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RIESZ MEANS OF THE EULER TOTIENT FUNCTION

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Abstract: Let ϕ denote the Euler totient function, defined by $\mathrm{id} * \mu$ where μ is the Möbius function. We shall consider the k-th Riesz mean of the arithmetical function $n/\phi(n)$ for any positive integer $k \geqslant 2$ on the assumption of the Riemann Hypothesis. Our result is a refinement of Theorem 2 in A. Sankaranarayanan and S. K. Singh [6]. We also improve it upon the assumption of the Gonek-Hejhal Hypothesis.

Keywords: Euler totient function, Riemann zeta-function, Riemann Hypothesis, Mertens Hypothesis, Gonek-Hejhal Hypothesis.

1. Statement of results

Let $s = \sigma + it$ be a complex variable, where σ and t are real, and let γ be Euler's constant. The arithmetical function ϕ denotes the Euler totient function, defined by $\phi(n) = \sum_{d|n} d\mu(n/d)$, where μ is the Möbius function. For any positive real number $x\,(\geqslant x_0)$, with x_0 being a sufficiently large positive number, R. Sitaramachandrarao [7] established the first Riesz mean of $n/\phi(n)$ and showed that the asymptotic relation

$$\sum_{n\leqslant x} \frac{n}{\phi(n)}\left(1-\frac{n}{x}\right) = \frac{315\zeta(3)}{4\pi^4}x - \frac{1}{2}\log x + \frac{1}{2}\left(1-\gamma-\log(2\pi)-\sum_p \frac{\log p}{p(p-1)}\right) + E_1(x) \tag{1.1}$$

holds with error term $E_1(x) = O\left(x^{-\frac{1}{5}}\right)$. He conjectured that $E_1(x) \ll x^{-\frac{3}{4}+\varepsilon}$ holds for any small fixed positive number ε. Recently, the asymptotic formulas for the general k-th Riesz mean related to the arithmetic function $n/\phi(n)$, for any positive integer $k \geqslant 2$, were studied by A. Sankaranarayanan and S. K. Singh [4]–[6].

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They showed that

$$\frac{1}{k!}\sum_{n\leqslant x}\frac{n}{\phi(n)}\left(1-\frac{n}{x}\right)^k = c_1(k)x + c_2(k)\log x + c_3(k) + E_k(x) \tag{1.2}$$

holds, where the function $E_k(x)$ is the error term. Here the constants $c_1(k)$, $c_2(k)$, and $c_3(k)$ are given by[1] $c_1(k) = \frac{\zeta(2)\zeta(3)}{(k+1)!\zeta(6)}$, $c_2(k) = -\frac{1}{2k!}$, and

$$c_3(k) = \frac{1}{k!}\left\{\frac{1}{2}\sum_{j=1}^{k}\frac{1}{j} - \frac{\gamma}{2} - \log\sqrt{2\pi} - \frac{1}{2}\sum_{p}\frac{\log p}{p(p-1)}\right\},$$

respectively. They showed [4] that $E_k(x) \ll x^{-\frac{1}{2}+\varepsilon}$ for any small fixed positive constant ε and that the implied constant is independent of k. Later, they refined in [5] certain arguments of [4], and with some extra inputs they established the following result in [5]. Let c^* be any real number $\geqslant 10$. Then

$$E_1(x) \ll x^{-\frac{1}{2}}(\log x)^{\frac{5}{4}}\log\log x$$

and

$$E_k(x) \ll \frac{\max(4^k, c^{*\frac{2}{3}+\varepsilon})\log x}{x^{c^*k-1}} + c^{*\frac{1}{2}}\frac{x^{-\frac{1}{2}}(\log x)^{\frac{1}{4}}\log\log x}{e^k}$$

for any positive integer $k \geqslant 2$, where the implied constants are independent of k. They [6] showed that there exists a computable constant c such that

$$E_k(x) \ll \frac{x^{-\frac{1}{2}}}{k}\exp\left(-c\frac{(\log x)^{\frac{1}{3}}}{(\log\log x)^{\frac{1}{3}}}\right)$$

for $x \geqslant x_0$, where x_0 is a sufficiently large positive number and k is any positive integer. Furthermore, assuming that the Riemann Hypothesis is true, they showed that the inequality

$$E_k(x) \ll \frac{x^{-\frac{3}{4}+\varepsilon}}{k} \tag{1.3}$$

holds for any integer $k \geqslant 2$ and any small fixed positive constant ε. Moreover, the implied constant is independent of k.

Before going into the introduction of our theorem, we denote the function $h_n(s)$ below (the Euler product (2.2) in Lemma 2.1) defined by

$$h_n(s) := \prod_p\left(1 + \sum_{m=1}^{2^n-1}\frac{(-1)^{m-1}}{p^{ms+m+1}\left(1-\frac{1}{p}\right)} - \frac{1}{p^{2^n s+2^n}\left(1-\frac{1}{p}\right)}\right),$$

[1] The correct coefficient of $c_1(k)$ is $\frac{\zeta(2)\zeta(3)}{(k+1)!\zeta(6)}$ since the coefficient (29) in [4] was given by $\frac{\zeta(2)\zeta(3)}{k!\zeta(6)}$, which might be mistyped.

which is absolutely and uniformly convergent in any compact set in the half-plane $\mathrm{Re}\, s > -1 + \frac{1}{2^n}$. The purpose of this paper is to consider exact representation of $E_k(x)$ of the general k-th Riesz mean related to the arithmetical function $n/\phi(n)$ on the assumption that the Riemann Hypothesis is true and all zeros of the Riemann zeta-function $\zeta(s)$ on the critical line are simple. We will prove the following theorem.

Theorem 1.1. *Suppose that the Riemann Hypothesis is true and all the zeros ρ on the critical line of the Riemann zeta-function $\zeta(s)$ are simple. Then, for any positive integers $k \geqslant 2$ and $n > \frac{\log\log x}{2\log 2}$, there exists a point T $(x^4 \leqslant T \leqslant x^4+1)$ such that*

$$E_k(x) = Y_{k,n}(x,T)x^{-\frac{3}{4}} + O\left(x^{-1+\frac{C}{\sqrt{\log x}}}\left(\frac{\sqrt{\log x}}{(k-1)!} + \frac{1}{k}\right)\right) \tag{1.4}$$

with an absolute constant $C > 0$, where

$$\begin{aligned} Y_{k,n}(x,T) := \mathrm{Re} \sum_{0<\gamma<T} \zeta\left(-\frac{3}{4} + i\frac{\gamma}{2}\right)\zeta\left(\frac{1}{4} + i\frac{\gamma}{2}\right) \frac{\zeta\left((2^n - 3\cdot 2^{n-2}) + i2^n\gamma\right)}{\zeta'\left(\frac{1}{2} + i\gamma\right)} \\ \times h_n\left(-\frac{3}{4} + i\frac{\gamma}{2}\right) \frac{x^{i\frac{\gamma}{2}}}{\left(-\frac{3}{4} + i\frac{\gamma}{2}\right)\left(\frac{1}{4} + i\frac{\gamma}{2}\right)\cdots\left(k - \frac{3}{4} + i\frac{\gamma}{2}\right)}. \end{aligned}$$

Remark 1.1. The inequality (1.3) implies $Y_{k,n}(x,T) \ll k^{-1}x^{\varepsilon}$ for any fixed small positive constant ε. Based on this observation we conjecture that the estimate $E_k(x) \ll x^{-\frac{3}{4}}$ holds. In fact, if the finite sum $Y_{k,n}(x,T)$ is estimated by $O(1)$, then the estimate $E_k(x) \ll x^{-\frac{3}{4}}$ holds.

We make use of the Gonek-Hejhal Hypothesis (S. M. Gonek [1] and D. Hejhal [2] independently conjectured), namely

$$J_{-\lambda}(T) := \sum_{0<\gamma\leqslant T} \frac{1}{|\zeta'(\rho)|^{2\lambda}} \asymp T(\log T)^{(\lambda-1)^2} \tag{1.5}$$

for real numbers $\lambda < \frac{3}{2}$ to improve the error term $E_k(x)$ (Theorem 2 (conditional) in [6]). Then we have the following theorem.

Theorem 1.2. *Suppose that the Riemann Hypothesis and the Gonek-Hejhal Hypothesis are true and all the zeros ρ on the critical line of the Riemann zeta-function $\zeta(s)$ are simple. Then the estimate $E_k(x) = O\left(x^{-\frac{3}{4}}\right)$ holds for any positive integer $k \geqslant 2$.*

In what follows, ε denotes any arbitrarily small positive number, not necessarily the same ones at each occurrence.

2. Some lemmas

In order to prove Theorems 1.1 and 1.2, we shall prepare three lemmas. We denote the function $F(s)$ defined by

$$\sum_{n=1}^{\infty} \frac{n}{\phi(n)n^s}$$

for Re $s > 1$. Then the product representation of $F(s)$ gives the following formula.

Lemma 2.1. *For any positive integer $n \geqslant 2$ and* Re $s > 1$*, we have*

$$F(s) = \zeta(s)\zeta(s+1)\frac{\zeta(2^n s + 2^n)}{\zeta(2s+2)}h_n(s) \tag{2.1}$$

with

$$h_n(s) := \prod_p \left(1 + \sum_{m=1}^{2^n-1} \frac{(-1)^{m-1}}{p^{ms+m+1}\left(1-\frac{1}{p}\right)} - \frac{1}{p^{2^n s+2^n}\left(1-\frac{1}{p}\right)}\right), \tag{2.2}$$

where $h_n(s)$ is absolutely and uniformly convergent in any compact set in the half-plane Re $s > -1 + \frac{1}{2^n}$*.*

Proof. We use induction on n. The lemma is true for $n = 2$ which is Lemma 3.1 in [4]. Now, we assume that the statement (2.1) holds for $2, 3, \ldots, n-1$. The induction assumption tells us

$$F(s) = \zeta(s)\zeta(s+1)\frac{\zeta(2^{n-1}s + 2^{n-1})}{\zeta(2s+2)}h_{n-1}(s)$$

with

$$h_{n-1}(s) = \prod_p \left(1 + \sum_{m=1}^{2^{n-1}-1} \frac{(-1)^{m-1}}{p^{ms+m+1}\left(1-\frac{1}{p}\right)} - \frac{1}{p^{2^{n-1}s+2^{n-1}}\left(1-\frac{1}{p}\right)}\right).$$

Using the induction hypothesis, for Re $s > 1$, we have

$$\begin{aligned} F(s) &= \zeta(s)\zeta(s+1)\frac{\zeta(2^{n-1}s + 2^{n-1})}{\zeta(2s+2)} \prod_p \frac{1 + \frac{1}{p^{2^{n-1}s+2^{n-1}}}}{1 + \frac{1}{p^{2^{n-1}s+2^{n-1}}}} \\ &\quad \times \prod_p \left(1 + \sum_{m=1}^{2^{n-1}-1} \frac{(-1)^{m-1}}{p^{ms+m+1}\left(1-\frac{1}{p}\right)} - \frac{1}{p^{2^{n-1}s+2^{n-1}}\left(1-\frac{1}{p}\right)}\right) \\ &= \zeta(s)\zeta(s+1)\frac{\zeta(2^n s + 2^n)}{\zeta(2s+2)} \prod_p \left(1 + \frac{1}{p^{2^{n-1}s+2^{n-1}}}\right) \\ &\quad \times \left(1 + \sum_{m=1}^{2^{n-1}-1} \frac{(-1)^{m-1}}{p^{ms+m+1}\left(1-\frac{1}{p}\right)} - \frac{1}{p^{2^{n-1}s+2^{n-1}}\left(1-\frac{1}{p}\right)}\right). \end{aligned}$$

Now, we find that

$$\prod_p \left(1+\frac{1}{p^{2^{n-1}s+2^{n-1}}}\right)\left(1+\frac{1}{1-\frac{1}{p}}\left(\sum_{m=1}^{2^{n-1}-1}\frac{(-1)^{m-1}}{p^{ms+m+1}}-\frac{1}{p^{2^{n-1}s+2^{n-1}}}\right)\right)$$

$$=\prod_p \left(1+\sum_{m=1}^{2^{n-1}-1}\frac{(-1)^{m-1}}{p^{ms+m+1}\left(1-\frac{1}{p}\right)}-\frac{1}{p^{2^{n-1}s+2^{n-1}+1}}-\frac{1}{p^{2^{n-1}s+2^{n-1}+2}}\right.$$

$$\left.+\cdots+\sum_{m=1}^{2^{n-1}-1}\frac{(-1)^{m-1}}{p^{(m+2^{n-1})s+(m+2^{n-1}+1)}\left(1-\frac{1}{p}\right)}-\frac{1}{p^{2^n s+2^n}\left(1-\frac{1}{p}\right)}\right)$$

$$=\prod_p \left(1+\sum_{m=1}^{2^n-1}\frac{(-1)^{m-1}}{p^{ms+m+1}\left(1-\frac{1}{p}\right)}-\frac{1}{p^{2^n s+2^n}\left(1-\frac{1}{p}\right)}\right)$$

holds. Substituting this into the above result we obtain the identity (2.1). ■

Lemma 2.2. *Let x be any sufficiently large real number and let $\delta = 1/\sqrt{\log x}$. For any positive integer $n(> \frac{\log(1/\delta)}{\log 2} > 8)$, we have*

$$h_n(-1+\delta+it) \ll \exp\left(\frac{C}{\delta}\right) \tag{2.3}$$

with an absolute constant $C > 0$.

Proof. We use (2.2) and the prime number theorem $\pi(u) \sim \frac{u}{\log u}$ for any positive number $u \geqslant 2$ to estimate an upper bound for the function $h_n(-1+\delta+it)$ for any positive integer $n(> \frac{\log(1/\delta)}{\log 2} > 8)$, namely

$$|h_n(-1+\delta+it)| \leqslant \prod_p \left(1+\frac{1}{p-1}\sum_{m=1}^{2^n}\frac{1}{p^{m\delta}}\right)$$

$$\ll \prod_p \left(1+\frac{1}{(p-1)(p^\delta-1)}\right) = \exp\left(\sum_p \log\left(1+\frac{1}{(p-1)(p^\delta-1)}\right)\right)$$

$$\leqslant \exp\left(C\sum_p \frac{1}{p^{1+\delta}}\right) = \exp\left(C(1+\delta)\int_2^\infty \frac{\pi(u)}{u^{2+\delta}}du\right)$$

$$\ll \exp\left(C(1+\delta)\int_2^\infty \frac{du}{u^{1+\delta}\log u}\right) \ll \exp\left(\frac{C}{\delta}\right)$$

with an absolute constant $C > 0$. This completes the proof of Lemma 2.2. ■

Lemma 2.3. *Assume that the Riemann Hypothesis is true. Then there exists a point* $t \in [T, T+1]$ *such that*

$$\zeta(\sigma+it) \ll t^{\varepsilon} \qquad \text{and} \qquad \frac{1}{\zeta(\sigma+it)} \ll t^{\varepsilon}$$

for every σ $(1/2 \leqslant \sigma \leqslant 2)$ *and any sufficiently large real number* $T > 0$.

Proof. The first and second terms of this lemma are given by (14.2.5), (14.14.1) and (14.16.2) in E. C. Titchmarsh [8], respectively. ■

3. Proof of Theorem 1.1

Proof. Suppose that the Riemann Hypothesis is true, and all the zeros ρ on the critical line of the Riemann zeta-function $\zeta(s)$ are simple. Let x be any sufficiently large real number and let $\delta = 1/\sqrt{\log x}$. Assume that there exists a T satisfying the condition $x^4 \leqslant T \leqslant x^4+1$. We set any positive integer n $(> \frac{\log(1/\delta)}{\log 2} > 8)$. We make use of Lemma 2.1 with

$$F(s) = \zeta(s)\zeta(s+1)\frac{\zeta(2^n s+2^n)}{\zeta(2s+2)}h_n(s)$$

and (5.19) in H. Montgomery and R. C. Vaughan [3] with $\sigma_0 := 1+\frac{1}{\log x}$ to obtain

$$\begin{aligned} S_k(x) &:= \frac{1}{k!}\sum_{l\leqslant x}\frac{l}{\phi(l)}\left(1-\frac{l}{x}\right)^k \\ &= \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} F(s)\frac{x^s}{s(s+1)(s+2)\cdots(s+k)}ds + O\left(xT^{-k+\varepsilon}\right). \end{aligned} \tag{3.1}$$

Now, we move the line of integration to $\mathrm{Re}\, s = -1+\delta$. In the rectangular contour formed by the line segments joining the points $\sigma_0 - iT$, $\sigma_0 + iT$, $-1+\delta+iT$, $-1+\delta-iT$, and $\sigma_0 - iT$ in the counter-clockwise sense, we observe that $s=1$ is a simple pole, $s=0$ is a double pole, and $s = -\frac{3}{4}+i\frac{\gamma}{2}$ is a simple pole of the integrand. Thus, we get the main term from the sum of the residue coming from the poles $s=1$, $s=0$, and $s=-\frac{3}{4}+i\frac{\gamma}{2}$. That is,

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} F(s)\frac{x^s}{s(s+1)(s+2)\cdots(s+k)}ds \\ &\quad= \frac{1}{2\pi i}\left\{\int_{-1+\delta+iT}^{\sigma_0+iT} + \int_{-1+\delta-iT}^{-1+\delta+iT} + \int_{\sigma_0-iT}^{-1+\delta-iT}\right\} F(s)\frac{x^s}{s(s+1)\cdots(s+k)}ds \\ &\qquad+ \operatorname*{Res}_{s=1}\left(F(s)\frac{x^s}{s(s+1)\cdots(s+k)}\right) + \operatorname*{Res}_{s=0}\left(F(s)\frac{x^s}{s(s+1)\cdots(s+k)}\right) \\ &\qquad+ \sum_{0<|\gamma|<T}\operatorname*{Res}_{s=-\frac{3}{4}+i\frac{\gamma}{2}}\left(F(s)\frac{x^s}{s(s+1)\cdots(s+k)}\right). \end{aligned} \tag{3.2}$$

The last two terms on the right-hand side of (3.2) were evaluated by A. Sankaranarayanan and S. K. Singh [4], who have shown that

$$\operatorname*{Res}_{s=1}\left(F(s)\frac{x^s}{s(s+1)\cdots(s+k)}\right)=\frac{\zeta(2)\zeta(3)}{(k+1)!\zeta(6)}x$$

and

$$\begin{aligned}&\operatorname*{Res}_{s=0}\left(F(s)\frac{x^s}{s(s+1)\cdots(s+k)}\right)\\&\qquad=\frac{1}{2k!}\left\{-\log x+\sum_{j=1}^{k}\frac{1}{j}-\gamma-\log 2\pi-\sum_{p}\frac{\log p}{p(p-1)}\right\},\end{aligned}$$

which denote the first and second terms of the above defined by $c_1(k)x$ and $c_2(k)\log x+c_3(k)$, respectively. Furthermore, we have

$$\begin{aligned}&\sum_{0<|\gamma|<T}\operatorname*{Res}_{s=-\frac{3}{4}+i\frac{\gamma}{2}}\left(F(s)\frac{x^s}{s(s+1)\cdots(s+k)}\right)\\&\quad=\sum_{0<\gamma<T}\operatorname{Re}\left(\zeta\left(-\frac{3}{4}+i\frac{\gamma}{2}\right)\zeta\left(\frac{1}{4}+i\frac{\gamma}{2}\right)\frac{\zeta\left((2^n-3\cdot 2^{n-2})+i2^{n-1}\gamma\right)}{\zeta'\left(\frac{1}{2}+i\gamma\right)}\right.\\&\qquad\left.\times h_n\left(-\frac{3}{4}+i\frac{\gamma}{2}\right)\frac{x^{i\frac{\gamma}{2}}}{\left(-\frac{3}{4}+i\frac{\gamma}{2}\right)\left(\frac{1}{4}+i\frac{\gamma}{2}\right)\left(\frac{5}{4}+i\frac{\gamma}{2}\right)\cdots\left(k-\frac{3}{4}+i\frac{\gamma}{2}\right)}\right)x^{-\frac{3}{4}}.\end{aligned}$$

Let $T\geqslant T_0$, where T_0 is a sufficiently large real number. Using (2.3), the second term (the left vertical line segment) of the integral on the right-hand side of (3.2) contributes the quantity

$$\begin{aligned}Q_k(x)&:=\frac{1}{2\pi}\int_{-T}^{T}\frac{F\left(-1+\delta+it\right)x^{-1+\delta+it}}{(-1+\delta+it)(\delta+it)\cdots(k-1+\delta+it)}dt\\&=\frac{x^{-1+\delta}}{2\pi}\left(\int_{|t|\leqslant T_0}+\int_{T_0\leqslant|t|\leqslant T}\right)\zeta(-1+\delta+it)\zeta(\delta+it)\\&\quad\times\frac{\zeta(2^n\delta+i2^nt)h_n(-1+\delta+it)x^{it}}{\zeta(2\delta+2it)(-1+\delta+it)(\delta+it)\cdots(k-1+\delta+it)}dt\\&\ll\frac{x^{-1+\delta}}{\delta(k-1)!}\exp\left(\frac{C}{\delta}\right)\\&\quad+x^{-1+\delta}\exp\left(\frac{C}{\delta}\right)\int_{T_0\leqslant|t|\leqslant T}\left|\frac{t^{\frac{3}{2}-\delta}\zeta(2-\delta-it)t^{\frac{1}{2}-\delta}\zeta(1-\delta-it)}{(2t)^{\frac{1}{2}-2\delta}\zeta(1-2\delta+2it)t^{k+1}}\right|dt,\end{aligned}\tag{3.3}$$

where $C>0$ is an absolute constant. Using Lemma 2.3 we have

$$Q_k(x)\ll x^{-1+\frac{C}{\sqrt{\log x}}}\left(\frac{\sqrt{\log x}}{(k-1)!}+\frac{1}{k}\right)\tag{3.4}$$

for any positive integer $k \geqslant 2$. Also, we can estimate the contributions coming from the upper horizontal line (the lower horizontal line is similar). We note that

$$\begin{aligned}\int_{-1+\delta+iT}^{\sigma_0+iT} & F(s)\frac{x^s}{s(s+1)\cdots(s+k)}ds \\ &= \left(\int_{-\frac{3}{4}+iT}^{\sigma_0+iT} + \int_{-1+\delta+iT}^{-\frac{3}{4}+iT}\right) F(s)\frac{x^s}{s(s+1)\cdots(s+k)}ds \\ &=: I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{-\frac{3}{4}+iT}^{\sigma_0+iT} F(s)\frac{x^s}{s(s+1)\cdots(s+k)}ds \\ &= \left(\int_{\frac{1}{2}}^{\sigma_0} + \int_0^{\frac{1}{2}} + \int_{-\frac{1}{2}}^{0} + \int_{-\frac{3}{4}}^{-\frac{1}{2}}\right) F(\sigma+iT)\frac{x^{\sigma+iT}}{(\sigma+iT)\cdots(\sigma+k+iT)}d\sigma \\ &=: J_1 + J_2 + J_3 + J_4.\end{aligned}$$

Now, we observe that the function J_1 is bounded above by

$$\begin{aligned}\int_{\frac{1}{2}}^{\sigma_0} & \left|\zeta(\sigma+iT)\zeta(\sigma+1+iT)\frac{\zeta(2^n\sigma+2^n+2^n iT)}{\zeta(2\sigma+2+2iT)}h_n(\sigma+iT)\frac{x^\sigma}{T^{k+1}}\right| d\sigma \\ &\ll \int_{\frac{1}{2}}^{\sigma_0} T^{\varepsilon}\frac{x}{T^{k+1}}d\sigma \ll xT^{-k+\varepsilon}\end{aligned}$$

using Lemmas 2.1 and 2.3. We use the functional equation of the Riemann zeta-function and Lemmas 2.1 and 2.3 to obtain

$$|J_2| \ll \int_0^{\frac{1}{2}} \left|T^{\frac{1}{2}-\sigma}\zeta(1-\sigma-iT)T^{\varepsilon}\frac{x^{\frac{1}{2}}}{T^{k+1}}\right| d\sigma \ll x^{\frac{1}{2}}T^{-k-\frac{1}{2}+\varepsilon}.$$

Similarly, we have $|J_3| \ll T^{-k+3\varepsilon}$ and $|J_4| \ll x^{-\frac{1}{2}}T^{-k+\frac{1}{2}+\varepsilon}$. Hence, combining the above estimates, we obtain

$$I_1 \ll x^{-\frac{1}{2}}T^{-k+\frac{1}{2}+\varepsilon} \tag{3.5}$$

for a positive number T ($x^4 \leqslant T \leqslant x^4+1$). Also, we use the functional equation of the Riemann zeta-function and Lemmas 2.1 and 2.3 to obtain

$$\begin{aligned}|I_2| &= \left|\int_{-1+\delta+iT}^{-\frac{3}{4}+iT} F(s)\frac{x^s}{s(s+1)\cdots(s+k)}ds\right| \\ &\ll \int_{-1+\delta}^{-\frac{3}{4}} \left|T^{\frac{1}{2}-\sigma}\zeta(1-\sigma-iT)T^{-\frac{1}{2}-\sigma}\zeta(-\sigma-iT)\frac{T^{\delta}}{(2T)^{-3/2-2\sigma}}\frac{x^\sigma}{T^{k+1}}\right| d\sigma \\ &\ll \int_{-1+\delta}^{-\frac{3}{4}} x^{-\frac{3}{4}}T^{-k+\frac{1}{2}+\delta} \ll x^{-\frac{3}{4}}T^{-k+\frac{1}{2}+\delta}.\end{aligned}$$

Hence using horizontal lines of height $\pm T$ to move the line of integration in (3.2), we find that the total contribution of the horizontal lines in absolute value is

$$\ll x^{-\frac{1}{2}} T^{-k+\frac{1}{2}+\delta}. \tag{3.6}$$

Taking the relation (3.2) and the error estimates (3.4), (3.5), (3.6) into (3.1), we obtain, for $k \geqslant 2$,

$$\begin{aligned} S_k(x) &= c_1(k)x + c_2(k)\log x + c_3(k) + Y_{k,n}(x,T)x^{-\frac{3}{4}} \\ &\quad + O\left(x^{-1+\frac{C}{\sqrt{\log x}}}\left(\frac{\sqrt{\log x}}{(k-1)!} + \frac{1}{k}\right)\right) \end{aligned} \tag{3.7}$$

with a positive number T ($x^4 \leqslant T \leqslant x^4+1$), where

$$\begin{aligned} Y_{k,n}(x,T) := \operatorname{Re} \sum_{0<\gamma<T} &\zeta\left(-\frac{3}{4}+i\frac{\gamma}{2}\right)\zeta\left(\frac{1}{4}+i\frac{\gamma}{2}\right)\frac{\zeta\left((2^n-3\cdot 2^{n-2})+i2^n\gamma\right)}{\zeta'\left(\frac{1}{2}+i\gamma\right)} \\ &\times h_n\left(-\frac{3}{4}+i\frac{\gamma}{2}\right)\frac{x^{i\frac{\gamma}{2}}}{\left(-\frac{3}{4}+i\frac{\gamma}{2}\right)\left(\frac{1}{4}+i\frac{\gamma}{2}\right)\cdots\left(k-\frac{3}{4}+i\frac{\gamma}{2}\right)}. \end{aligned} \tag{3.8}$$

This completes the proof of the identity (1.4). ■

4. Proof of Theorem 1.2

Suppose that the Riemann Hypothesis and the Gonek-Hejhal Hypothesis are true and all the zeros ρ on the critical line of the Riemann zeta-function $\zeta(s)$ are simple. We use (3.8), Lemma 2.3, functional equation of the Riemann zeta-function, and Stirling's formula for $\chi(s)$ to obtain

$$\begin{aligned} &\sum_{0<\gamma<T} \operatorname{Re}\Bigg(\chi\left(-\frac{3}{4}+i\frac{\gamma}{2}\right)\chi\left(\frac{1}{4}+i\frac{\gamma}{2}\right)\zeta\left(\frac{7}{4}-i\frac{\gamma}{2}\right)\zeta\left(\frac{3}{4}-i\frac{\gamma}{2}\right) \\ &\quad\times\frac{\zeta\left(2^n-3\cdot 2^{n-2}+i2^n\gamma\right)}{\zeta'\left(\frac{1}{2}+i\gamma\right)}\frac{h_n\left(-\frac{3}{4}+i\frac{\gamma}{2}\right)x^{i\frac{\gamma}{2}}}{\left(-\frac{3}{4}+i\frac{\gamma}{2}\right)\left(\frac{1}{4}+i\frac{\gamma}{2}\right)\cdots\left(k-\frac{3}{4}+i\frac{\gamma}{2}\right)}\Bigg) \\ &\ll \sum_{0<\gamma<T}\frac{1}{\gamma^{k-\frac{1}{2}-\varepsilon}|\zeta'(\frac{1}{2}+i\gamma)|}. \end{aligned}$$

It suffices to show that $Y_{k,n}(x,T)$ converges for $k=2$. Using (1.5) and partial summation we obtain

$$\sum_{0<\gamma<T}\frac{1}{\gamma^{\frac{3}{2}-\varepsilon}|\zeta'(\frac{1}{2}+i\gamma)|} \ll \left[\frac{J_{-\frac{1}{2}}(t)}{t^{\frac{3}{2}-\varepsilon}}\right]_{14}^{T} + \int_{14}^{T}\frac{J_{-\frac{1}{2}}(t)}{t^{\frac{5}{2}-\varepsilon}}dt \ll 1,$$

which implies that the estimate $E_k(x) = O\left(x^{-\frac{3}{4}}\right)$ holds for any positive integer $k \geqslant 2$. ■

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