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ON THE DISCREPANCY BETWEEN BEST AND UNIFORM APPROXIMATION

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Abstract: For ζ a transcendental real number, we consider the classical Diophantine exponents $w_n(\zeta)$ and $\widehat{w}_n(\zeta)$. They measure how small $|P(\zeta)|$ can be for an integer polynomial P of degree at most n and naive height bounded by X, for arbitrarily large and all large X, respectively. The discrepancy between the exponents $w_n(\zeta)$ and $\widehat{w}_n(\zeta)$ has attracted interest recently. Studying parametric geometry of numbers, W. Schmidt and L. Summerer were the first to refine the trivial inequality $w_n(\zeta) \geqslant \widehat{w}_n(\zeta)$. Y. Bugeaud and the author found another estimation provided that the condition $w_n(\zeta) > w_{n-1}(\zeta)$ holds. In this paper we establish an unconditional version of the latter result, which can be regarded as a proper extension. Unfortunately, the new contribution involves an additional exponent and is of interest only in certain cases.

Keywords: Diophantine inequalities, exponents of Diophantine approximation, U-numbers.

1. Introduction

Let n be a positive integer and ζ be a transcendental real number. For a polynomial P as usual let $H(P)$ denote its height, which is the maximum modulus among the coefficients of P. We want to investigate relations between the two classical exponents of Diophantine approximation $w_n(\zeta)$ and $\widehat{w}_n(\zeta)$ introduced below. Define $w_n(\zeta)$ as the supremum of $w \in \mathbb{R}$ such that

$$H(P) \leqslant X, \qquad 0 < |P(\zeta)| \leqslant X^{-w}, \tag{1}$$

has a solution $P \in \mathbb{Z}[T]$ of degree at most n for arbitrarily large values of X. Similarly, let $\widehat{w}_n(\zeta)$ be the supremum of w such that (1) has a solution $P \in \mathbb{Z}[T]$ of degree at most n for *all* large X. The interest of the exponent in (1) and the derived exponents arises partly from the relation to approximation to a real number by algebraic numbers of bounded degree. Indeed, when α is an algebraic number very close to ζ, then the evaluation $P_\alpha(\zeta)$ is also very small by absolute value, for P_α the irreducible minimal polynomial of α over $\mathbb{Z}$. More precisely

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$|P_{\alpha}(\zeta)| \leqslant C(n,\zeta)H(P_{\alpha})|\zeta-\alpha|$ for a constant that depends only on ζ and the degree n of α. The converse is a delicate problem, at least for certain numbers ζ, related to the famous problem of Wirsing posed in [20]. We do not further discuss this issue here and only affirm that results involving the exponents $w_n, \widehat{w}_n$ typically imply comparable results concerning approximation by algebraic numbers in an obvious way. For any real number ζ, our exponents clearly satisfy the relations

$$w_1(\zeta) \leqslant w_2(\zeta) \leqslant \cdots, \qquad \widehat{w}_1(\zeta) \leqslant \widehat{w}_2(\zeta) \leqslant \cdots. \tag{2}$$

Dirichlet's box principle further implies

$$w_n(\zeta) \geqslant \widehat{w}_n(\zeta) \geqslant n. \tag{3}$$

The value $w_n(\zeta)$ can be infinity. In this case ζ is called a U-number, more precisely ζ is called U_n-number if n is the smallest such index. The existence of U_n-numbers for any $n \geqslant 1$ was first proved by LeVeque [11]. On the other hand, the quantities $\widehat{w}_n(\zeta)$ can be effectively bounded. For $n=1$, it is not hard to see that we always have $\widehat{w}_1(\zeta)=1$, see [10]. For $n=2$, Davenport and Schmidt [9] showed

$$\widehat{w}_2(\zeta) \leqslant \frac{3+\sqrt{5}}{2} = 2.6180\ldots. \tag{4}$$

Roy [13] proved that for certain numbers he called *extremal numbers* there is equality, so (4) is sharp. For an overview of the results on the values $\widehat{w}_2(\zeta)$ attained for real ζ, see [6]. For $n \geqslant 3$, little is known about the exponents $\widehat{w}_n(\zeta)$. The supremum of the values $\widehat{w}_n(\zeta)$ over all real real numbers ζ remains unknown in this case, in fact even the existence of a real number ζ with the property $\widehat{w}_n(\zeta) > n$ is open. The first result in this direction was due to Davenport and Schmidt [9], who showed $\widehat{w}_n(\zeta) \leqslant 2n-1$ for any real ζ. Recently this bound has been refined in [8] and further in [16], in the latter paper the currently best known bound

$$\widehat{w}_n(\zeta) \leqslant \frac{3(n-1)+\sqrt{n^2-2n+5}}{2} \tag{5}$$

was established. The right hand side is of order $2n-2+o(1)$ as $n \to \infty$. Conditionally on a conjecture of Schmidt and Summerer [19], small improvements of (5) can be obtained with the method in [8]. In particular for $n \geqslant 10$ it would imply $\widehat{w}_n(\zeta) \leqslant 2n-2$, see [15, Theorem 3.1].

2. The discrepancy between $w_n(\zeta)$ and $\widehat{w}_n(\zeta)$

2.1. The main result

Investigating parametric geometry of numbers introduced by them, Schmidt and Summerer [18] found the estimate

$$w_n(\zeta) \geqslant \frac{(n-1)\widehat{w}_n(\zeta)(\widehat{w}_n(\zeta)-1)}{1+(n-2)\widehat{w}_n(\zeta)},$$

for the minimum discrepancy between $w_n(\zeta)$ and $\widehat{w}_n(\zeta)$, for any transcendental real ζ. Rearrangements lead to the equivalent formulation

$$\widehat{w}_n(\zeta) \leqslant \frac{1}{2}\left(1+\frac{n-2}{n-1}w_n(\zeta)\right)+\sqrt{\frac{1}{4}\left(\frac{n-2}{n-1}w_n(\zeta)+1\right)^2+\frac{w_n(\zeta)}{n-1}}. \tag{6}$$

In fact analogous estimates were established in the more general context of linear forms with respect to any given real vector $(\zeta_1,\ldots,\zeta_n)$ that is $\mathbb{Q}$-linearly independent together with $\{1\}$, and in this case they are sharp for any dimension and parameter, see Roy [12]. The above estimates yield a proper improvement of the obvious left inequality in (3) unless $w_n(\zeta)=\widehat{w}_n(\zeta)=n$. A non-trivial identity case occurs for $n=2$ and ζ an extremal number as defined in Section 1. See also [19] for an improvement of (6) when $n=3$, and a conjecture concerning the optimal bound for arbitrary n. A special case of the recent [8, Theorem 2.2] complements (6).

Theorem 2.1 (Bugeaud, Schleischitz). *Let $n\geqslant 2$ be integers and ζ be a transcendental real number. Then in case of*

$$w_n(\zeta) > w_{n-1}(\zeta), \tag{7}$$

we have

$$\widehat{w}_n(\zeta) \leqslant \frac{nw_n(\zeta)}{w_n(\zeta)-n+1}. \tag{8}$$

Observe that in contrast to (6), the bound in (8) for $\widehat{w}_n(\zeta)$ decreases as $w_n(\zeta)$ increases. For $n=2$ and ζ any Sturmian continued fraction, see [7] for a definition, there is equality in (8). This can be verified by inserting the exact values of $w_2(\zeta)$ and $\widehat{w}_2(\zeta)$ determined in the main result of [7]. In particular, for extremal numbers mentioned above we have equality in both (6) and (8) when $n=2$.

The condition (7) in Theorem 2.1 was used predominately to guarantee that the polynomials in the definition of w_n have degree precisely n (in the special case of [8, Theorem 2.2] reproduced in Theorem 2.1). In other words, for any $\epsilon>0$, there are arbitrarily large irreducible integer polynomials P of degree exactly n for which the estimate

$$|P(\zeta)| \leqslant H(P)^{-w_n(\zeta)+\epsilon} \tag{9}$$

holds. This was a crucial observation for the proof. We point out that this property does not hold in general, i.e. when we drop the condition (7). Indeed, using continued fraction expansion, one can even construct real numbers for which the degree of any P which satisfies (9) equals one, when ϵ is sufficiently small. This can be deduced from the proof of [5, Corollary 1]. It is unknown whether (8) still holds when we drop the condition (7). The purpose of this paper is to provide a weaker but unconditioned relation. We will agree on $w_0(\zeta)=0$ in our following main result.

Theorem 2.2. *Let n be a positive integer and ζ be a real transcendental number. Let $l \in \{1, 2, \ldots, n\}$ be the smallest integer such that $w_l(\zeta) = w_n(\zeta)$. Then the estimation*

$$\widehat{w}_n(\zeta) \leqslant \min\left\{n + l - 1, \frac{nw_n(\zeta)}{w_n(\zeta) - l + 1} + w_{n-l}(\zeta) \cdot \left(1 - \frac{n}{w_n(\zeta) - l + 1}\right)\right\} \tag{10}$$

holds.

Remark 1. In case of $w_n(\zeta) \leqslant n + l - 1$, the trivial estimate in (3) implies (10). More generally, when $w_n(\zeta)$ does not exceed $n + l - 1$ by much, the Schmidt-Summerer bound (6) is even smaller than both bounds in (10).

The left bound in (10) will be an easy consequence of Theorem 3.1 from [8] reproduced below, the main new contribution is the right bound. When $l \leqslant \lfloor n/2 \rfloor$, by definition of l we have $w_{n-l}(\zeta) = w_n(\zeta)$ and the right bound in the minimum in (10) becomes $w_n(\zeta)$, which is trivial in view of (3). Thus Theorem 2.2 is of interest primarily when $l > n/2$ and $w_n(\zeta) > n+l-1$. However, if these relations hold and $w_{n-l}(\zeta)$ does not exceed $n - l$ by much, then one checks that the right expression in the minimum in (10) is the smaller one. In general, the new right bound in (10) is of interest when l is rather close to n and $w_{n-l}(\zeta)$ is relatively small, whereas $w_n(\zeta)$ is large. For $l = n$, the bound in (10) becomes (8), and we recover Theorem 2.1. The expression $w_{n-l}(\zeta)$ involved in (10) is unpleasant, as it can be arbitrarily close to $w_n(\zeta)$. We would like to replace it by $\widehat{w}_{n-l}(\zeta)$, which could be effectively bounded with (5) by roughly $2(n - l)$. The proof will suggest that such improvements are realistic.

2.2. U_m-numbers

The claim of Theorem 2.2 is of particular interest when ζ is a U_m-number (see Section 1). In that case, in [8, Corollary 2.5] it was deduced essentially from the generalization [8, Theorem 2.3] of Theorem 2.1 that $\widehat{w}_m(\zeta) = m$, and moreover

$$\widehat{w}_n(\zeta) \leqslant n + m - 1, \qquad n \geqslant 1. \tag{11}$$

We remark that [8, Theorem 2.3] rephrased in Theorem 3.1 below provides another proof of (11). We can now refine this estimate when n is roughly between m and $2m$.

Corollary 2.3. *Let $n > m \geqslant 1$ be integers and ζ be a U_m-number. Then*

$$\widehat{w}_n(\zeta) \leqslant n + \min\{m - 1, w_{n-m}(\zeta)\}.$$

Proof. By assumption we have $w_{m-1} < w_m = w_{m+1} = \cdots = w_n = \infty$, where we agree on $w_0(\zeta) = 0$ if $m = 1$. Thus we may apply Theorem 2.2 with $l = m$, which yields the claimed bound. ■

As indicated, the possible gain by the replacement of $m-1$ by $w_{n-m}(\zeta)$ in the minimum can only occur when n is not too large compared to m. More precisely $n < 2m-1$ is a necessary condition by (3). On the other hand, when ζ is a U_m-number and $w_l(\zeta)$ is small for some $l < m-1$, then Corollary 2.3 yields a significant improvement for $n = m + l$. It is reasonable that even U_m-numbers with the property $w_l(\zeta) = l$ for $1 \leqslant l \leqslant m-1$ exist. For $m = 2$ this is true, using continued fraction expansion one can even construct a U_2-number ζ with any prescribed value $w_1(\zeta) \in [1, \infty)$. See [4, Theorem 7.6] and its preceding remarks. However, for U_2-numbers we do not get any new insight from Theorem 2.2. Concerning U-numbers of larger index, Alniaçik [1] showed the existence of uncountably many U_m-numbers of arbitrary index $m \geqslant 2$ with the property $w_1(\zeta) = 1$. (In [2] the analogous result for T-numbers is proposed, however as pointed out by Bugeaud in [4, Section 7.10] crucial estimates in [2] are not carried out properly and serious revision of the paper is required.) For such U_m-numbers, the succeeding uniform exponent $\widehat{w}_{m+1}(\zeta)$ can be bounded with Corollary 2.3 as

$$\widehat{w}_{m+1}(\zeta) \leqslant m + 2.$$

For large m, this leads to a reasonable improvement compared to the trivial bound $(m+1) + m - 1 = 2m$ from (11). Moreover, Alniaçiks main theorem in [1] seems to allow one to construct U_m-numbers with arbitrary prescribed value $w_1(\zeta) = w_1 \in [1, \infty)$, thus extending the result for U_2-numbers above. Indeed, it suffices to take $b_{s_n+1} = \nu_{s_n+1} = \lfloor q_{s_n+1}^{w_1-1} \rfloor$ (in fact rather $b_{s_n+1} = \nu_{s_n+1} = \lfloor q_{s_n}^{w_1-1} \rfloor$ in the classical notation of continued fractions where $q_{n+1} = a_{n+1}q_n + q_{n-1}$ for a_j the partial quotients and p_n/q_n the convergents, this seems to be a minor inaccuracy in [1]) and let the remaining (i.e. j not of the form $s_n + 1$) $b_j = a_j$ in the formulation of the theorem. Elementary facts on continued fractions and Roth's Theorem imply $w_1(\zeta) = w_1$. Strangely, this observation seems not to have been previously mentioned. As soon as $w_1(\zeta) = w_1 < m - 1$, the resulting bound $\widehat{w}_{m+1}(\zeta) \leqslant m + w_1 + 1$ again improves the trivial upper bound $2m$.

On the other hand, the larger intermediate exponents $w_2(\zeta), w_3(\zeta), \ldots, w_{m-1}(\zeta)$ are hard to control for a U_m-number. A construction of Schmidt [17] shows that it is possible to obtain $w_l(\zeta) \leqslant m + l - 1$ simultaneously for $1 \leqslant l \leqslant m-1$ for some U_m-number ζ. This refined an earlier result of Alniaçik, Avci and Bugeaud [3]. However, this estimation is not sufficient to improve the previously known bound $\widehat{w}_n(\zeta) \leqslant m + n - 1$ with Corollary 2.3. Finally, we point out that U_m-numbers which allow arbitrarily good irreducible polynomial evaluations $|P(\zeta)|$ of some degree $n > m$ as well satisfy $\widehat{w}_n(\zeta) = n$. By the condition more precisely we mean that the exponent w in (1) can be chosen arbitrarily large among polynomials of degree m and additionally among irreducible polynomials P of degree $n > m$. Indeed, the second expression in the right hand estimate in (10) can be dropped in this case, as can be seen from the proof below.

3. Proof of Theorem 2.2

We reproduce some results from [8] for the proof. The first is [8, Theorem 2.3], which essentially implies the left bound in (10).

Theorem 3.1 (Bugeaud, Schleischitz). *Let m, n be positive integers and ζ be a transcendental real number. Then*

$$\min\{w_m(\zeta), \widehat{w}_n(\zeta)\} \leqslant m+n-1.$$

Theorem 3.1 was recently refined [14] by replacing the right hand side by $1/\widehat{\lambda}_{m+n-1}(\zeta)$, for $\widehat{\lambda}_{m+n-1}(\zeta) \geqslant 1/(m+n-1)$ the classic exponent of uniform simultaneous rational approximation to $(\zeta, \zeta^2, \ldots, \zeta^{m+n-1})$ for a real number ζ. See for example [5] for an exact definition. We will further directly apply the following [8, Lemma 3.1], a generalization of [9, Lemma 8] by Davenport and Schmidt, which was the core of the proof of both Theorem 2.1 and Theorem 3.1. We write $A \ll_. B$ in the sequel when B exceeds A at most by a constant that depends on the subscript variables.

Lemma 3.2 (Bugeaud, Schleischitz). *Let P, Q be two coprime integer polynomials of degree m and n, respectively. Further let ζ be any real number. Then at least one of the estimates*

$$|P(\zeta)| \gg_{m,n,\zeta} H(P)^{-n+1}H(Q)^{-m}, \qquad |Q(\zeta)| \gg_{m,n,\zeta} H(P)^{-n}H(Q)^{-m+1}$$

holds. In particular

$$\max\{|P(\zeta)|, |Q(\zeta)\} \gg_{m,n,\zeta} H(P)^{-n+1}H(Q)^{-m+1}\min\{H(P)^{-1}, H(Q)^{-1}\}.$$

In the formulation of the lemma we dropped the condition $\zeta P(\zeta)Q(\zeta) \neq 0$ stated in [8], which is not required as pointed out to me by D. Roy. We further point out that Wirsing [20] showed that for any $n \geqslant 1$ there exists a constant $K(n) > 1$, such that uniformly for all polynomials $P, Q \in \mathbb{Z}[T]$ of degree at most n

$$K(n)^{-1}H(P)H(Q) \leqslant H(PQ) \leqslant K(n)H(P)H(Q) \tag{12}$$

holds. He deduced that the polynomials within the definition of $w_n(\zeta)$ can be chosen irreducible. Moreover it follows from the definition of the exponents that in the case of $w_n(\zeta) > w_{n-1}(\zeta)$ these irreducible polynomials have degree precisely n. This fact was already an essential ingredient in the proof of [8, Theorem 2.1], which is our Theorem 2.1.

Proof of Theorem 2.2. First assume $w_n(\zeta) \leqslant l+n-1$. Then by (3) clearly $\widehat{w}_n(\zeta) \leqslant l+n-1$ as well. Moreover, it is easy to check that the right bound in (10) exceeds $n+l-1$, see also Remark 1. Hence we can restrict to $w_n(\zeta) > l+n-1$. We will further assume $w_n(\zeta) < \infty$ for simplicity. The case $w_n(\zeta) = \infty$ can be treated very similarly by considering the polynomials P for which $-\log|P(\zeta)|/\log H(P)$ tends to infinity.

By the choice of l we have $w_n(\zeta) = w_l(\zeta) > w_{l-1}(\zeta)$. Hence, as carried out above, for any $\epsilon > 0$ there exist infinitely many irreducible integer polynomials P of degree precisely l such that

$$H(P)^{-w_n(\zeta)-\epsilon} \leqslant |P(\zeta)| \leqslant H(P)^{-w_n(\zeta)+\epsilon}. \tag{13}$$

Fix one such irreducible P of large height and small $\delta > 0$ to be chosen later and let

$$\theta = \frac{w_n(\zeta) - l + 1}{n}, \qquad X_\delta = H(P)^{\theta-\delta}. \tag{14}$$

We want to give a lower bound on $|Q(\zeta)|$ for Q an arbitrary integer polynomial of degree at most n and height $H(Q) \leqslant X_\delta$. We distinguish two cases.

Case 1. The polynomial Q is not a polynomial multiple of P. Then P, Q are coprime as P is irreducible, and thus we may apply Lemma 3.2. First assume $|Q(\zeta)| \geqslant |P(\zeta)|$. Then we infer

$$-\frac{\log |Q(\zeta)|}{\log X_\delta} \leqslant -\frac{\log |P(\zeta)|}{\log X_\delta} \leqslant \frac{w_n(\zeta) + \epsilon}{\theta - \delta}. \tag{15}$$

The upper bound follows for such Q as we may choose ϵ and δ arbitrarily small, and doing so the right hand side in (15) tends to $nw_n(\zeta)/(w_n(\zeta) - l + 1)$, whereas the remaining expression in (10) is non-negative. Now assume $|Q(\zeta)| < |P(\zeta)|$. Then (13) yields

$$\max\{|P(\zeta)|, |Q(\zeta)|\} = |P(\zeta)| \leqslant H(P)^{-w_n(\zeta)+\epsilon}. \tag{16}$$

First assume $H(Q) \leqslant H(P)$. Then Lemma 3.2 yields

$$\max\{|P(\zeta)|, |Q(\zeta)|\} \gg_{n,\zeta} H(P)^{-l} H(Q)^{-n+1} \geqslant H(P)^{-n-l+1}.$$

This contradicts (16) for sufficiently large $H(P)$ and sufficiently small $\epsilon > 0$, by our hypothesis $w_n(\zeta) > l + n - 1$. Hence $H(Q) > H(P)$ must hold. Then Lemma 3.2 implies

$$\max\{|P(\zeta)|, |Q(\zeta)|\} \gg_{n,\zeta} H(P)^{-l+1} H(Q)^{-n} \geqslant H(Q)^{-\frac{l-1}{\tau}} H(Q)^{-n},$$

where $\tau = \log H(Q)/\log H(P) > 1$. Combination with (16) yields

$$\frac{w_n(\zeta)}{\tau} - \epsilon \leqslant \frac{w_n(\zeta) - \epsilon}{\tau} \leqslant n + \frac{l-1}{\tau},$$

hence

$$\tau \geqslant \frac{w_n(\zeta) - l + 1}{n + \epsilon} = \theta \cdot \frac{n}{n + \epsilon}.$$

This contradicts our assumption $H(Q) \leqslant X_\delta$, which is equivalent to $\tau \leqslant \theta - \delta$, when ϵ is chosen small enough compared to δ. This contraction finishes the proof of case 1.

Case 2. The integer polynomial Q is of the form $Q = PV$ for some integer polynomial V. The degree of V is at most $n-l$ since Q has degree at most n and P has degree precisely l. Moreover from Wirsing's estimate (12) we infer

$$H(V) \ll_n \frac{H(Q)}{H(P)} \leqslant \frac{X_\delta}{H(P)} = H(P)^{\theta-1-\delta}.$$

Let $\tilde{\varepsilon} > 0$ be small. By definition of w_{n-l}, for $\varepsilon > 0$ a variation of $\tilde{\varepsilon}$ (that tends to 0 as $\tilde{\varepsilon}$ does) and for sufficiently large $H(P)$, all but finitely many V satisfy

$$|V(\zeta)| \geqslant H(V)^{-w_{n-l}(\zeta)-\tilde{\varepsilon}} \geqslant H(P)^{-w_{n-l}(\zeta)(\theta-1-\delta)-\varepsilon}. \tag{17}$$

We briefly discuss the possible exceptions $V \in \{V_1, \ldots, V_h\}$ for the given $\tilde{\epsilon}$. By the finiteness and transcendence of ζ we infer an absolute lower bound $\max_{1\leqslant j\leqslant h} |V_j(\zeta)| \gg 1$. Thus for $V \in \{V_1, \ldots, V_h\}$ we have $|Q(\zeta)| = |P(\zeta)| \cdot |V(\zeta)| \gg |P(\zeta)|$. The bound $nw_n(\zeta)/(w_n(\zeta) - l + 1)$ follows similarly to (15) as $H(P) \to \infty$ and $\tilde{\epsilon} \to 0$. Now we treat the main case of V that satisfy (17). Together with (13) and Wirsing's estimate we obtain

$$|Q(\zeta)| = |P(\zeta)| \cdot |V(\zeta)| \geqslant H(P)^{-w_n(\zeta)-w_{n-l}(\zeta)(\theta-1-\delta)-(\epsilon+\varepsilon)}.$$

We conclude

$$-\frac{\log|Q(\zeta)|}{\log X_\delta} \leqslant \frac{w_n(\zeta) + w_{n-l}(\zeta)(\theta-1-\delta)}{\theta-\delta} + \frac{\epsilon+\varepsilon}{\theta-\delta}.$$

As we may choose δ and ϵ, ε arbitrarily small, we obtain

$$-\frac{\log|Q(\zeta)|}{\log X_\delta} \leqslant \frac{w_n(\zeta) + w_{n-l}(\zeta)(\theta-1)}{\theta} + \epsilon',$$

for arbitrarily small $\epsilon' > 0$. Inserting the value of θ from (14) we obtain

$$-\frac{\log|Q(\zeta)|}{\log X_\delta} \leqslant \frac{nw_n(\zeta)}{w_n(\zeta)-l+1} + w_{n-l}(\zeta) \cdot \frac{w_n(\zeta)-n-l+1}{w_n(\zeta)-l+1} + \epsilon'.$$

The right bound in (10) follows with elementary rearrangements. Since this holds for any polynomial multiple Q of P of height $H(Q) \leqslant X_\delta$, the proof of case 2 is finished as well. ∎

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