)^2}.$$

Hence,

$$\sum_{\omega \text{ trivial}} \Big(\frac{1}{|\alpha+1-\omega|^2} + \frac{1}{|\alpha+1+it-\omega|^2} \Big) \leqslant 2W_1(\alpha) \cdot r_1 + W_2(\alpha) \cdot 2r_2. \tag{2.8}$$

For the non-trivial zeros $\rho = \beta + i\gamma$, we combine the inequality

$$0 \leqslant -\mathrm{Re}\Big\{ \frac{\zeta_L'}{\zeta_L}(\alpha+1) + \frac{\zeta_L'}{\zeta_L}(\alpha+1+it) \Big\}$$

with Lemmas 2.1 and 2.4 to deduce that

$$\begin{aligned} 0 \leqslant \log d_L + G_1(\alpha; |t|) \cdot r_1 + G_2(\alpha; |t|) \cdot 2r_2 + \mathrm{Re}\Big\{ \frac{1}{\alpha+it} + \frac{1}{\alpha+1+it} \Big\} \\ - \sum_{\rho} \mathrm{Re}\Big\{ \frac{1}{\alpha+1-\rho} + \frac{1}{\alpha+1+it-\rho} \Big\} + \frac{1}{\alpha} + \frac{1}{\alpha+1}. \end{aligned} \tag{2.9}$$

Observe, as $\beta \in (0,1)$,

$$\mathrm{Re}\Big\{ \frac{1}{\alpha+1+it-\rho} \Big\} = \frac{\alpha+1-\beta}{|\alpha+1+it-\rho|^2} \geqslant \frac{\alpha}{|\alpha+1+it-\rho|^2}$$

and

$$\mathrm{Re}\Big\{ \frac{1}{\alpha+it} + \frac{1}{\alpha+1+it} \Big\} \leqslant \frac{1}{\alpha} + \frac{1}{\alpha+1}.$$

We rearrange (2.9) and employ these observations to find that

$$\begin{aligned} \sum_{\rho} \Big(\frac{1}{|\alpha+1-\rho|^2} + \frac{1}{|\alpha+1+it-\rho|^2} \Big) \\ \leqslant \frac{1}{\alpha} \big(\log d_L + G_1(\alpha; |t|) \cdot r_1 + G_2(\alpha; |t|) \cdot 2r_2 \big) + \frac{2}{\alpha^2} + \frac{2}{\alpha+\alpha^2}. \end{aligned} \tag{2.10}$$

Combining with (2.8) yields the desired bound. ■

2.4. Choice of weights

In the proof of Theorem 1.1, we will need to select a suitable weight function so we describe our choice and its properties here.

Lemma 2.6. *For real numbers $A, B > 0$ and positive integer $\ell \geqslant 1$ satisfying $B > 2\ell A$, there exists a real-variable function $f(t) = f_\ell(t)$ such that:*

(i) $0 \leqslant f(t) \leqslant A^{-1}$ *for all* $t \in \mathbb{R}$.
(ii) *The support of f is contained in $[B - 2\ell A, B]$.*
(iii) *Its Laplace transform $F(z) = F_\ell(z) = \int_{\mathbb{R}} f_\ell(t) e^{-zt} dt$ is given by*

$$F(z) = e^{-(B-2\ell A)z} \Big(\frac{1 - e^{-Az}}{Az} \Big)^{2\ell}. \tag{2.11}$$

(iv) *Let $\mathscr{L} \geqslant 1$ be arbitrary. Suppose $s = \sigma + it \in \mathbb{C}$ satisfies $\sigma < 1$ and $t \in \mathbb{R}$. Write $\sigma = 1 - \frac{x}{\mathscr{L}}$ and $t = \frac{y}{\mathscr{L}}$. If $0 \leqslant \alpha \leqslant 2\ell$ then*

$$\begin{aligned} |F((1-s)\mathscr{L})| &\leqslant e^{-(B-2\ell A)x} \Big(\frac{2}{A\sqrt{x^2+y^2}} \Big)^{\alpha} \\ &= e^{-(B-2\ell A)(1-\sigma)\mathscr{L}} \Big(\frac{2}{A|s-1|\mathscr{L}} \Big)^{\alpha}. \end{aligned}$$

Furthermore,

$$|F((1-s)\mathscr{L})| \leqslant e^{-(B-2\ell A)x} \qquad \text{and} \qquad F(0) = 1.$$

Remark. Heath-Brown [HB95] used the weight f_ℓ with $\ell = 1$ for his computation of Linnik's constant for the least rational prime in an arithmetic progression.

Proof.

- For parts (i)–(iii), let $\mathbf{1}_S(\,\cdot\,)$ be an indicator function for the set $S \subseteq \mathbb{R}$. For $j \geqslant 1$, define

$$w_0(t) := \frac{1}{A}\mathbf{1}_{[-A/2,A/2]}(t), \qquad \text{and} \qquad w_j(t) := (w * w_{j-1})(t).$$

Since $\int_{\mathbb{R}} w_0(t)dt = 1$, it is straightforward verify that $0 \leqslant w_{2\ell}(t) \leqslant A^{-1}$ and $w_{2\ell}(t)$ is supported in $[-\ell A, \ell A]$. Observe the Laplace transform $W(z)$ of w_0 is given by

$$W(z) = \frac{e^{Az/2} - e^{-Az/2}}{Az} = e^{Az/2} \cdot \Big(\frac{1 - e^{-Az}}{Az} \Big),$$

so the Laplace transform $W_{2\ell}(z)$ of $w_{2\ell}$ is given by

$$W_{2\ell}(z) = \Big(\frac{e^{Az/2} - e^{-Az/2}}{Az} \Big)^{2\ell} = e^{\ell A z} \Big(\frac{1 - e^{-Az}}{Az} \Big)^{2\ell}.$$

The desired properties for f follow upon choosing $f(t) = w_{2\ell}(t - B + \ell A)$.

- For part (iv), we see by (iii) that

$$|F((1-s)\mathscr{L})| \leqslant e^{-(B-2\ell A)x}\Big|\frac{1-e^{-A(x+iy)}}{A(x+iy)}\Big|^{2\ell}. \tag{2.12}$$

To bound the above quantity, we observe that for $w = a+ib$ with $a > 0$ and $b \in \mathbb{R}$,

$$\Big|\frac{1-e^{-w}}{w}\Big|^2 \leqslant \Big(\frac{1-e^{-a}}{a}\Big)^2 \leqslant 1.$$

This observation can be checked in a straightforward manner (cf. Lemma 2.7). It follows that

$$\begin{aligned}\Big|\frac{1-e^{-A(x+iy)}}{A(x+iy)}\Big|^{2\ell} &= \Big|\frac{1-e^{-A(x+iy)}}{A(x+iy)}\Big|^{\alpha}\cdot\Big|\frac{1-e^{-A(x+iy)}}{A(x+iy)}\Big|^{2\ell-\alpha} \\ &\leqslant \Big(\frac{2}{A\sqrt{x^2+y^2}}\Big)^{\alpha}.\end{aligned}$$

In the last step, we noted $|1-e^{-A(x+iy)}| \leqslant 2$ since $x > 0$ by assumption. Combining this with (2.12) yields the desired bound. The additional estimate for $|F((1-s)\mathscr{L})|$ is the case when $\alpha = 0$. One can verify $F(0) = 1$ by straightforward calculus arguments. ■

Lemma 2.7. *For $z = x+iy$ with $x > 0$ and $y \in \mathbb{R}$,*

$$\Big|\frac{1-e^{-z}}{z}\Big|^2 \leqslant \Big(\frac{1-e^{-x}}{x}\Big)^2.$$

Proof. We need only consider $y \geqslant 0$ by conjugate symmetry. Define

$$\Phi_x(y) := \Big|\frac{1-e^{-z}}{z}\Big|^2 = \frac{1+e^{-2x}-2e^{-x}\cos y}{x^2+y^2} \qquad \text{for } y \geqslant 0,$$

which is a non-negative smooth function of y. Since $\Phi_x(y) \to 0$ as $y \to \infty$, we may choose $y_0 \geqslant 0$ such that $\Phi_x(y)$ has a global maximum at $y = y_0$. Suppose, for a contradiction, that

$$\Phi_x(y_0) > \Big(\frac{1-e^{-x}}{x}\Big)^2. \tag{2.13}$$

By calculus, one can show $(1-e^{-x})/x \geqslant e^{-x/2}$ for $x > 0$. With this observation, notice that

$$\begin{aligned}\Phi_x'(y_0) &= \frac{2e^{-x}\cdot\sin y_0}{x^2+y_0^2} - \frac{2\Phi_x(y_0)\cdot y_0}{x^2+y_0^2} \\ &< \frac{2e^{-x}\cdot\sin y_0}{x^2+y_0^2} - \frac{2\big(\frac{1-e^{-x}}{x}\big)^2\cdot y_0}{x^2+y_0^2} \qquad \text{by (2.13)} \\ &\leqslant \frac{2e^{-x}\cdot\sin y_0}{x^2+y_0^2} - \frac{2e^{-x}\cdot y_0}{x^2+y_0^2} \\ &\leqslant 0,\end{aligned}$$

since $\sin y \leqslant y$ for $y \geqslant 0$. On the other hand, $\Phi_x(y)$ has a global max at $y = y_0$ implying $\Phi_x'(y_0) = 0$, a contradiction. ■

3. Deuring-Heilbronn phenomenon

In this section, we prove Theorems 1.3 and 3.1. Notice Theorem 1.2 is contained in Theorem 3.1 below.

Theorem 3.1. *Let $T \geqslant 1$ be fixed. Suppose $\zeta_L(s)$ has a real zero β_1 and let $\rho' = \beta' + i\gamma'$ be another zero of $\zeta_L(s)$ satisfying*

$$\tfrac{1}{2} \leqslant \beta' < 1 \qquad \text{and} \qquad |\gamma'| \leqslant T.$$

Then for d_L sufficiently large

$$\beta' \leqslant 1 - \frac{\log\Big(\dfrac{c}{(1-\beta_1)\log d_L}\Big)}{C \log d_L},$$

where $c = c(T) > 0$ and $C = C(T) > 0$ are absolute effective constants. In particular, one may take T and $C = C(T)$ according to the table below.

T	1	3.5	8.7	22	54	134	332	825	2048	5089	12646
C	35.8	37.0	39.3	42.5	46.1	50.0	53.8	57.6	61.4	65.2	69.0

Remarks.

(i) This result for general $T \geqslant 1$ follows from [LMO79, Theorem 5.1] but our primary concern is verifying the table of values for T and C. The choices of T in the given table are obviously not special; one can compute C for any fixed T by a simple modification to our argument below. We made these selections primarily for their application in the proof of Theorem 1.1 in section 5.

(ii) If $n_L = o(\log d_L)$ then one can take $C = 24.01$ for any fixed T.

3.1. Proof of Theorem 3.1

Let m be a positive integer and $\alpha \geqslant 1$. From [LMO79, Equation (5.4)] with $s = \alpha + 1 + i\gamma'$, it follows that

$$\mathrm{Re}\Big\{\sum_{n=1}^{\infty} z_n^m\Big\} \leqslant \frac{1}{\alpha^m} - \frac{1}{(\alpha+1-\beta_1)^{2m}} + \mathrm{Re}\Big\{\frac{1}{(\alpha+i\gamma')^{2m}} - \frac{1}{(\alpha+i\gamma'+1-\beta_1)^{2m}}\Big\}, \tag{3.1}$$

where $z_n = z_n(\gamma')$ satisfies $|z_1| \geqslant |z_2| \geqslant \dots$ and runs over the multiset

$$\{(\alpha+1-\omega)^{-2}, (\alpha+1+i\gamma'-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } \zeta_L(s)\}. \tag{3.2}$$

Note that the multiset includes trivial zeros of $\zeta_L(s)$. With this choice, we have that

$$(\alpha+1-\beta')^{-2} \leqslant |z_1| \leqslant \alpha^{-2}. \tag{3.3}$$

Since

$$\Big|\frac{1}{(\alpha+it)^{2m}} - \frac{1}{(\alpha+it+1-\beta_1)^{2m}}\Big| \leqslant \alpha^{-2m}\Big|1 - \frac{1}{(1+\frac{1-\beta_1}{\alpha+it})^{2m}}\Big|$$
$$\ll \alpha^{-2m-1}m(1-\beta_1),$$

equation (3.1) implies

$$\mathrm{Re}\Big\{\sum_{n=1}^{\infty} z_n^m\Big\} \ll \alpha^{-2m-1}m(1-\beta_1). \tag{3.4}$$

On the other hand, by Theorem 2.3, for $\epsilon > 0$, there exists some $m_0 = m_0(\epsilon)$ with $1 \leqslant m_0 \leqslant (12+\epsilon)M$ such that

$$\mathrm{Re}\Big\{\sum_{n=1}^{\infty} z_n^{m_0}\Big\} \geqslant \tfrac{\epsilon}{50}|z_1|^{m_0} \geqslant \tfrac{\epsilon}{50}(\alpha+1-\beta')^{-2m_0} \geqslant \tfrac{\epsilon}{50}\alpha^{-2m_0}\exp(-\tfrac{2m_0}{\alpha}(1-\beta')),$$

where $M = |z_1|^{-1}\sum_{n=1}^{\infty}|z_n|$ according to our parameters $z_n = z_n(\gamma')$ in (3.2). Comparing with (3.4) for $m = m_0$, it follows that

$$\exp(-(24+2\epsilon)\tfrac{M}{\alpha}(1-\beta')) \ll_\epsilon \tfrac{M}{\alpha}(1-\beta_1). \tag{3.5}$$

Therefore, it suffices to bound M/α and optimize over $\alpha \geqslant 1$. By Lemma 2.5 and (3.3), notice that

$$\begin{aligned}\frac{M}{\alpha} \leqslant \frac{(\alpha+1-\beta')^2}{\alpha}\cdot\Big\{\frac{1}{\alpha}\log d_L + \Big(\frac{G_1(\alpha;|\gamma'|)}{\alpha} + 2W_1(\alpha)\Big)\cdot r_1 \\ + \Big(\frac{G_2(\alpha;|\gamma'|)}{\alpha} + W_2(\alpha)\Big)\cdot 2r_2 + \frac{2}{\alpha^2} + \frac{2}{\alpha+\alpha^2}\Big\}\end{aligned} \tag{3.6}$$

for $\alpha \geqslant 1$. To simplify the above, we note $1-\beta' \leqslant 1/2$ by assumption and $G_j(\alpha;|\gamma'|) \leqslant G_j(\alpha;T)$ for $j = 1,2$ by the remark following Lemma 2.4. Also in (3.6), if a coefficient of r_1 or r_2 is positive, we employ an estimate of Odlyzko [Odl77] which implies

$$(\log 60)\cdot r_1 + (\log 22)\cdot 2r_2 \leqslant \log d_L \tag{3.7}$$

for d_L sufficiently large. With these observations, it follows that

$$\begin{aligned}\frac{M}{\alpha} &\leqslant \frac{(\alpha+1/2)^2}{\alpha} \\ &\quad\times\Big[\Big(\frac{1}{\alpha} + \max\Big\{\frac{G_1(\alpha;T)+2\alpha W_1(\alpha)}{\alpha\log 60}, \frac{G_2(\alpha;T)+\alpha W_2(\alpha)}{\alpha\log 22}, 0\Big\}\Big)\log d_L \\ &\quad + \frac{2}{\alpha^2} + \frac{2}{\alpha+\alpha^2}\Big].\end{aligned}$$

Seeking to minimize the coefficient of $\log d_L$, after some numerical calculations, we choose $\alpha = \alpha(T)$ according to the following table:

T	1	3.5	8.7	22	54	134	332	825	2048	5089	12646
α	3.07	4.06	5.68	7.73	9.43	10.7	11.7	12.7	13.7	14.7	15.7

To complete the proof for $T = 1$, say, the corresponding choice of $\alpha = 3.07$ implies

$$\frac{M}{\alpha} \leqslant 1.4883 \log d_L$$

for d_L sufficiently large. Substituting this bound into (3.5) and fixing $\epsilon > 0$ sufficiently small yields the desired result since $24 \times 1.4883 < 35.8$. The other cases follow similarly. ■

Remark.

- To clarify remark (ii) following Theorem 3.1, notice that if $n_L = o(\log d_L)$ then the coefficients of r_1 and r_2 in (3.6) can be made arbitrary small for d_L sufficiently large depending on $\alpha \geqslant 1$. Fixing α sufficiently large (depending on T) gives
$$M/\alpha \leqslant 1.0001 \log d_L$$
for d_L sufficiently large. As $24 \times 1.0001 < 24.01$ the remark follows.
- All computations were performed using Maple. Relevant code can be obtained either on the author's personal webpage or by email request.

3.2. Proof of Theorem 1.3

The proof is very similar to the above proof for Theorem 3.1 with a few differences which we outline here. Recall β' is now a real zero of $\zeta_L(s)$ distinct from β_1 (counting with multiplicity). As before, let m be a positive integer and $\alpha \geqslant 1$. From [LMO79, Equation (5.4)] with $s = \alpha + 1$ instead, it follows that

$$\mathrm{Re}\Big\{ \sum_{n=1}^{\infty} z_n^m \Big\} \leqslant \frac{1}{\alpha^m} - \frac{1}{(\alpha + 1 - \beta_1)^{2m}}, \tag{3.8}$$

where z_n satisfies $|z_1| \geqslant |z_2| \geqslant \dots$ and runs over the multiset

$$\{(\alpha + 1 - \omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } \zeta_L(s)\}.$$

If ω is a trivial zero (and hence a non-positive integer by (2.4)) then $(\alpha + 1 - \omega)^{-2} \geqslant 0$. Thus, for any z_n in (3.8) corresponding to a trivial zero, we have $z_n^m \geqslant 0$ so we may discard such z_n. It follows that

$$\mathrm{Re}\Big\{ \sum_{n=1}^{\infty} \tilde{z}_n^m \Big\} \leqslant \frac{1}{\alpha^m} - \frac{1}{(\alpha + 1 - \beta_1)^{2m}}, \tag{3.9}$$

where $\tilde{z}_n$ satisfies $|\tilde{z}_1| \geqslant |\tilde{z}_2| \geqslant \ldots$ and runs over the new (smaller) multiset

$$\{(\alpha+1-\rho)^{-2} : \rho \neq \beta_1 \text{ is any non-trivial zero of } \zeta_L(s)\}. \tag{3.10}$$

For this new choice of $\tilde{z}_n$, the analogue of (3.3) still holds for $\tilde{z}_1$ and we argue similarly to deduce (3.5) holds for the new $\tilde{M} = |\tilde{z}_1|^{-1}\sum_n |\tilde{z}_n|$. Thus, by the proof of Lemma 2.5 (namely by (2.10) with $t = 0$), we deduce that

$$\frac{\tilde{M}}{\alpha} \leqslant \frac{(\alpha+1-\beta')^2}{2\alpha} \cdot \left\{ \frac{1}{\alpha} \log d_L + \frac{G_1(\alpha;0)}{\alpha} \cdot r_1 + \frac{G_2(\alpha;0)}{\alpha} \cdot 2r_2 + \frac{2}{\alpha^2} + \frac{2}{\alpha+\alpha^2} \right\} \tag{3.11}$$

for $\alpha \geqslant 1$. Comparing with (3.6), notice the additional factor of 2 in the denominator and the lack of $W_1(\alpha)$ and $W_2(\alpha)$ terms. Continuing to argue analogously, we simplify the above by noting $1-\beta' < 1$ and applying Odlyzko's bound (3.7) to conclude

$$\frac{M}{\alpha} \leqslant \frac{(\alpha+1)^2}{2\alpha} \left[\left(\frac{1}{\alpha} + \max\left\{ \frac{G_1(\alpha;0)}{\alpha \log 60}, \frac{G_2(\alpha;0)}{\alpha \log 22}, 0 \right\} \right) \log d_L + \frac{2}{\alpha^2} + \frac{2}{\alpha+\alpha^2} \right]$$

for d_L sufficiently large. Selecting $\alpha = 5.8$ gives

$$\frac{\tilde{M}}{\alpha} \leqslant 0.6882 \log d_L$$

for d_L sufficiently large. As $24 \times 0.6882 < 16.6$, we similarly conclude the desired result. ■

4. Weighted sum of prime ideals

4.1. Setup

For the remainder of the paper, denote

$$\mathscr{L} = \log d_L.$$

Suppose the integer $\ell \geqslant 2$ and real numbers $A, B > 0$ satisfy $B - 2\ell A > 0$. Select the weight function f from Lemma 2.6 according to these parameters. Assume $2 \leqslant B \leqslant 100$ henceforth, while ℓ and A remain arbitrary.

Recall K is a number field with ring of integers $\mathcal{O}_K$ and L/K is a finite Galois extension. Let C be a conjugacy class of $G := \mathrm{Gal}(L/K)$. Define

$$S := \sum_{\substack{\mathfrak{p} \text{ unramified in } L \\ \mathrm{N}\mathfrak{p} = p \text{ rational prime}}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} f\Big(\frac{\log \mathrm{N}\mathfrak{p}}{\mathscr{L}}\Big) \cdot \mathbf{1}\Big\{ \Big[\frac{L/K}{\mathfrak{p}}\Big] = C \Big\}, \tag{4.1}$$

where $\mathrm{N} = \mathrm{N}^K_{\mathbb{Q}}$ is the absolute norm of K, $\mathbf{1}\{\cdot\}$ is an indicator function, and $\big[\frac{L/K}{\mathfrak{p}}\big]$ is the Artin symbol of $\mathfrak{p}$. To prove Theorem 1.1, we claim it suffices to show $S > 0$ for d_L sufficiently large and a suitable choice of parameters A, B

and ℓ; in particular, we must take $B \leqslant 40$. By our choice of f, it would follow that there exists an unramified prime ideal $\mathfrak{p}$ of degree 1 with $\left[\frac{L/K}{\mathfrak{p}}\right] = C$ satisfying $\mathrm{N}_{\mathbb{Q}}^{K}\mathfrak{p} \leqslant d_L^B$ for d_L sufficiently large. For all values of d_L which are not sufficiently large, the result follows from (1.1) (that is, [LMO79, Theorem 1.1]). This proves the claim.

Now, we wish to transform S into a contour integral by using the logarithmic derivatives of certain Artin L-functions. One is naturally led to consider the contour

$$I := \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Psi_C(s)F((1-s)\mathscr{L})ds \tag{4.2}$$

with

$$\Psi_C(s) := -\frac{|C|}{|G|}\sum_{\psi}\bar{\psi}(g)\frac{L'}{L}(s,\psi,L/K), \tag{4.3}$$

where $g \in C$, the sum runs over irreducible characters ψ of $\mathrm{Gal}(L/K)$, and $L(s,\psi,L/K)$ is the Artin L-function attached to ψ. By orthogonality of characters (see [Hei67, Section 3]), observe that

$$\Psi_C(s) = \sum_{\mathfrak{p}\subseteq\mathcal{O}_K}\sum_{m=1}^{\infty}\frac{\log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p}^m)^s}\cdot\Theta_C(\mathfrak{p}^m) \qquad \text{for } \mathrm{Re}\{s\} > 1, \tag{4.4}$$

where, for prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$ unramified in L,

$$\Theta_C(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } \left[\frac{L/K}{\mathfrak{p}}\right]^m \in C, \\ 0 & \text{else,}\end{cases} \tag{4.5}$$

and $0 \leqslant \Theta_C(\mathfrak{p}^m) \leqslant 1$ for prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$ ramified in L. Comparing (4.2) and (4.4), it follows by Mellin inversion that

$$I = \mathscr{L}^{-1}\sum_{\mathfrak{p}\subseteq\mathcal{O}_K}\sum_{m=1}^{\infty}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^m}f\Big(\frac{\log \mathrm{N}\mathfrak{p}^m}{\mathscr{L}}\Big)\cdot\Theta_C(\mathfrak{p}^m). \tag{4.6}$$

Comparing (4.6) and (4.1), it is apparent that the integral I and quantity $\mathscr{L}^{-1}S$ should be equal up to a neglible contribution from: (i) ramified prime ideals, (ii) prime ideals whose norm is not a rational prime, and (iii) prime ideal powers. In the following lemma, we prove exactly this by showing that the collective contribution of (i), (ii), and (iii) in (4.6) is bounded by $O(A^{-1}\mathscr{L}^2e^{-(B-2\ell A)\mathscr{L}/2})$.

Lemma 4.1. *In the above notation,*

$$\mathscr{L}^{-1}S = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Psi_C(s)F((1-s)\mathscr{L})ds + O(A^{-1}\mathscr{L}^2e^{-(B-2\ell A)\mathscr{L}/2}).$$

Proof. Denote $Q_1 = e^{(B-2\ell A)\mathscr{L}}$ and $Q_2 = e^{B\mathscr{L}}$.

Ramified prime ideals

Since the product of ramified prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$ divides the different $\mathfrak{D}_{L/K}$, it follows that

$$\sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \text{ramified in } L}} \log \mathrm{N}\mathfrak{p} \leqslant \log d_L = \mathscr{L}.$$

Therefore, by Lemma 2.6,

$$\begin{aligned}\sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \text{ramified in } L}} \sum_{m=1}^{\infty} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^m} f\Big(\frac{\log \mathrm{N}\mathfrak{p}^m}{\mathscr{L}}\Big)\Theta_C(\mathfrak{p}^m) &\ll A^{-1} \sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \text{ramified in } L}} \log \mathrm{N}\mathfrak{p} \sum_{\substack{m \geqslant 1 \\ \mathrm{N}\mathfrak{p}^m > Q_1}} \frac{1}{\mathrm{N}\mathfrak{p}^m} \\ &\ll A^{-1} \sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \text{ramified in } L \\ \mathrm{N}\mathfrak{p} > Q_1}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \\ &\ll A^{-1} \mathscr{L} e^{-(B-2\ell A)\mathscr{L}}.\end{aligned}$$

Prime ideals with norm not equal to a rational prime

For a given integer q, there are at most n_K prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$ satisfying $\mathrm{N}\mathfrak{p} = q$. Thus, by Lemma 2.6,

$$\begin{aligned}\sum_{p \text{ prime}} \sum_{k \geqslant 2} \sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \mathrm{N}\mathfrak{p} = p^k}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} f\Big(\frac{\log \mathrm{N}\mathfrak{p}}{\mathscr{L}}\Big) \cdot \Theta_C(\mathfrak{p}) &\ll A^{-1} n_K \mathscr{L} \sum_{p \text{ prime}} \sum_{\substack{k \geqslant 2 \\ Q_1 < p^k < Q_2}} \frac{1}{p^k} \\ &\ll A^{-1} n_K \mathscr{L} Q_1^{-1/2} \\ &\ll A^{-1} \mathscr{L}^2 e^{-(B-2\ell A)\mathscr{L}/2}.\end{aligned}$$

Note in the last step we used the fact that $n_K \leqslant n_L \ll \mathscr{L}$ by a theorem of Minkowski.

Prime ideal powers

Arguing similar to the previous case, one may again see that

$$\sum_{p \text{ prime}} \sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \mathrm{N}\mathfrak{p} = p}} \sum_{m \geqslant 2} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^m} f\Big(\frac{\log \mathrm{N}\mathfrak{p}^m}{\mathscr{L}}\Big) \cdot \Theta_C(\mathfrak{p}^m) \ll A^{-1} \mathscr{L}^2 e^{-(B-2\ell A)\mathscr{L}/2}.$$

The desired result follows after comparing (4.1), (4.2) and (4.6) with the three estimates above. ∎

4.2. Deuring's reduction

Equipped with Lemma 4.1, the natural next step is to move the contour to the left of $\mathrm{Re}\{s\} = 1$ but this poses a difficulty. Artin L-functions are not yet in general known to have analytic continuation in the left halfplane $\mathrm{Re}\{s\} \leqslant 1$. Thus, we employ a reduction due to Deuring [Deu35] as argued in Lagarias-Montgomery-Odlyzko [LMO79, Section 3] whose argument we repeat here for the sake of clarity.

For $g \in C$, define the cyclic subgroup $H = \langle g \rangle$ of $\mathrm{Gal}(L/K)$ and let E be the fixed field of H. Then by [Hei67, Lemma 4],

$$\Psi_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \frac{L'}{L}(s, \chi, L/E), \tag{4.7}$$

where the sum runs over irreducible characters χ of H. These characters are necessarily 1-dimensional since H is abelian. By class field theory, the Artin L-function $L(s, \chi, L/E)$ is actually a certain Hecke L-function $L(s, \chi, E)$ since L/E is abelian. Further, χ is a primitive Hecke character satisfying

$$\chi(\mathfrak{P}) = \chi\Big(\Big[\frac{L/E}{\mathfrak{P}}\Big]\Big)$$

for all prime ideals $\mathfrak{P} \subseteq \mathcal{O}_E$ unramified in L. Therefore, (4.7) becomes

$$\Psi_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \frac{L'}{L}(s, \chi, E), \tag{4.8}$$

where χ are certain primitive Hecke characters of E. Note that, from [Hei67] for example,

$$\zeta_L(s) = \prod_{\chi} L(s, \chi, L/E) \tag{4.9}$$

and the conductor-discriminant formula states

$$\log d_L = \sum_{\chi} \log(d_E \mathrm{N}^E_{\mathbb{Q}} \mathfrak{f}_\chi), \tag{4.10}$$

where $\mathfrak{f}_\chi \subseteq \mathcal{O}_E$ is the conductor of χ.

4.3. A sum over low-lying zeros

In light of (4.8), we are now in a position to use the analytic properties of Hecke L-functions and shift the contour in Lemma 4.1. We will reduce the analysis to a careful consideration of contribution coming from zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$ which are "low-lying".

Lemma 4.2. *Let $T^\star \geqslant 1$ be fixed. In the above notation,*

$$\Big|\tfrac{|G|}{|C|}\mathscr{L}^{-1}S - F(0)\Big| \leqslant \sum_{\substack{\rho \\ |\gamma|<T^\star}} |F((1-\rho)\mathscr{L})| + O\Big(\mathscr{L}\Big(\frac{2}{AT^\star\mathscr{L}}\Big)^{2\ell} + \frac{\mathscr{L}^2}{A}e^{-(B-2\ell A)\mathscr{L}/2}\Big) + O\Big(\mathscr{L}\Big(\frac{1}{A\mathscr{L}}\Big)^{2\ell}e^{-(B-2\ell A)\mathscr{L}} + \mathscr{L}\Big(\frac{2}{A\mathscr{L}}\Big)^{2\ell}e^{-3(B-2\ell A)\mathscr{L}/2}\Big), \tag{4.11}$$

where the sum is over non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$.

Proof. Consider the contour in Lemma 4.1. Using (4.8), we shift the line of integration to $\mathrm{Re}\{s\} = -\frac{1}{2}$. From (4.9), this picks up exactly the non-trivial zeros of $\zeta_L(s)$, its simple pole at $s = 1$, and its trivial zero at $s = 0$ of order $r_1 + r_2 - 1$. For $\mathrm{Re}\{s\} = -1/2$, we have by Lemma 2.6 that

$$F((1-s)\mathscr{L}) \ll e^{-3(B-2\ell A)\mathscr{L}/2} \cdot \Big(\frac{2}{A\mathscr{L}|s|}\Big)^{2\ell} \tag{4.12}$$

and, from [LO77, Lemma 6.2] and (4.10),

$$\begin{aligned}\sum_\chi \Big|\frac{L'}{L}(s,\chi,E)\Big| &\ll \sum_\chi \big\{\log(d_E \mathrm{N}^E_{\mathbb{Q}}\mathfrak{f}_\chi) + n_E\log(|s|+2)\big\} \\ &\ll \mathscr{L} + [L:E]\cdot n_E\log(|s|+2) \\ &\ll \mathscr{L} + n_L\log(|s|+2).\end{aligned}$$

It follows that

$$\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} \Psi_C(s)F((1-s)\mathscr{L})ds \ll \mathscr{L}\Big(\frac{2}{A\mathscr{L}}\Big)^{2\ell}e^{-3(B-2\ell A)\mathscr{L}/2}$$

as $n_L \ll \mathscr{L}$. For the zero at $s = 0$ of $\Psi_C(s)$, we may bound its contribution using (2.11) to deduce that

$$(r_1 + r_2 - 1)F(\mathscr{L}) \ll \mathscr{L}\Big(\frac{1}{A\mathscr{L}}\Big)^{2\ell}e^{-(B-2\ell A)\mathscr{L}},$$

since $r_1 + 2r_2 = n_L \ll \mathscr{L}$. These observations and Lemma 4.1 therefore yield

$$\Big|\tfrac{|G|}{|C|}\mathscr{L}^{-1}S - F(0)\Big| \leqslant \sum_\rho |F((1-\rho)\mathscr{L})| + O\Big(\frac{\mathscr{L}^2}{A}e^{-(B-2\ell A)\mathscr{L}/2} + \mathscr{L}\Big(\frac{1}{A\mathscr{L}}\Big)^{2\ell}e^{-(B-2\ell A)\mathscr{L}}\Big) + O\Big(\mathscr{L}\Big(\frac{2}{A\mathscr{L}}\Big)^{2\ell}e^{-3(B-2\ell A)\mathscr{L}/2}\Big), \tag{4.13}$$

where the sum is over all non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$. By [LMO79, Lemma 2.1] and Lemma 2.6, we have that

$$\sum_{k=0}^{\infty} \sum_{\substack{\rho \\ T^\star+k \leqslant |\gamma| < T^\star+k+1}} |F((1-\rho)\mathscr{L})| \ll \Big(\frac{2}{A\mathscr{L}}\Big)^{2\ell} \sum_{k=0}^{\infty} \frac{\mathscr{L} + n_L \log(T^\star + k)}{(T^\star + k)^{2\ell}}$$
$$\ll \mathscr{L}\Big(\frac{2}{AT^\star \mathscr{L}}\Big)^{2\ell},$$

as $n_L \ll \mathscr{L}$, $T^\star$ is fixed, and $\ell \geqslant 2$. The result follows from (4.13) and the above estimate. ∎

For the sum over low-lying zeros in Lemma 4.2, we bound zeros far away from the line $\mathrm{Re}\{s\} = 1$ using Lemma 4.3 below. In the non-exceptional case, this could have been done in a fairly simple manner but when a Siegel zero exists, we will need to partition the zeros according to their height. This will amount to applying a coarse version of partial summation, allowing us to exploit the Deuring-Heilbronn phenomenon more efficiently.

Lemma 4.3. *Let $J \geqslant 1$ be given and $T^\star \geqslant 1$ be fixed. Suppose*

$$1 \leqslant R_1 \leqslant R_2 \leqslant \ldots \leqslant R_J \leqslant \mathscr{L}, \qquad 0 = T_0 < T_1 \leqslant T_2 \leqslant \ldots \leqslant T_J = T^\star.$$

Then

$$\sum_{\substack{\rho \\ |\gamma| < T^\star}} |F((1-\rho)\mathscr{L})| = \sum_{\rho}{}' |F((1-\rho)\mathscr{L})| + O\Big(\min\Big\{\Big(\frac{2}{A}\Big)^{2\ell}, \mathscr{L}\Big\} e^{-(B-2\ell A)R_1}\Big)$$
$$+ \sum_{j=2}^{J} O\Big(\mathscr{L}\Big(\frac{2}{AT_{j-1}\mathscr{L}}\Big)^{2\ell} e^{-(B-2\ell A)R_j}\Big), \tag{4.14}$$

where the marked sum $\sum'$ indicates a restriction to zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$ satisfying

$$\beta > 1 - \frac{R_j}{\mathscr{L}}, \qquad T_{j-1} \leqslant |\gamma| < T_j \quad \textit{for some } 1 \leqslant j \leqslant J.$$

If $J = 1$ then the secondary error term in (4.14) vanishes.

Remark. To prove Theorem 1.1, we will apply the above lemma with $J = 10$ when a Siegel zero exists. One could use higher values of J or a more refined version of Lemma 4.3 to obtain some improvement on the final result.

Proof. Recall $\ell \geqslant 2$ for our choice of weight f. Let $1 \leqslant j \leqslant J$ be arbitrary. Define the multiset

$$\mathcal{Z}_j := \{\rho : \zeta_L(\rho) = 0,\ \beta \leqslant 1 - \frac{R_j}{\mathscr{L}},\ T_{j-1} \leqslant |\gamma| < T_j\}$$

and denote $S_j := \sum_{\rho\in\mathcal{Z}_j} |F((1-\rho)\mathscr{L})|$. Since

$$\sum_{\substack{\rho \\ |\gamma|<T^\star}} |F((1-\rho)\mathscr{L})| = \sum_{\rho}{}' |F((1-\rho)\mathscr{L})| + \sum_{j=1}^{J} S_j,$$

it suffices to show

$$S_1 \ll \min\Big\{\Big(\frac{2}{A}\Big)^{2\ell}, \mathscr{L}\Big\} e^{-(B-2\ell A)R_1}$$

and

$$S_j \ll \mathscr{L}\Big(\frac{2}{AT_{j-1}\mathscr{L}}\Big)^{2\ell} e^{-(B-2\ell A)R_j}, \qquad \text{for } 2 \leqslant j \leqslant J.$$

Assume $2 \leqslant j \leqslant J$. As $T_j \leqslant T^\star$ and $T^\star$ is fixed, it follows $\#\mathcal{Z}_j \ll \mathscr{L}$ by [LMO79, Lemma 2.1]. Hence, by Lemma 2.6 and the definition of $\mathcal{Z}_j$,

$$S_j \ll e^{-(B-2\ell A)R_j} \sum_{\rho\in\mathcal{Z}_j} \Big(\frac{2}{A|\gamma|\mathscr{L}}\Big)^{2\ell} \ll \mathscr{L}\Big(\frac{2}{AT_{j-1}\mathscr{L}}\Big)^{2\ell} e^{-(B-2\ell A)R_j},$$

as desired. It remains to consider S_1. On one hand, we similarly have $\#\mathcal{Z}_1 \ll \mathscr{L}$ by [LMO79, Lemma 2.1]. Thus, by Lemma 2.6 and the definition of S_1,

$$S_1 \ll \mathscr{L} e^{-(B-2\ell A)R_1}. \tag{4.15}$$

On the other hand, we may give an alternate bound for S_1. For integers $1 \leqslant m, n \leqslant \mathscr{L}$, consider the rectangles

$$\mathcal{R}_{m,n} := \Big\{s = \sigma + it \in \mathbb{C} : 1 - \frac{m+1}{\mathscr{L}} \leqslant \sigma \leqslant 1 - \frac{m}{\mathscr{L}}, \quad \frac{n-1}{\mathscr{L}} \leqslant |t| \leqslant \frac{n}{\mathscr{L}}\Big\}.$$

We bound the contribution of zeros ρ lying in $\mathcal{R}_{m,n}$ when $m \geqslant R_1$. If a zero $\rho \in \mathcal{R}_{m,n}$ then

$$|F((1-\rho)\mathscr{L})| \ll e^{-(B-2\ell A)m}\Big(\frac{2}{A\sqrt{m^2+(n-1)^2}}\Big)^{2\ell},$$

by Lemma 2.6 with $\alpha = 2\ell$. Further, by [LMO79, Lemma 2.2],

$$\#\{\rho \in \mathcal{R}_{m,n} : \zeta_L(\rho) = 0\} \ll \sqrt{(m+1)^2+n^2} \ll \sqrt{m^2+(n-1)^2}.$$

The latter estimate follows since $m, n \geqslant 1$. Adding up these contributions and using the conjugate symmetry of zeros, we find that

$$\begin{aligned} S_1 &\ll \sum_{\substack{m\geqslant R_1 \\ n\geqslant 1}} \sum_{\substack{\rho\in\mathcal{R}_{m,n} \\ \zeta_L(\rho)=0}} |F((1-\rho)\mathscr{L})| \\ &\ll \Big(\frac{2}{A}\Big)^{2\ell} \sum_{\substack{m\geqslant R_1 \\ n\geqslant 1}} e^{-(B-2\ell A)m}\big(\sqrt{m^2+(n-1)^2}\big)^{-2\ell+1} \\ &\ll \Big(\frac{2}{A}\Big)^{2\ell} e^{-(B-2\ell A)R_1}, \end{aligned}$$

since $\ell \geqslant 2$. Taking the minimum of the above and (4.15) gives the desired bound for S_1. ∎

If a Siegel zero exists then we shall choose the parameters in Lemma 4.3 so that the restricted sum over zeros is actually empty. Otherwise, if a Siegel zero does not exist then Lemma 4.3 will be applied with $J = 1$ and $T_1 = T^\star = 1$ so we must handle the remaining restricted sum over zeros in the final arguments.

Lemma 4.4. *Let $\eta > 0$ and $R \geqslant 1$ be arbitrary. For $A > 0$ and $\ell \geqslant 1$, define*

$$\tilde{F}_\ell(z) := \Big(\frac{1-e^{-Az}}{Az}\Big)^{2\ell}.$$

Suppose $\zeta_L(s)$ is non-zero in the region

$$\mathrm{Re}\{s\} \geqslant 1 - \frac{\lambda}{\mathscr{L}}, \qquad |\Im\{s\}| \leqslant 1,$$

for some $0 < \lambda \leqslant 10$. Then, provided d_L is sufficiently large depending on η, R, and A,

$$\sum_\rho{}' |\tilde{F}_\ell((1-\rho)\mathscr{L})| \leqslant \Big(\frac{1-e^{-A\lambda}}{A\lambda}\Big)^{2(\ell-1)} \cdot \Big\{\phi\Big(\frac{1-e^{-2A\lambda}}{A^2\lambda}\Big) + \frac{2A\lambda - 1 + e^{-2A\lambda}}{2A^2\lambda^2} + \eta\Big\}, \tag{4.16}$$

where $\phi = \frac{1}{2}(1-\frac{1}{\sqrt{5}})$ and the marked sum $\sum'$ indicates a restriction to zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$ satisfying

$$\beta \geqslant 1 - \frac{R}{\mathscr{L}}, \qquad |\gamma| \leqslant 1.$$

In particular, as $\lambda \to 0$, the bound in (4.16) becomes $\frac{2\phi}{A} + 1 + \eta$.

Proof. This result is motivated by [HB95, Lemma 13.3]. Define

$$h(t) := \begin{cases} A^{-2} \cdot \sinh\big((A-t)\lambda\big) & \text{if } 0 \leqslant t \leqslant A, \\ 0 & \text{if } t \geqslant A, \end{cases}$$

so

$$H(z) = \int_0^\infty e^{-zt} h(t) dt = \frac{1}{2A^2}\Big\{\frac{e^{A\lambda}}{\lambda+z} + \frac{e^{-A\lambda}}{\lambda - z} - \frac{2\lambda e^{-Az}}{\lambda^2 - z^2}\Big\}.$$

As per the argument in [HB95, Lemma 13.3],

$$|\tilde{F}_1(\lambda + z)| \leqslant \frac{2e^{-A\lambda}}{\lambda} \cdot \mathrm{Re}\{H(z)\} \tag{4.17}$$

for $\mathrm{Re}\{z\} \geqslant 0$. Combining the above with Lemma 2.7 and noting $(1-e^{-x})/x$ is decreasing for $x > 0$, it follows that

$$|\tilde{F}_\ell(\lambda + z)| \leqslant \Big(\frac{1-e^{-A\lambda}}{A\lambda}\Big)^{2(\ell-1)} \cdot \frac{2e^{-A\lambda}}{\lambda} \cdot \mathrm{Re}\{H(z)\}$$

for $\mathrm{Re}\{z\} \geqslant 0$. Setting $\sigma = 1 - \frac{\lambda}{\mathscr{L}} \in \mathbb{R}$, this implies

$$\sideset{}{'}\sum_{\rho} |\tilde{F}_\ell((1-\rho)\mathscr{L})| \leqslant \Big(\frac{1-e^{-A\lambda}}{A\lambda}\Big)^{2(\ell-1)} \cdot \frac{2e^{-A\lambda}}{\lambda} \sideset{}{'}\sum_{\rho} \mathrm{Re}\{H((\sigma-\rho)\mathscr{L})\},$$

so it suffices to bound the sum on the RHS. Since h and H satisfy Conditions 1 and 2 of [KN12], we apply [KN12, Theorem 3] to bound the sum $\sum'$ on the RHS yielding

$$\begin{aligned}\sideset{}{'}\sum_{\rho} \mathrm{Re}\{H((\sigma-\rho)\mathscr{L})\} &\leqslant h(0)(\phi+\eta) + H((\sigma-1)\mathscr{L}) \\ &\quad - \mathscr{L}^{-1} \sum_{\mathfrak{N} \subseteq \mathcal{O}_L} \frac{\Lambda_L(\mathfrak{N})}{(\mathrm{N}^L_{\mathbb{Q}}\mathfrak{N})^\sigma} h\Big(\frac{\log \mathrm{N}^L_{\mathbb{Q}}\mathfrak{N}}{\mathscr{L}}\Big) \\ &\leqslant h(0)(\phi+\eta) + H((\sigma-1)\mathscr{L}),\end{aligned}$$

for d_L sufficiently large depending on η, R, and A. Using the definitions of h and H and rescaling η appropriately, we obtain the desired result. ■

5. Proof of Theorem 1.1

Let $\mathcal{Z}$ be the multiset consisting of zeros of $\zeta_L(s)$ in the rectangle

$$0 < \mathrm{Re}\{s\} < 1, \qquad |\Im\{s\}| \leqslant 1.$$

Choose $\rho_1 \in \mathcal{Z}$ such that $\mathrm{Re}\{\rho_1\} = \beta_1 = 1 - \frac{\lambda_1}{\mathscr{L}} \in (0,1)$ is maximal. If $\lambda_1 < 0.0784$ then ρ_1 is exceptional; that is, ρ_1 is a simple real zero of $\zeta_L(s)$ as shown by Kadiri [Kad12]. We divide our arguments according to this exceptional case. Recall that our goal is to show the quantity S, defined by (4.1), is strictly positive for d_L sufficiently large and $B \leqslant 40$.

5.1. Non-exceptional case ($\lambda_1 \geqslant 0.0784$)

Choose

$$\ell = 2, \qquad B = 7.41, \qquad \text{and} \qquad A = 1.5$$

to give a corresponding f and its Laplace transform F defined by Lemma 2.6. Observe that $B - 2\ell A = 1.41$ for the above choices.

Let $\epsilon > 0$. Apply Lemma 4.2 with $T^\star = 1$. Then employ Lemma 4.3 with $J = 1, T_1 = T^\star = 1$ and $R_1 = R = R(\epsilon)$ sufficiently large so that

$$\frac{|G|}{|C|}\mathscr{L}^{-1} S \geqslant 1 - \sideset{}{'}\sum_{\rho} |F((1-\rho)\mathscr{L})| - \epsilon$$

for d_L sufficiently large depending on ϵ. Here the restricted sum is over zeros $\rho = \beta + i\gamma$ satisfying

$$\beta > 1 - \frac{R}{\mathscr{L}}, \qquad |\gamma| < 1.$$

It suffices to prove the sum over zeros ρ is $< 1 - \epsilon/2$ for fixed sufficiently small ϵ. Observe by the definition of $\tilde{F}_2$ in Lemma 4.2 and our choice of ρ_1 that

$$\sum_{\rho}{}' |F((1-\rho)\mathscr{L})| = \sum_{\rho}{}' e^{-1.41\lambda}|\tilde{F}_2((1-\rho)\mathscr{L})| \leqslant e^{-1.41\lambda_1} \sum_{\rho}{}' |\tilde{F}_2((1-\rho)\mathscr{L})|.$$

Since $\lambda_1 \geqslant 0.0784$, we may bound the remaining sum using Lemma 4.4 with $\lambda = 0.0784$. Hence, the above is

$$\leqslant e^{-1.41\lambda_1} \times 1.1166 \leqslant e^{-1.41\times 0.0784} \times 1.1166 = 0.9997\cdots < 1,$$

as required.

5.2. Exceptional case ($\lambda_1 < 0.0784$)

For this subsection, let $0 < \eta < 0.0784$ be an absolute parameter which will be specified later.

λ_1 small ($0.0784 > \lambda_1 \geqslant \eta$)

Again, choose the weight function f from Lemma 2.6 with

$$\ell = 2, \qquad B = 2.63, \qquad \text{and} \qquad A = 0.1$$

so $B - 2\ell A = 2.23$. The argument is similar to the previous case but we take special care of the real zero β_1. By the same choices as the non-exceptional case, we deduce that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geqslant 1 - |F((1-\beta_1)\mathscr{L})| - \sum_{\rho\neq\beta_1}{}' |F((1-\rho)\mathscr{L})| - \epsilon \tag{5.1}$$

for d_L sufficiently large depending on ϵ. Observe that, since ρ_1 is real and $(1 - e^{-t})/t \leqslant 1$ for $t > 0$,

$$|F((1-\rho_1)\mathscr{L})| = e^{-2.23\lambda_1}\Big(\frac{1-e^{-0.1\lambda_1}}{0.1\lambda_1}\Big)^4 \leqslant e^{-2.23\lambda_1}.$$

Now, select another zero $\rho' \in \mathcal{Z}$ of $\zeta_L(s)$ such that $\rho' \neq \rho_1$ (counting with multiplicity in $\mathcal{Z}$) and $\text{Re}\{\rho'\} = \beta' = 1 - \frac{\lambda'}{\mathscr{L}}$ is maximal. In the exceptional case, ρ_1 is a simple real zero so ρ' is indeed genuinely distinct from ρ_1. By our choice of ρ', Lemma 2.6, and a subsequent application of Lemma 4.4 with $\lambda = 0$, we gave that

$$\sum_{\rho\neq\rho_1}{}' |F((1-\rho)\mathscr{L})| \leqslant e^{-2.23\lambda'} \sum_{\rho\neq\rho_1}{}' |\tilde{F}_2((1-\rho)\mathscr{L})| \leqslant e^{-2.23\lambda'} \times 6.5279.$$

As $\lambda_1 \geqslant \eta$, it follows that $\lambda' \geqslant 0.6546 \log \lambda_1^{-1}$ from [KN12, Theorem 4] for d_L sufficiently large depending on η. Hence, the above is

$$\leqslant 6.5279 \times \lambda_1^{2.23\times 0.6546} \leqslant 6.5279 \times \lambda_1^{1.4597}.$$

Thus, (5.1) implies

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geqslant 1 - e^{-2.23\lambda_1} - 6.5279 \times \lambda_1^{1.4597} - \epsilon$$
$$\geqslant \big(2.23 - 6.5279 \times \lambda_1^{0.4597} - 2.4865\lambda_1\big)\lambda_1 - \epsilon,$$

since $1 - e^{-t} \geqslant t - t^2/2$ for $t > 0$. The quantity in the brackets is clearly decreasing with λ_1 so, since $\lambda_1 < 0.0784$, we conclude that the above is

$$\geqslant \big(2.23 - 6.5279 \times 0.0784^{0.4597} - 2.4865 \times 0.0784\big)\lambda_1 - \epsilon$$
$$\geqslant 0.0097\lambda_1 - \epsilon.$$

As $\lambda_1 \geqslant \eta > 0$ by assumption, the result follows after taking $\epsilon = 10^{-6}\eta$.

$\boldsymbol{\lambda_1}$ very small ($\mathscr{L}^{-200} \leqslant \lambda_1 < \eta$)

Choose the weight function f from Lemma 2.6 with

$$\ell = 101, \qquad B = 36.5, \qquad \text{and} \qquad A = \tfrac{1}{404},$$

so $B - 2\ell A = 36$. Applying Lemma 4.2 with $T^\star = 1$, it follows that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geqslant 1 - |F((1-\beta_1)\mathscr{L})| - \sum_{\substack{\rho \neq \beta_1 \\ |\gamma| < 1}} |F((1-\rho)\mathscr{L})| + O(\mathscr{L}^{-201}).$$

Similar to the previous subcase, we have that $|F((1-\beta_1)\mathscr{L})| \leqslant e^{-36\lambda_1}$. For the remaining sum over zeros, we apply Lemma 4.3 with $J = 1, T^\star = T_1 = 1$, and $R_1 = \frac{1}{35.8}\log(c_1/\lambda_1)$ with $c_1 > 0$ absolute and sufficiently small. As $\lambda_1 \geqslant \mathscr{L}^{-200}$, we may assume without loss that $R_1 < \frac{1}{4}\mathscr{L}$ for $\mathscr{L}$ sufficiently large[2]. Therefore,

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geqslant 1 - e^{-36\lambda_1} - \sideset{}{'}\sum_{\rho \neq \beta_1} |F((1-\rho)\mathscr{L})| + O\big(\mathscr{L}^{-201} + \lambda_1^{36/35.8}\big), \tag{5.2}$$

where the sum $\sum'$ is defined as per Lemma 4.3. By our choice of parameters T_1 and R_1, it follows from Theorem 3.1 that the restricted sum over zeros in (5.2) is actually empty. As $1 - e^{-t} \geqslant t - t^2/2$ for $t > 0$, we conclude that

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S \geqslant 36\lambda_1 + O(\mathscr{L}^{-201} + \lambda_1^{36/35.8}).$$

Since $\mathscr{L}^{-200} \leqslant \lambda_1 < \eta$ by assumption and η is sufficiently small, we conclude that the RHS is $\gg \lambda_1$ after fixing η.

[2] This implies that the zero $1 - \beta_1$ is already discarded in the error term arising from Lemma 4.3. This minor point will be relevant when λ_1 is extremely small.

λ_1 extremely small ($\lambda_1 < \mathscr{L}^{-200}$)

Choose the weight function f from Lemma 2.6 with

$$\ell = \lceil 1.1\mathscr{L} \rceil, \qquad B = 39.5, \qquad \text{and} \qquad A = \frac{0.9}{\mathscr{L}},$$

so $B - 2\ell A > 37.5$ for d_L sufficiently large. Applying Lemma 4.2 with $T^\star = 12646$, it follows that

$$\begin{aligned}\left|\tfrac{|G|}{|C|}\mathscr{L}^{-1}S - F(0)\right| &\leqslant \sum_{\substack{\rho \\ |\gamma|<12646}} |F((1-\rho)\mathscr{L})| \\ &\quad + O\Big(\mathscr{L}e^{2.2\log\left(\frac{2}{0.9\times 12646}\right)\mathscr{L}} + \mathscr{L}^3 e^{-37.5\mathscr{L}/2}\Big) \\ &\quad + O\Big(\mathscr{L}e^{-37.5\mathscr{L}+2.2\log\left(\frac{1}{0.9}\right)\mathscr{L}} + \mathscr{L}e^{-\frac{3}{2}\times 37.5\mathscr{L}+2.2\log\left(\frac{2}{0.9}\right)\mathscr{L}}\Big) \\ &\leqslant \sum_{\substack{\rho \\ |\gamma|<12646}} |F((1-\rho)\mathscr{L})| + O(e^{-18\mathscr{L}}). \qquad (5.3)\end{aligned}$$

For the remaining sum, we use Lemma 4.3 with $J = 10$ selecting T_j and $R_j = \frac{\log(c_j/\lambda_1)}{C_j}$ according to the table below. Note $c_j = c(T_j) > 0$ is the absolute constant in Theorem 3.1.

j	1	2	3	4	5	6	7	8	9	10
T_j	3.5	8.7	22	54	134	332	825	2048	5089	12646
C_j	37.0	39.3	42.5	46.1	50.0	53.8	57.6	61.4	65.2	69.0

Therefore,

$$\begin{aligned}\frac{|G|}{|C|}\mathscr{L}^{-1}S &\geqslant 1 - |F((1-\beta_1)\mathscr{L})| \\ &\qquad - \sideset{}{'}\sum_{\rho\neq\beta_1,1-\beta_1} |F((1-\rho)\mathscr{L})| - |F(\beta_1\mathscr{L})| + O(e^{-18\mathscr{L}}) \\ &\qquad + O\big(\mathscr{L}\lambda_1^{37.5/37.0}\big) + \sum_{j=2}^{10} O\Big(\mathscr{L}e^{2.2\log\left(\frac{2}{0.9T_{j-1}}\right)\mathscr{L}}\lambda_1^{37.5/C_j}\Big),\end{aligned} \qquad (5.4)$$

where the sum $\sum'$ is defined as per Lemma 4.3. Since the zeros of $\zeta_L(s)$ are permuted under the map $\rho \mapsto 1-\rho$, it follows from Theorem 3.1 and our choice of parameters T_j and C_j that the restricted sum over zeros in (5.4) is actually empty. For the zeros $1-\beta_1$ and β_1, notice

$$|F((1-\beta_1)\mathscr{L})| \leqslant e^{-37.5\lambda_1} \qquad \text{and} \qquad F(\beta_1\mathscr{L}) \leqslant e^{-37.5(\mathscr{L}-\lambda_1)} = O(e^{-37.5\mathscr{L}})$$

as $\lambda_1 < 0.0784$. Moreover, as $\lambda_1 < \mathscr{L}^{-200}$ and $\frac{37.5}{37.0} > 1.01$, we observe that

$$\mathscr{L} \cdot \lambda_1^{37.5/37.0} \ll \lambda_1^{-1/200} \cdot \lambda_1^{1.01} \ll \lambda_1^{1.005}.$$

To bound the sum over error terms in (5.4), notice $\lambda_1 \gg \mathscr{L} e^{-16.6\mathscr{L}}$ by Corollary 1.4, which implies

$$\mathscr{L} e^{2.2 \log\left(\frac{2}{0.9T_j}\right)\mathscr{L}} \lambda_1^{37.5/C_j} \ll \lambda_1 \cdot \mathscr{L}^2 e^{2.2 \log\left(\frac{2}{0.9T_{j-1}}\right)\mathscr{L} + 16.6(1-37.5/C_j)\mathscr{L}}.$$

Substituting the prescribed values for C_j and T_{j-1}, the above is $\ll \lambda_1 e^{-0.2\mathscr{L}}$ for all $2 \leqslant j \leqslant 10$. Incorporating all of these observations into (5.4) yields

$$\begin{aligned} \frac{|G|}{|C|} \mathscr{L}^{-1} S &\geqslant 1 - e^{-37.5\lambda_1} + O\Big(\lambda_1^{1.005} + \lambda_1 e^{-0.2\mathscr{L}} + e^{-18\mathscr{L}}\Big) \\ &\geqslant 37.5\lambda_1 + O\Big(\lambda_1^{1.005} + \lambda_1 e^{-0.2\mathscr{L}} + e^{-18\mathscr{L}}\Big), \end{aligned}$$

since $1 - e^{-t} \geqslant t - t^2/2$ for $t > 0$. Noting $\lambda_1 \gg \mathscr{L} e^{-16.6\mathscr{L}}$ by Corollary 1.4, we finally conclude that the RHS is positive for d_L sufficiently large and $\lambda_1 < \mathscr{L}^{-200}$.

Remark. We outline the minor modifications required to justify the remark following Theorem 1.1.

- If there is a sequence of fields $\mathbb{Q} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_r = L$ such that L_j is normal over L_{j-1} for $1 \leqslant j \leqslant r$ then, by [Sta74, Lemmas 10, 11], it follows that $\lambda_1 \gg \mathscr{L} e^{-0.5\mathscr{L}}$. For λ_1 extremely small, one may therefore select

$$\ell = \lceil 0.05\mathscr{L} \rceil, \qquad B = 36.4, \qquad \text{and} \qquad A = \frac{3.53}{\mathscr{L}},$$

  and apply Lemma 4.2 with $T^\star = 149$. Afterwards, employ Lemma 4.3 with T_j and $R_j = \frac{\log(c_j/\lambda_1)}{C_j}$ chosen according to the table below.

j	1	2	3
T_j	1	12.2	149
C_j	35.8	40.3	50.4

  and follow the same arguments. This requires additional instances of Theorem 3.1 with $T = 12.2$ and 149 yielding $C(T) = 40.3$ amd 50.4 respectively.
- If $n_L = o(\log d_L)$ then, by the remark following Theorem 1.3, we have that $\lambda_1 \gg \mathscr{L} e^{-12.01\mathscr{L}}$. Moreover, by remark (ii) following Theorem 3.1, one can use

$$J = 1, \qquad T_1 = T^\star = e^{64}, \qquad \text{and} \qquad R_1 = \frac{\log(c/\lambda_1)}{24.01}$$

  in the application of Lemmas 4.2 and 4.3. One may then modify the subcase λ_1 very small to consider $\mathscr{L}^{-1000} \leqslant \lambda_1 < \eta$ and take

$$\ell = 1000, \qquad B = 24.1, \qquad A = 1/10^6.$$

Similarly, one may modify the subcase λ_1 extremely small to consider $\lambda_1 < \mathscr{L}^{-1000}$ and take

$$\ell = \lceil 0.1\mathscr{L} \rceil, \qquad B = 24.1, \qquad \text{and} \qquad A = \frac{0.2}{\mathscr{L}}.$$

Following the same arguments yields the claimed result.

- If $\zeta_L(s)$ does not have a Siegel zero then $\lambda_1 \gg 1$ so the subcases when $\lambda_1 < \eta$ are unnecessary.

Remark. For λ_1 extremely small, the selection of parameters A, B, ℓ, T_1 and T_2 was primarily based on numerical experimentation.

References

[AK14] J.-H. Ahn and S.-H. Kwon, *Some explicit zero-free regions for Hecke L-functions*, J. Number Theory **145** (2014), 433–473.

[Deu35] M. Deuring, *Über den Tschebotareffschen Dichtigkeitssatz*, Math. Ann. **110**(1) (1935), 414–415.

[HB95] D.R. Heath-Brown, *Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progressions*, Proc. London Math. Soc. **64**(2) (1995), 265–338.

[Hei67] H. Heilbronn, *Zeta functions and L-functions*, in J.W.S Cassels and A. Fröhlich, editors, *Algebraic Number Theory*, pages 204–230, Academic Press, 1967.

[Kad12] H. Kadiri, *Explicit zero-free regions for Dedekind zeta functions*, Int. J. Number Theory **8**(1) (2012), 1–23.

[KM00] V.K. Murty, *The least prime in a conjugacy class*, C. R. Math. Acad. Sci. Soc. R. Can. **22**(4) (2000), 129–146.

[KN] H. Kadiri and N. Ng, *The least prime ideal in the Chebotarev Density Theorem*, unpublished.

[KN12] H. Kadiri and N. Ng, *Explicit zero density theorems for Dedekind zeta functions*, J. Number Theory **132** (2012), 748–775.

[Lin44a] Y.V. Linnik, *On the least prime in an arithmetical progression. I. The basic theorem*, Rec. Math. [Mat. Sbornik] **15** (1944), 139–178.

[Lin44b] Y.V. Linnik, *On the least prime in an arithmetical progression. II. The Deuring-Heilbronn phenomenon*, Rec. Math. [Mat. Sbornik] **15** (1944), 347–368.

[LMO79] J.C. Lagarias, H.L. Montgomery, and A.M. Odlyzko, *A bound for the least prime ideal in the Chebotarev density theorem*, Invent. Math. **54**(3) (1979), 271–296.

[LO77] J.C. Lagarias and A.M. Odlyzko, *Effective versions of the Chebotarev density theorem*, in *Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975)*, pages 409–464, Academic Press, London, 1977.

[Odl77] A.M. Odlyzko, *Lower bounds for discriminants of number fields. II*, Tôhoku Math. J. **29**(2) (1977), 209–216.

[Sta74] H. Stark, *Some effective cases of the Brauer-Siegel theorem*, Invent. Math. **23** (1974), 135–152.

[Tsc26] N.Tschebotareff, *Die Bestimmung der Dichtigkeit einer Menge von Primzahlen, welche zu einer gegebenen Substitutionsklasse gehören*, Math. Ann. **95**(1) (1926), 191–228.

[Zam17] A. Zaman, *Analytic estimates for the Chebotarev Density Theorem and their Applications*, PhD thesis, University of Toronto, 2017.

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