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EXPLICIT VERSIONS OF THE PRIME IDEAL THEOREM FOR DEDEKIND ZETA FUNCTIONS UNDER GRH, II

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Abstract: We have recently proved several explicit versions of the prime ideal theorem under GRH. Here we further explore the method, in order to deduce its strongest consequence for the case where x diverges.

Keywords: prime ideal theorem, Dedekind functions, explicit bounds, GRH.

1. Introduction

For a number field $\mathbb{K}$ we denote by

- $n_{\mathbb{K}}$ its dimension,
- $\Delta_{\mathbb{K}}$ the absolute value of its discriminant,
- $\delta_{\mathbb{K}} := (\Delta_{\mathbb{K}})^{(1/n_{\mathbb{K}})}$ its root discriminant,
- r_1 the number of its real places,
- r_2 the number of its imaginary places,
- $d_{\mathbb{K}} := r_1 + r_2 - 1$.

In [2] we use a two step process to prove explicit versions of the prime ideal theorem under GRH: first we prove a bound for $|\psi_{\mathbb{K}}(x) - x|$ depending on a parameter T to be fixed later ([2, Theorem 1.1]), then we prove several formulas based on some choices for T ([2, Corollaries 1.2 and 1.3]). A scheme to produce explicit versions of the prime ideal theorem for number fields has been proved by Lagarias and Odlyzko in [3] and recently Winckler computed the effective constants in [7]. In this paper, we reuse Theorem 1.1 of [2] with an additional parameter called κ producing the general result in Theorem 2.5, we then choose κ and T in such a way as to obtain the best possible asymptotic expansion for $x \to +\infty$. By doing so we obtain a formula that is not too far from the best possible bound for $|\psi_{\mathbb{K}}(x) - x|$ which can be proved by using this method.

We recall that the Lambert-W function is the function such that $\forall x \geqslant 0$, $W(x)e^{W(x)} = x$.

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Theorem 1.1. *Assume GRH. Let $x \geqslant 3$,*

$$
\begin{aligned}
w &:= W\Big(\frac{e^{\sqrt{5}}}{2\pi}\delta_{\mathbb{K}}\Big[\frac{(\sqrt{5}-1)\pi\sqrt{x}}{2n_{\mathbb{K}}} + 21.3270 + \frac{33.3542}{n_{\mathbb{K}}}\Big]\Big),\\
T &:= 8.2822 + \frac{1}{w}\Big[\frac{(\sqrt{5}-1)\pi\sqrt{x}}{2n_{\mathbb{K}}} + 21.3270 + \frac{33.3542}{n_{\mathbb{K}}}\Big],\\
\epsilon_{\mathbb{K}}(x,T) &:= \max\Big(0, d_{\mathbb{K}}\log x - 3.6133 n_{\mathbb{K}}\frac{\sqrt{x}}{T}\Big).
\end{aligned}
$$

Then

$$
\begin{aligned}
|\psi_{\mathbb{K}}(x) - x| \leqslant \frac{\sqrt{x}}{\pi}\Big[\Big(&\frac{1}{2}\log^2\big(e^{w+1} + 33.5251\delta_{\mathbb{K}}\big) - \frac{1}{2}\log^2\delta_{\mathbb{K}}\\
&+ 3.9792\log\delta_{\mathbb{K}} - 3.4969\Big)n_{\mathbb{K}} + 25.5362\Big]\\
&+ 1.0155\log\Delta_{\mathbb{K}} - 2.1042 n_{\mathbb{K}} + 8.8590 + \epsilon_{\mathbb{K}}(x,T). \qquad (1.1)
\end{aligned}
$$

Moreover we also have

$$
\begin{aligned}
|\psi_{\mathbb{K}}(x) - x| \leqslant (2.2543\sqrt{x} + 1.0155)\log\Delta_{\mathbb{K}} &+ (0.9722\sqrt{x} - 2.1042)n_{\mathbb{K}}\\
&+ \frac{x}{10} + 9.0458\sqrt{x} + 7.0320 + \epsilon_{\mathbb{K}}(x,10). \qquad (1.2)
\end{aligned}
$$

The choice of T we have made to deduce Theorem 1.1 from Theorem 2.5 gives the best coefficients for all terms in the asymptotic expansion, down to the term of order $\sqrt{x}$. This choice is not too far from the best our method can achieve, even for finite x; in other words, the T we choose in Theorem 1.1 is not too far from the optimal T for Theorem 2.5. Note that the values of the other parameters we fix in the proofs of Theorem 1.1 and Theorem 2.5 affect the term of order $\sqrt{x}$ of the asymptotic expansion.

Inequality (1.2) is a kind of Chebyshev bound, which has the interesting property that the linear term has a coefficient independent of the field. It is better than (1.1) when x is very small with respect to $\delta_{\mathbb{K}}$.

Asymptotic expansions

We discuss the asymptotic expansions of (1.1) when x diverges and $\mathbb{K}$ is fixed. We have, as $t \to +\infty$

$$
W(t) = \log t - \log\log t + \frac{\log\log t}{\log t} + O\Big(\Big(\frac{\log\log t}{\log t}\Big)^2\Big)
$$

and, even though we will not use it,

$$
\forall t \geqslant e, \quad W(t) \leqslant \log t - \log\log t + 1.024\frac{\log\log t}{\log t}
$$

so that the asymptotic expansion we are computing is not too far from an upper bound. From the first expansion, we deduce

$$\frac{1}{2}W(t)^2 + W(t) = \frac{1}{2}\log^2 t - \log t\log\log t + \log t + \frac{1}{2}(\log\log t)^2 + o(1).$$

We have $w \sim \frac{1}{2}\log x$ when x diverges, thus

$$\begin{aligned}\log^2\left(e^{w+1} + 33.5251\delta_{\mathbb{K}}\right) &= \Big(w + 1 + \frac{33.5251\delta_{\mathbb{K}}}{e^{w+1}} + O\Big(\frac{1}{e^{2w}}\Big)\Big)^2 \\ &= w^2 + 2w + 1 + O\Big(\frac{\log x}{\sqrt{x}}\Big).\end{aligned}$$

We thus have, taking $\nu := \frac{\sqrt{5}-1}{2}e^{\sqrt{5}}$

$$\begin{aligned}&\frac{1}{2}\log^2\left(e^{w+1} + 33.5251\delta_{\mathbb{K}}\right) - \frac{1}{2}\log^2\delta_{\mathbb{K}} \\ &\quad= \frac{1}{2}w^2 + w + \frac{1}{2} - \frac{1}{2}\log^2\delta_{\mathbb{K}} + o(1) \\ &\quad= \frac{1}{2}\Big[\log\delta_{\mathbb{K}} + \frac{1}{2}\log x + \log\Big(\frac{\nu}{2n_{\mathbb{K}}}\Big)\Big]^2 - \Big[\log\delta_{\mathbb{K}} + \frac{1}{2}\log x + \log\Big(\frac{\nu}{2n_{\mathbb{K}}}\Big)\Big] \\ &\quad\quad\times\Big[\log\log x - \log 2 + \frac{2\log\delta_{\mathbb{K}}}{\log x} + \frac{2}{\log x}\log\Big(\frac{\nu}{2n_{\mathbb{K}}}\Big)\Big] \\ &\quad\quad+ \log\delta_{\mathbb{K}} + \frac{1}{2}\log x + \log\Big(\frac{\nu}{2n_{\mathbb{K}}}\Big) + \frac{1}{2}\Big[\log\log x - \log 2\Big]^2 \\ &\quad\quad+ \frac{1}{2} - \frac{1}{2}\log^2\delta_{\mathbb{K}} + o(1) \\ &\quad= \frac{1}{8}\log^2 x - \frac{1}{2}\log x\log\log x + \frac{1}{2}\Big[\log\delta_{\mathbb{K}} + \log\Big(\frac{e\nu}{n_{\mathbb{K}}}\Big)\Big]\log x \\ &\quad\quad+ \frac{1}{2}(\log\log x)^2 - \Big[\log\delta_{\mathbb{K}} + \log\Big(\frac{\nu}{n_{\mathbb{K}}}\Big)\Big]\log\log x \\ &\quad\quad+ \frac{1}{2}\log^2\Big(\frac{\nu}{n_{\mathbb{K}}}\Big) + \log\Big(\frac{\nu}{n_{\mathbb{K}}}\Big)\log\delta_{\mathbb{K}} + \frac{1}{2} + o(1).\end{aligned}$$

Thus the right hand side of (1.1) is

$$\begin{aligned}n_{\mathbb{K}}\frac{\sqrt{x}}{2\pi}\Big[&\frac{1}{4}\log^2 x - \log x\log\log x + \Big[\log\delta_{\mathbb{K}} + \log\Big(\frac{e\nu}{n_{\mathbb{K}}}\Big)\Big]\log x + (\log\log x)^2 \\ &- 2\Big[\log\delta_{\mathbb{K}} + \log\Big(\frac{\nu}{n_{\mathbb{K}}}\Big)\Big]\log\log x + \Big(2\log\Big(\frac{\nu}{n_{\mathbb{K}}}\Big) + 7.9584\Big)\log\delta_{\mathbb{K}} \\ &+ \log^2\Big(\frac{\nu}{n_{\mathbb{K}}}\Big) - 5.9938\Big] + 25.5362\frac{\sqrt{x}}{\pi} + o(\sqrt{x}).\end{aligned}$$

Since $e\nu \simeq 15.7187\ldots$ the coefficient of $n_{\mathbb{K}}\frac{\sqrt{x}}{2\pi}\log x$ is lower than $\log\delta_{\mathbb{K}}$ if $n_{\mathbb{K}} \geqslant 16$.

We have verified that the first five terms in the asymptotic expansion cannot be improved by any choice of the parameters. On the other hand, the sixth term contains the constants 7.9584, −5.9938 and 25.5362 which can be changed acting on the parameters.

The constants hidden in the $o(.)$ terms are unfortunately not uniform in $\mathbb{K}$ and are not even controlled by a linear bound in $n_\mathbb{K}$ and $\log\Delta_\mathbb{K}$: for instance, the rather innocent looking $\frac{n_\mathbb{K}\sqrt{x}}{T}$ is asymptotic to $\frac{2}{(\sqrt{5}-1)\pi}n_\mathbb{K}^2\log x$.

To facilitate the comparison with earlier results, we reorganize this asymptotic expansion in a form similar to Lagarias and Odlyzko's results. In this form, the right hand side of (1.1) is

$$\begin{aligned}
&\frac{\sqrt{x}}{2\pi}\Big[\log x - 2\log\log x + 2\log\Big(\frac{\nu}{n_\mathbb{K}}\Big) + 7.9584\Big]\log\Delta_\mathbb{K}\\
&+\frac{\sqrt{x}}{8\pi}\Big[\log^2 x - 4\log x\log\log x + 4\log\Big(\frac{e\nu}{n_\mathbb{K}}\Big)\log x + 4(\log\log x)^2\\
&-8\log\Big(\frac{\nu}{n_\mathbb{K}}\Big)\log\log x + 4\log^2\Big(\frac{\nu}{n_\mathbb{K}}\Big) + 4 - 23.9752\Big]n_\mathbb{K} + 25.5362\frac{\sqrt{x}}{\pi} + o(\sqrt{x})\\
=&\frac{\sqrt{x}}{2\pi}\Big[\log\Big(\frac{e^2\nu^2}{n_\mathbb{K}^2}\frac{x}{\log^2 x}\Big) + 5.9584\Big]\log\Delta_\mathbb{K}\\
&+\frac{\sqrt{x}}{8\pi}\Big[\log^2\Big(\frac{x}{\log^2 x}\Big) + 4\log\Big(\frac{e\nu}{n_\mathbb{K}}\Big)\log\Big(\frac{x}{\log^2 x}\Big) + 4\log^2\Big(\frac{e\nu}{n_\mathbb{K}}\Big)\\
&-8\log\Big(\frac{\nu}{n_\mathbb{K}\log x}\Big) - 27.9752\Big]n_\mathbb{K} + 25.5362\frac{\sqrt{x}}{\pi} + o(\sqrt{x})\\
=&\frac{\sqrt{x}}{2\pi}\Big[\log\Big(\frac{e^2\nu^2}{n_\mathbb{K}^2}\frac{x}{\log^2 x}\Big) + 5.9584\Big]\log\Delta_\mathbb{K}\\
&+\frac{\sqrt{x}}{8\pi}\Big[\log^2\Big(\frac{e^2\nu^2}{n_\mathbb{K}^2}\frac{x}{\log^2 x}\Big) - 4\log\Big(\frac{e^2\nu^2}{n_\mathbb{K}^2}\frac{1}{\log^2 x}\Big) - 19.9752\Big]n_\mathbb{K}\\
&+25.5362\frac{\sqrt{x}}{\pi} + o(\sqrt{x}).
\end{aligned}$$

As $\delta_\mathbb{K}$ diverges (1.1) is not very efficient, but still gives something similar to (1.2).

Numerical experiments

In [2] we prove the following results. First in Corollary 1.2:

$$\forall x \geqslant 100,\quad |\psi_\mathbb{K}(x) - x| \leqslant \sqrt{x}\Big[\Big(\frac{\log x}{2\pi} + 2\Big)\log\Delta_\mathbb{K} + \Big(\frac{\log^2 x}{8\pi} + 2\Big)n_\mathbb{K}\Big]. \tag{1.3}$$

Then in Corollary 1.3:

$$\begin{aligned}
\forall x \geqslant 3,\qquad |\psi_\mathbb{K}(x) - x| \leqslant \sqrt{x}\Big[&\Big(\frac{1}{2\pi}\log\Big(\frac{18.8\,x}{\log^2 x}\Big) + 2.3\Big)\log\Delta_\mathbb{K}\\
&+\Big(\frac{1}{8\pi}\log^2\Big(\frac{18.8\,x}{\log^2 x}\Big) + 1.3\Big)n_\mathbb{K} + 0.3\log x + 14.6\Big]
\end{aligned}\tag{1.4}$$

and

$$\forall x \geqslant 2000, \quad |\psi_{\mathbb{K}}(x) - x| \leqslant \sqrt{x}\Big[\Big(\frac{1}{2\pi}\log\Big(\frac{x}{\log^2 x}\Big) + 1.8\Big)\log\Delta_{\mathbb{K}} \qquad (1.5)$$
$$+ \Big(\frac{1}{8\pi}\log^2\Big(\frac{x}{\log^2 x}\Big) + 1.1\Big)n_{\mathbb{K}} + 1.2\log x + 10.2\Big].$$

We compare the upper bound (1.1) to these three formulas for several values of $n_{\mathbb{K}}$ and four discriminants for each $n_{\mathbb{K}}$. We test totally real and totally imaginary fields for the minimal discriminants allowed by Odlyzko's Table 3 in [4] and for their squares. In each table we indicate the minimal x after which Formula (1.1) is better than the corresponding formula. One observes that (1.1) is always better than (1.4), nearly always better than (1.3) (except for quadratic fields) and most of the times better than (1.5). The best between (1.1) and (1.2) is always better than (1.3–1.5) except for the case of quadratic fields in Formula (1.3).

From when does (1.1) get better than (1.3–1.5)

		real				imaginary			
	$n_{\mathbb{K}}$	$\Delta_{\mathbb{K}}$	(1.3)	(1.4)	(1.5)	$\Delta_{\mathbb{K}}$	(1.3)	(1.4)	(1.5)
minimal	2	4.9535	187929	3	2000	2.9633	445897	3	2000
	6	$2.9169 \cdot 10^{5}$	107	3	2000	$9.3896 \cdot 10^{3}$	106	3	2000
	10	$2.3927 \cdot 10^{11}$	100	3	2000	$1.8967 \cdot 10^{8}$	100	3	2000
	20	$6.5601 \cdot 10^{27}$	100	3	2000	$1.7076 \cdot 10^{20}$	100	3	2000
	50	$7.1245 \cdot 10^{81}$	100	3	2425	$2.8528 \cdot 10^{59}$	100	3	2306
	100	$1.5472 \cdot 10^{177}$	100	3	2713	$3.0629 \cdot 10^{128}$	100	3	2663
	200	$8.0911 \cdot 10^{374}$	100	3	2851	$2.1888 \cdot 10^{271}$	100	3	2843
	$n_{\mathbb{K}}$	$\Delta_{\mathbb{K}}$	(1.3)	(1.4)	(1.5)	$\Delta_{\mathbb{K}}$	(1.3)	(1.4)	(1.5)
square	2	$2.4538 \cdot 10^{1}$	25000	3	2000	8.7813	81922	3	2000
	6	$8.5086 \cdot 10^{10}$	100	3	2000	$8.8164 \cdot 10^{7}$	100	3	2000
	10	$5.7250 \cdot 10^{22}$	100	3	2000	$3.5975 \cdot 10^{16}$	100	3	2000
	20	$4.3035 \cdot 10^{55}$	100	3	2074	$2.9158 \cdot 10^{40}$	100	3	2000
	50	$5.0759 \cdot 10^{163}$	100	3	2597	$8.1386 \cdot 10^{118}$	100	3	2532
	100	$2.3937 \cdot 10^{354}$	100	3	2757	$9.3814 \cdot 10^{256}$	100	3	2745
	200	$6.5467 \cdot 10^{749}$	100	3	2830	$4.7910 \cdot 10^{542}$	100	3	2844

From when does the best of (1.1) and (1.2) get better than (1.3–1.5)

		real				imaginary			
	$n_{\mathbb{K}}$	$\Delta_{\mathbb{K}}$	(1.3)	(1.4)	(1.5)	$\Delta_{\mathbb{K}}$	(1.3)	(1.4)	(1.5)
minimal	2	4.9535	187929	3	2000	2.9633	445897	3	2000
	6	$2.9169 \cdot 10^{5}$	100	3	2000	$9.3896 \cdot 10^{3}$	100	3	2000
	10	$2.3927 \cdot 10^{11}$	100	3	2000	$1.8967 \cdot 10^{8}$	100	3	2000
	20	$6.5601 \cdot 10^{27}$	100	3	2000	$1.7076 \cdot 10^{20}$	100	3	2000
	50	$7.1245 \cdot 10^{81}$	100	3	2000	$2.8528 \cdot 10^{59}$	100	3	2000
	100	$1.5472 \cdot 10^{177}$	100	3	2000	$3.0629 \cdot 10^{128}$	100	3	2000
	200	$8.0911 \cdot 10^{374}$	100	3	2000	$2.1888 \cdot 10^{271}$	100	3	2000
	$n_{\mathbb{K}}$	$\Delta_{\mathbb{K}}$	(1.3)	(1.4)	(1.5)	$\Delta_{\mathbb{K}}$	(1.3)	(1.4)	(1.5)
square	2	$2.4538 \cdot 10^{1}$	25000	3	2000	8.7813	81922	3	2000
	6	$8.5086 \cdot 10^{10}$	100	3	2000	$8.8164 \cdot 10^{7}$	100	3	2000
	10	$5.7250 \cdot 10^{22}$	100	3	2000	$3.5975 \cdot 10^{16}$	100	3	2000
	20	$4.3035 \cdot 10^{55}$	100	3	2000	$2.9158 \cdot 10^{40}$	100	3	2000
	50	$5.0759 \cdot 10^{163}$	100	3	2000	$8.1386 \cdot 10^{118}$	100	3	2000
	100	$2.3937 \cdot 10^{354}$	100	3	2000	$9.3814 \cdot 10^{256}$	100	3	2000
	200	$6.5467 \cdot 10^{749}$	100	3	2000	$4.7910 \cdot 10^{542}$	100	3	2000

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2. Reproofs

We reprove the results of [1] and [2] adding an additional parameter to the main result of [2] and with a couple more digits. The methods of proof are the same and will thus not be repeated.

We recall that $r_{\mathbb{K}}$ is defined as the constant such that

$$\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s) = \frac{r_1 + r_2 - 1}{s} + r_{\mathbb{K}} + O(s) \qquad \text{as} \quad s \to 0.$$

Lemma 2.1. *Assume GRH. One has*

$$|r_{\mathbb{K}}| \leqslant 1.0155 \log \Delta_{\mathbb{K}} - 2.1042 n_{\mathbb{K}} + 8.3423 - e_{\mathbb{K}}$$

where

$$e_{\mathbb{K}} := \begin{cases} 4.4002 & \text{if } (r_1, r_2) = (1, 0) \\ 0.6931 & \text{if } (r_1, r_2) = (0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof of Lemma 3.1 in [1] gives the general case. Moreover, for $\mathbb{Q}$ we know that $r_{\mathbb{Q}} = \log 2\pi$. For imaginary quadratic fields, an $r_2 \log 2$ term can be restored in the proof of the aforementioned lemma. ■

Lemma 2.2. *Assume GRH. One has*

$$\sum_{|\gamma|\leqslant 5}\frac{1}{|\rho|}\leqslant 1.0111\log\Delta_{\mathbb{K}}-1.6550n_{\mathbb{K}}+7.0320.$$

Proof. see Lemma 3.1 in [2]. ■

As in [2] we will denote $W_{\mathbb{K}}(T):=\log\Delta_{\mathbb{K}}+n_{\mathbb{K}}\log\left(\frac{T}{2\pi}\right)$; this is obviously not the Lambert W function, and we believe that there is no risk of confusion.

Lemma 2.3. *We have, for all $T\geqslant 5$,*

$$\sum_{|\gamma|\leqslant T}1\leqslant\frac{T}{\pi}\Big(1+\frac{1.4427}{T}\Big)W_{\mathbb{K}}(T)-\frac{T}{\pi}\Big(1-\frac{8.9250}{T}\Big)n_{\mathbb{K}}+\frac{8.6542}{\pi},\tag{2.1a}$$

$$\sum_{|\gamma|\geqslant T}\frac{1}{|\rho|^2}\leqslant\Big(1+\frac{2.8854}{T}\Big)\frac{W_{\mathbb{K}}(T)}{\pi T}+\Big(1+\frac{18.6019}{T}\Big)\frac{n_{\mathbb{K}}}{\pi T}+\frac{17.3084}{\pi T^2},\tag{2.1b}$$

$$\sum_{\substack{\rho\\|\gamma|\leqslant T}}\frac{\pi}{|\rho|}\leqslant\Big(\log\Big(\frac{T}{2\pi}\Big)+3.9792\Big)\log\Delta_{\mathbb{K}}+\Big(\frac{1}{2}\log^2\Big(\frac{T}{2\pi}\Big)-1.4969\Big)n_{\mathbb{K}}+25.5362.\tag{2.1c}$$

Proof. These are just **First sum**, **Second sum** and **Third sum**, in [2] with more digits. To prove (2.1c), we use Lemma 2.2 instead of Lemma 3.1 of [2]. ■

Lemma 2.4. *Let*

$$f_1(x):=\sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)},\qquad f_2(x):=\sum_{r=2}^{\infty}\frac{x^{2-2r}}{(2r-1)(2r-2)},$$

$$R_{r_1,r_2}(x):=-(r_1+r_2-1)(x\log x-x)+r_2(\log x+1)-(r_1+r_2)f_1(x)-r_2f_2(x).$$

Let $x\geqslant 3$, then

$$-(r_1+r_2-1)\log x\leqslant R'_{r_1,r_2}(x)\leqslant-\delta_{(r_1,r_2),(1,0)}\log(1-x^{-2})-\delta_{(r_1,r_2),(0,1)}\log(1-x^{-1})$$

where $\delta_{(r_1,r_2),(a,b)}$ is 1 if and only if both indices are equal and 0 otherwise.

Proof. See Lemma 2.2 in [2]. ■

We now restate Theorem 1.1 of [2].

Theorem 2.5. *For every $x \geqslant 3$, $T \geqslant 5$ and $0 < \kappa \leqslant 2$ we have:*

$$\begin{aligned}
&\Big|\psi_{\mathbb{K}}(x) - x + \sum_{|\gamma|<T} \frac{x^\rho}{\rho}\Big| \\
&\qquad \leqslant \frac{\sqrt{x}}{\pi}\Big[\frac{2}{\kappa} + \frac{\kappa}{2} + \frac{1.4427\kappa^2 + 3\kappa + 11.5416}{2\kappa T} + \frac{0.5915\kappa + 4.3282}{T^2}\Big] W_{\mathbb{K}}(T) \\
&\qquad + \frac{\sqrt{x}}{\pi}\Big[\frac{2}{\kappa} - \frac{\kappa}{2} + \frac{8.9250\kappa^2 + 3\kappa + 74.4076}{2\kappa T} + \frac{1.7702\kappa + 27.9029}{T^2}\Big] n_{\mathbb{K}} \\
&\qquad + \frac{\kappa x}{2T} + \frac{\sqrt{x}}{\pi}\Big[\frac{(1.3774\kappa^2 + 11.0190)\pi}{\kappa T} + \frac{(0.4133\kappa + 8.2643)\pi}{T^2}\Big] \\
&\qquad + |r_{\mathbb{K}}| + \tilde{\epsilon}_{\mathbb{K}}(x,T)
\end{aligned}$$

where

$$\tilde{\epsilon}_{\mathbb{K}}(x,T) := \begin{cases} -\log(1 - x^{-2}) & \text{if } (r_1, r_2) = (1,0) \\ -\log(1 - x^{-1}) & \text{if } (r_1, r_2) = (0,1) \\ \max\big(0, d_{\mathbb{K}} \log x - 3.6133 n_{\mathbb{K}} \frac{\sqrt{x}}{T}\big) & \text{otherwise.} \end{cases}$$

Proof. The proof proceeds as for [2, Theorem 1.1], but now we choose $h = \pm\frac{\kappa x}{T}$ with $\kappa \in (0,2]$ instead of $h = \pm\frac{2x}{T}$. We start from [2, Inequality (4.2)] which, given the small modification of Lemma 2.4 above, now reads:

$$\begin{aligned}
-(r_1 + r_2 - 1)\log x \leqslant{}& \frac{\psi^{(1)}_{\mathbb{K}}(x+h) - \psi^{(1)}_{\mathbb{K}}(x)}{h} \\
&- \Big(x + \frac{h}{2} - \sum_{\rho} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h\rho(\rho+1)} - r_{\mathbb{K}}\Big) \\
\leqslant{}& -\delta_{(r_1,r_2),(1,0)} \log(1 - x^{-2}) - \delta_{(r_1,r_2),(0,1)} \log(1 - x^{-1}).
\end{aligned}$$

As seen in [2, Section 4], under GRH we have

$$\Big|\sum_{|\gamma|\geqslant T} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h\rho(\rho+1)}\Big| \leqslant \sum_{|\gamma|\geqslant T} x^{\frac{3}{2}} \frac{\big(1 + \frac{h}{x}\big)^{\frac{3}{2}} + 1}{h|\rho(\rho+1)|} \leqslant A\frac{x^{\frac{3}{2}}}{h} \sum_{|\gamma|\geqslant T} \frac{1}{|\rho^2|},$$

with $A := 1 + \big(1 + \frac{h}{x}\big)^{\frac{3}{2}}$ while

$$\sum_{|\gamma|<T} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h\rho(\rho+1)} = \sum_{|\gamma|<T} \frac{x^\rho}{\rho} + hx^{-1/2} \sum_{|\gamma|<T} w_\rho x^{i\gamma}$$

with

$$w_\rho := \frac{\big(1 + \frac{h}{x}\big)^{\rho+1} - 1 - (\rho+1)\frac{h}{x}}{\rho(\rho+1)\big(\frac{h}{x}\big)^2}.$$

For $h > 0$, we take $h = \frac{\kappa x}{T}$ thus

$$A = 1 + \Big(1 + \frac{\kappa}{T}\Big)^{\frac{3}{2}} \leqslant 2 + \frac{3\kappa}{2T} + \frac{3\kappa^2}{8T^2}.$$

From Lemma 2.1 in [2] we know that $|w_\rho| \leqslant \frac{1}{2}$ hence by (2.1b) and (2.1a)

$$\begin{aligned}
&\frac{\pi}{\sqrt{x}}\Big|\sum_\rho \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h\rho(\rho+1)} - \sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big| \\
&\quad\leqslant \Big[\frac{2}{\kappa} + \frac{3}{2T} + \frac{3\kappa}{8T^2}\Big]\Big[\Big(1 + \frac{2.8854}{T}\Big)\frac{W_{\mathbb{K}}(T)}{\pi T} + \Big(1 + \frac{18.6019}{T}\Big)\frac{n_{\mathbb{K}}}{\pi T} + \frac{17.3084}{T}\Big] \\
&\quad\quad + \frac{\kappa}{2}\Big[\Big(1 + \frac{1.4427}{T}\Big)W_{\mathbb{K}}(T) - \Big(1 - \frac{8.9250}{T}\Big)n_{\mathbb{K}} + \frac{8.6542}{T}\Big] \\
&\quad= \Big[\Big(\frac{2}{\kappa} + \frac{3}{2T} + \frac{3\kappa}{8T^2}\Big)\Big(1 + \frac{2.8854}{T}\Big) + \frac{\kappa}{2}\Big(1 + \frac{1.4427}{T}\Big)\Big]W_{\mathbb{K}}(T) \\
&\quad\quad + \Big[\Big(\frac{2}{\kappa} + \frac{3}{2T} + \frac{3\kappa}{8T^2}\Big)\Big(1 + \frac{18.6019}{T}\Big) - \frac{\kappa}{2}\Big(1 - \frac{8.9250}{T}\Big)\Big]n_{\mathbb{K}} \\
&\quad\quad + \Big(\frac{2}{\kappa} + \frac{3}{2T} + \frac{3\kappa}{8T^2}\Big)\frac{17.3084}{T} + 4.3271\frac{\kappa}{T}.
\end{aligned}$$

After some simplifications it becomes, for $T \geqslant 5$,

$$\begin{aligned}
&\frac{\pi}{\sqrt{x}}\Big|\sum_\rho \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h\rho(\rho+1)} - \sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big| \\
&\quad\leqslant \Big[\frac{2}{\kappa} + \frac{\kappa}{2} + \frac{1.4427\kappa^2 + 3\kappa + 11.5416}{2\kappa T} + \frac{0.5915\kappa + 4.3281}{T^2}\Big]W_{\mathbb{K}}(T) \\
&\quad\quad + \Big[\frac{2}{\kappa} - \frac{\kappa}{2} + \frac{8.9250\kappa^2 + 3\kappa + 74.4076}{2\kappa T} + \frac{1.7702\kappa + 27.9029}{T^2}\Big]n_{\mathbb{K}} \\
&\quad\quad + \frac{4.3271\kappa^2 + 34.6168}{\kappa T} + \frac{1.2982\kappa + 25.9626}{T^2},
\end{aligned} \tag{2.2}$$

which is an analogous of [2, Equation (4.3)].

For $h < 0$, we take $h = -\frac{\kappa x}{T}$, we then have $x + h > 1$ if $\kappa \leqslant 3$, $x \geqslant 3$ and $T \geqslant 5$. We slightly modify the bound for A in that case and take

$$A = 1 + \Big(1 - \frac{\kappa}{T}\Big)^{\frac{3}{2}} \leqslant 2 - \frac{3\kappa}{2T} + \frac{\kappa^2}{2T^2}.$$

We still have $|w_\rho| \leqslant \frac{1}{2} + \frac{\kappa}{6T}$ and Equation (4.4) of [2] becomes for $T \geqslant 5$

$$\begin{aligned}\frac{\pi}{\sqrt{x}}\Big|\sum_\rho \frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)} - \sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big| \qquad\qquad (2.3)\\ \leqslant \Big[\frac{2}{\kappa}+\frac{\kappa}{2}+\frac{\kappa^3/3+1.4427\kappa^2-3\kappa+11.5416}{2\kappa T}\\ +\frac{0.2405\kappa^2+0.7886\kappa-4.3281}{T^2}\Big]W_{\mathbb{K}}(T)\\ +\Big[\frac{2}{\kappa}-\frac{\kappa}{2}+\frac{-\kappa^3/3+8.9250\kappa^2-3\kappa+74.4076}{2\kappa T}\\ +\frac{1.4875\kappa^2+2.3602\kappa-27.9028}{T^2}\Big]n_{\mathbb{K}}\\ +\frac{4.3271\kappa^2+34.6168}{\kappa T}+\frac{1.4424\kappa^2+1.7309\kappa-25.9626}{T^2}.\end{aligned}$$

Let $M_{W,\pm}(T)$, $M_{n,\pm}(T)$ and $M_{c,\pm}(T)$ be the functions of T such that the right hand side of (2.2) and (2.3) respectively are

$$\begin{gathered}M_{W,+}(T)W_{\mathbb{K}}(T)+M_{n,+}(T)n_{\mathbb{K}}+M_{c,+}(T),\\ M_{W,-}(T)W_{\mathbb{K}}(T)+M_{n,-}(T)n_{\mathbb{K}}+M_{c,-}(T),\end{gathered}$$

and their differences let be denoted as

$$\begin{aligned}D_W(T) &:= M_{W,+}(T)-M_{W,-}(T) = \frac{18-\kappa^2}{6T}+\frac{8.6562-0.1971\kappa-0.2405\kappa^2}{T^2},\\ D_n(T) &:= M_{n,+}(T)-M_{n,-}(T) = \frac{18+\kappa^2}{6T}+\frac{55.8057-0.5900\kappa-1.4875\kappa^2}{T^2},\\ D_c(T) &:= M_{c,+}(T)-M_{c,-}(T) = \frac{51.9252-0.4327\kappa-1.4424\kappa^2}{T^2}.\end{aligned}$$

We then have

$$\begin{aligned}&\Big|\psi_{\mathbb{K}}(x)-x+\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\Big|\\ &\quad\leqslant \frac{\sqrt{x}}{\pi}\big(M_{W,+}(T)W_{\mathbb{K}}(T)+M_{n,+}(T)n_{\mathbb{K}}+M_{c,+}(T)\big)\\ &\qquad+\frac{\kappa x}{2T}+|r_{\mathbb{K}}|+\delta_{(r_1,r_2),(1,0)}\log(1-x^{-2})+\delta_{(r_1,r_2),(0,1)}\log(1-x^{-1})\\ &\qquad+\max\Big(0,(r_1+r_2-1)\log x-\frac{\sqrt{x}}{\pi}\big(D_W(T)W_{\mathbb{K}}(T)+D_n(T)n_{\mathbb{K}}+D_c(T)\big)\Big).\end{aligned}$$

The claim follows if $d_{\mathbb{K}}=0$ since $D_W(T)$, $D_n(T)$ and $D_c(T)\geqslant 0$ (D_c and the coefficients of $\frac{1}{T^2}$ in D_n and D_W are positive $\forall\kappa \in [-6,4]$). If $d_{\mathbb{K}} > 0$ we have $\frac{1}{n_{\mathbb{K}}}\log\Delta_{\mathbb{K}} \geqslant \frac{1}{2}\log 5$ thus

$$\begin{aligned} D_W(T)\frac{W_{\mathbb{K}}(T)}{n_{\mathbb{K}}} + D_n(T) &\geqslant \Big(\frac{18-\kappa^2}{6T} + \frac{8.6562 - 0.1971\kappa - 0.2405\kappa^2}{T^2}\Big)\log\Big(\frac{\sqrt{5}T}{2\pi}\Big) \\ &\quad + \frac{18+\kappa^2}{6T} + \frac{55.8057 - 0.5900\kappa - 1.4874\kappa^2}{T^2} \\ &\geqslant \frac{3.6133\pi}{T} \end{aligned}$$

when $T \geqslant 5$ and $0 \leqslant \kappa \leqslant 2$. ∎

3. Proof of Theorem 1.1

By Equation (2.1c) we have for $T \geqslant 5$

$$\Big|\sum_{\substack{\rho \\ |\gamma|<T}} \frac{x^\rho}{\rho}\Big| \leqslant \frac{\sqrt{x}}{\pi}\Big[\Big(\log\Big(\frac{T}{2\pi}\Big) + \alpha\Big)\log\Delta_{\mathbb{K}} + \Big(\frac{1}{2}\log^2\Big(\frac{T}{2\pi}\Big) + \beta\Big)n_{\mathbb{K}} + \gamma\Big] \tag{3.1}$$

with $\alpha = 3.9792$, $\beta = -1.4969$ and $\gamma = 25.5362$. Recalling the upper bound for $|r_{\mathbb{K}}|$ in Lemma 2.1, from the result in Theorem 2.5 and (3.1) we deduce that for $x \geqslant 3$ and $T \geqslant 5$,

$$|\psi_{\mathbb{K}}(x) - x| \leqslant \Big(\frac{\sqrt{x}}{\pi}F(T) + 1.0155\Big)\log\Delta_{\mathbb{K}} + \Big(\frac{\sqrt{x}}{\pi}G(T) - 2.1042\Big)n_{\mathbb{K}} + H(x,T) \tag{3.2}$$

with

$$\begin{aligned} F(T) &:= \log\Big(\frac{T}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2} + \frac{1.4427\kappa^2 + 3\kappa + 11.5416}{2\kappa T} \\ &\quad + \frac{0.5915\kappa + 4.3282}{T^2} + \alpha, \\ G(T) &:= \frac{1}{2}\log^2\Big(\frac{T}{2\pi}\Big) \\ &\quad + \Big(\frac{2}{\kappa} + \frac{\kappa}{2} + \frac{1.4427\kappa^2 + 3\kappa + 11.5416}{2\kappa T} + \frac{0.5915\kappa + 4.3282}{T^2}\Big)\log\Big(\frac{T}{2\pi}\Big) \\ &\quad + \beta + \frac{2}{\kappa} - \frac{\kappa}{2} + \frac{8.9250\kappa^2 + 3\kappa + 74.4076}{2\kappa T} + \frac{1.7702\kappa + 27.9029}{T^2}, \\ H(x,T) &:= \frac{\kappa x}{2T} + \frac{\sqrt{x}}{\pi}\Big[\gamma + \frac{(1.3774\kappa^2 + 11.0190)\pi}{\kappa T} + \frac{(0.4133\kappa + 8.2643)\pi}{T^2}\Big] \\ &\quad + 8.3423 + \epsilon_{\mathbb{K}}(x,T). \end{aligned} \tag{3.3}$$

We need to choose T to get the lowest possible bound for $|\psi_{\mathbb{K}}(x) - x|$, thus we choose the best T by looking for an approximate zero of

$$\frac{\partial}{\partial T}\Big(\frac{\sqrt{x}}{\pi}F(T)\log\Delta_{\mathbb{K}} + \frac{\sqrt{x}}{\pi}G(T)n_{\mathbb{K}} + H(x,T)\Big)$$

above 5. Unfortunately we are not able to find T as an explicit function of x.

Both $T \mapsto F(T)$ and $T \mapsto G(T)$ have a unique minimum while $T \mapsto H(x,T)$ is decreasing for any $x > 0$. The main increasing terms are $\frac{\sqrt{x}}{2\pi}\big[\log\big(\frac{T}{2\pi}\big)+\frac{\kappa}{2}+\frac{2}{\kappa}\big]^2 n_{\mathbb{K}}$ from $G(T)$, and $\frac{\sqrt{x}}{\pi}\big[\log\big(\frac{T}{2\pi}\big)+\frac{\kappa}{2}+\frac{2}{\kappa}\big]\log\Delta_{\mathbb{K}}$ from $F(T)$, while the main decreasing term is $\frac{\kappa x}{2T}$ from $H(x,T)$. The derivative of the sum of these three terms is zero for $T\log\big(\frac{e^{\frac{2}{\kappa}+\frac{\kappa}{2}}\delta_{\mathbb{K}}T}{2\pi}\big) = \frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}}}$, we should thus choose

$$
\begin{aligned}
T = T_W &:= \frac{2\pi}{\delta_{\mathbb{K}} e^{\frac{2}{\kappa}+\frac{\kappa}{2}}} e^{W\left(\frac{\kappa e^{\frac{2}{\kappa}+\frac{\kappa}{2}}\delta_{\mathbb{K}}\sqrt{x}}{4n_{\mathbb{K}}}\right)} \\
&= \frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}} W\left(\frac{\kappa e^{\frac{2}{\kappa}+\frac{\kappa}{2}}\delta_{\mathbb{K}}\sqrt{x}}{4n_{\mathbb{K}}}\right)}.
\end{aligned}
$$

However, for $\delta_{\mathbb{K}} \to \infty$ we have $T_W \to 0$. We thus slightly complicate the expression we are trying to minimize: this will have the effect to give a minimum that is both more precise and above 5. The expressions contain the parameter κ, which has to be fixed. To find a good value for κ, we computed the asymptotic expansion of the result with optimal T and κ unevaluated but independent of x. This is

$$
\frac{\sqrt{x}}{2\pi}\Big(\frac{\log^2 x}{4} - \log x \log\log x + \Big(\log\delta_{\mathbb{K}} + \log\Big(\frac{\kappa e^{\frac{2}{\kappa}+\frac{\kappa}{2}}}{2\pi n_{\mathbb{K}}}\Big) + 1\Big)\log x + o(\log x)\Big) n_{\mathbb{K}},
$$

so that the best value for κ is the one minimizing $\kappa e^{\frac{2}{\kappa}+\frac{\kappa}{2}}$, i.e. $\sqrt{5}-1$. Thus, to ease a little bit the computations, we set $\kappa = \sqrt{5}-1$ right now, and we retain the symbol κ only in those terms which will contribute to the main part of the result. Notice that $\frac{2}{\kappa}+\frac{\kappa}{2} = \sqrt{5}$ and $\frac{2}{\kappa}-\frac{\kappa}{2} = 1$ and that (3.2) and (3.3) give

$$
\begin{aligned}
F(T) &\leqslant \log\Big(\frac{T}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2} + \frac{7.0604}{T} + \frac{5.0593}{T^2} + 3.9792, \\
G(T) &\leqslant \frac{1}{2}\Big(\log\Big(\frac{T}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2}\Big)^2 + \Big(\frac{7.0604}{T} + \frac{5.0593}{T^2}\Big)\log\Big(\frac{T}{2\pi}\Big) \\
&\qquad - 2.9969 + \frac{37.1145}{T} + \frac{30.0910}{T^2}, \\
H(x,T) &\leqslant \frac{\kappa x}{2T} + \frac{\sqrt{x}}{\pi}\Big(25.5362 + \frac{33.3542}{T} + \frac{27.5673}{T^2}\Big) + 8.3423 + \epsilon_{\mathbb{K}}(x,T).
\end{aligned}
$$

We have kept κ in all terms which will contribute to the highest order terms of the asymptotic expansion in x, in order to make explicit the role of this parameter on the final quality of the result.

Since $\epsilon_{\mathbb{K}}$ is small with respect to most other parameters and not differentiable, we remove it from the optimization process. Let then

$$\begin{aligned}
E_0(x,T) &:= F(T)\log\delta_{\mathbb{K}} + G(T) + \frac{\pi}{n_{\mathbb{K}}\sqrt{x}}(H(x,T) - \epsilon_{\mathbb{K}}(x,T)) \\
&\leqslant \Big(\log\Big(\frac{T}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2} + \frac{7.0604}{T} + \frac{5.0593}{T^2} + 3.9792\Big)\log\delta_{\mathbb{K}} \\
&\quad + \frac{1}{2}\Big(\log\Big(\frac{T}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2}\Big)^2 + \Big(\frac{7.0604}{T} + \frac{5.0593}{T^2}\Big)\log\Big(\frac{T}{2\pi}\Big) \\
&\quad - 2.9969 + \frac{37.1145}{T} + \frac{30.0910}{T^2} + \frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}}T} \\
&\quad + \frac{1}{n_{\mathbb{K}}}\Big(25.5362 + \frac{33.3542}{T} + \frac{27.5673}{T^2}\Big) + \frac{8.3423\pi}{n_{\mathbb{K}}\sqrt{x}} \\
&\leqslant \frac{1}{2}\Big(\log\Big(\frac{T}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2} + \log\delta_{\mathbb{K}}\Big)^2 - \frac{1}{2}\log^2\delta_{\mathbb{K}} \\
&\quad + \Big(\frac{7.0604}{T} + \frac{5.0594}{T^2}\Big)\Big(\log\Big(\frac{T}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2} + \log\delta_{\mathbb{K}}\Big) \\
&\quad + \frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}}T} + \frac{21.3270}{T} + \frac{18.7781}{T^2} + \frac{1}{n_{\mathbb{K}}}\Big(\frac{33.3542}{T} + \frac{27.5673}{T^2}\Big) \\
&\quad + 3.9792\log\delta_{\mathbb{K}} - 2.9969 + \frac{25.5362}{n_{\mathbb{K}}} + \frac{8.3423\pi}{n_{\mathbb{K}}\sqrt{x}} \\
&=: E(x,T).
\end{aligned} \tag{3.4}$$

It is obvious that $\lim_{T\to\infty} E(x,T) = \infty$. We have

$$\begin{aligned}
\frac{\partial E(x,T)}{\partial T} &= \Big(\frac{1}{T} - \frac{7.0604}{T^2} - \frac{10.1186}{T^3}\Big)\Big(\log\Big(\frac{T}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2} + \log\delta_{\mathbb{K}}\Big) \\
&\quad - \Big[\frac{14.2666}{T^2} + \frac{32.4969}{T^3} + \frac{1}{n_{\mathbb{K}}}\Big(\frac{33.3542}{T^2} + \frac{55.1346}{T^3}\Big) + \frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}}T^2}\Big].
\end{aligned}$$

Let $T_F = 8.282137\ldots$ be the positive root of $T^2 - 7.0604T - 10.1186$ (which is where the estimate of F reaches its minimum). We obviously have

$$\frac{\partial E(x,T)}{\partial T}\Big|_{T=T_F} < 0 \qquad \text{and} \qquad \frac{\partial E(x,T)}{\partial T} = 0$$

when

$$\begin{aligned}
&\Big(T - 7.0604 - \frac{10.1186}{T}\Big)\Big(\log\Big(\frac{T}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2} + \log\delta_{\mathbb{K}}\Big) \\
&\qquad = \frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}}} + 14.2666 + \frac{32.4969}{T} + \frac{1}{n_{\mathbb{K}}}\Big(33.3542 + \frac{55.1346}{T}\Big).
\end{aligned} \tag{3.5}$$

The left hand side of this equation is increasing for $T > T_F$ and maps $[T_F, +\infty)$ onto $[0, +\infty)$ while the right hand side is decreasing for $T > 0$ thus the equation has a single solution for $T > T_F$. Thus for given $\mathbb{K}$ and x, $E(x,T)$ has a single local minimum for some $T > T_F$ and this minimum is reached for the unique $T > T_F$ satisfying (3.5). The solutions (in T) of (3.5) can unfortunately not be expressed with standard analytic functions. We thus slightly modify (3.5) to have a solution with a nice expression in terms of the Lambert-W function. We will discuss in Remark 1 below the effect of the change we made to the equation.
Suppose we have found a T_0 satisfying

$$\Big(T_0 - 7.0604 - \frac{10.1186}{T_0}\Big)\Big(\log\Big(\frac{T_0}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2} + \log\delta_{\mathbb{K}}\Big) = \frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}}} + 21.3270 + \frac{33.5251}{n_{\mathbb{K}}}. \tag{3.6}$$

Denote

$$w := \log\Big(\frac{T_0}{2\pi}\Big) + \frac{2}{\kappa} + \frac{\kappa}{2} + \log\delta_{\mathbb{K}}$$

so that

$$\frac{\kappa\pi\sqrt{x}}{2T_0 n_{\mathbb{K}}} + \frac{21.3270}{T_0} + \frac{33.3542}{T_0 n_{\mathbb{K}}} = \Big(1 - \frac{7.0604}{T_0} - \frac{10.1186}{T_0^2}\Big)w.$$

Then

$$\begin{aligned} E(x,T_0) = {}& \frac{1}{2}w^2 + w - \frac{1}{2}\log^2\delta_{\mathbb{K}} + \frac{18.7781 + \frac{27.5673}{n_{\mathbb{K}}} - 5.0593w}{T_0^2} \\ & + 3.9792\log\delta_{\mathbb{K}} - 2.9969 + \frac{25.5362}{n_{\mathbb{K}}} + \frac{8.3423\pi}{n_{\mathbb{K}}\sqrt{x}} \end{aligned} \tag{3.7}$$

and so, according to Lemma 3.1, see below, it is

$$\begin{aligned} &\leqslant \frac{1}{2}(w+1)^2 - \frac{1}{2}\log^2\delta_{\mathbb{K}} + 3.9792\log\delta_{\mathbb{K}} - 3.4969 + \frac{25.5362}{n_{\mathbb{K}}} + \frac{8.8590\pi}{n_{\mathbb{K}}\sqrt{x}} \\ &= \frac{1}{2}\Big(\log\Big(\delta_{\mathbb{K}} e^{\frac{2}{\kappa}+\frac{\kappa}{2}}\frac{T_0}{2\pi}\Big) + 1\Big)^2 - \frac{1}{2}\log^2\delta_{\mathbb{K}} \\ &\quad + 3.9792\log\delta_{\mathbb{K}} - 3.4969 + \frac{25.5362}{n_{\mathbb{K}}} + \frac{8.8590\pi}{n_{\mathbb{K}}\sqrt{x}}. \end{aligned} \tag{3.8}$$

To have an upper-bound for $E(x,T_0)$, we can substitute T_0 in (3.8) by anything greater than T_0. We define T_W and redefine w by

$$\begin{aligned} a &:= \frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}}} + 21.3270 + \frac{33.3542}{n_{\mathbb{K}}}, \\ w &:= W\Big(\frac{e^{\frac{2}{\kappa}+\frac{\kappa}{2}}}{2\pi}\delta_{\mathbb{K}} a\Big), \\ T_W &:= \frac{2\pi e^w}{\delta_{\mathbb{K}} e^{\frac{2}{\kappa}+\frac{\kappa}{2}}} = \frac{a}{w}, \end{aligned}$$

which means that T_W is the solution of the equation

$$T\Big(\log\Big(\frac{T}{2\pi}\Big)+\frac{2}{\kappa}+\frac{\kappa}{2}+\log\delta_{\mathbb{K}}\Big)=\frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}}}+21.3270+\frac{33.3542}{n_{\mathbb{K}}}. \tag{3.9}$$

Recalling the constant T_F defined above, T_F+T_W is larger than T_0. Indeed, if we replace T_0 by T_F+T_W in (3.6), the first factor is bigger than T_W while the second is bigger than the one in (3.9) so that the left hand side of (3.6) is bigger than its right hand side; since the left hand side is increasing this proves that $T_F+T_W\geqslant T_0$. We now replace T_0 by T_F+T_W in (3.8) obtaining

$$\begin{aligned}E_0(x,T_0)&\leqslant\frac{1}{2}\Big(\log\Big(\delta_{\mathbb{K}}e^{\frac{2}{\kappa}+\frac{\kappa}{2}}\frac{T_W+T_F}{2\pi}\Big)+1\Big)^2+3.9792\log\delta_{\mathbb{K}}\\&\quad-3.4969+\frac{25.5362}{n_{\mathbb{K}}}+\frac{8.8590\pi}{n_{\mathbb{K}}\sqrt{x}}\\&\leqslant\frac{1}{2}\log^2\Big(e^{w+1}+33.5251\delta_{\mathbb{K}}\Big)+3.9792\log\delta_{\mathbb{K}}\\&\quad-3.4969+\frac{25.5362}{n_{\mathbb{K}}}+\frac{8.8590\pi}{n_{\mathbb{K}}\sqrt{x}}\end{aligned}$$

which is exactly the first claim in Theorem 1.1.

We now proceed for the second inequality (1.2). We fix a value for T, postponing to Remark 2 the reason for this choice. The minimal value for F is reached when $(\kappa,T)\simeq(2.141,7.2773)$ and the actual value is $\leqslant 2.2367\pi$. We make a slightly different choice, which is $\kappa=2$ and $T=10$, which increases slightly the coefficient of $\sqrt{x}\log\Delta_{\mathbb{K}}$ but decreases the coefficient of $\sqrt{x}\,n_{\mathbb{K}}$ and x, and makes the formula slightly nicer. We then have

$$\begin{aligned}F(T)&\leqslant 2.2543\pi\\G(T)&\leqslant 0.9722\pi\end{aligned}$$

which gives

$$\begin{aligned}|\psi_{\mathbb{K}}(x)-x|&\leqslant(2.2543\sqrt{x}+1.0155)\log\Delta_{\mathbb{K}}+(0.9722\sqrt{x}-2.1042)n_{\mathbb{K}}\\&\quad+\frac{x}{10}+9.0458\sqrt{x}+7.0320+\epsilon_{\mathbb{K}}(x,10)\end{aligned}$$

proving the second claim in Theorem 1.1.

Remark 1. We discuss some choices we made for the first bound. Let us call $T_{\min}$ the zero of (3.5) above T_F. When we define T_0 from (3.6) we obviously have $T_0\neq T_{\min}$. However the difference between the functions appearing on the right hand side of (3.6) and (3.5) is

$$7.0604-\frac{32.4969}{T}-\frac{55.1346}{n_{\mathbb{K}}T}.$$

This means that, to obtain $T_{\min}$, we should remove from the right hand side of the equation defining T_0 a quantity that is asymptotic to 7.0604. Hence, to the first order for $x \to +\infty$, $T_0 - T_{\min} \sim \frac{7.0604}{\log T_0}$. Thus $\log T_0 - \log T_{\min} \sim \frac{7.0604}{T_0 \log T_0}$, and $\frac{1}{2}\log^2 T_0 - \frac{1}{2}\log^2 T_{\min} \sim \frac{7.0604}{T_0}$. This difference produces a term of order $\frac{n_{\mathbb{K}}\sqrt{x}}{T_0} \asymp n_{\mathbb{K}}^2 \log x$ which is already much smaller than the main terms of the upper bound. On the other hand, when we substitute

$$\frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}}T_{\min}} + 21.3270 + \frac{33.3542}{n_{\mathbb{K}}}$$

in (3.4) to obtain the equivalent of (3.7), a term $\frac{7.0604}{T_{\min}} \sim \frac{7.0604}{T_0}$ would remain. This term cancels out the previous one, so that the final effect of the replacement of $T_{\min}$ by T_0 is even smaller than $n_{\mathbb{K}}^2 \log x$.

The situation where $\delta_{\mathbb{K}} \to +\infty$ with x and $n_{\mathbb{K}}$ fixed is slightly different. In this case $T_{\min}$ and T_0 both tend to T_F so that the result is changed by a term of the order of $\frac{1}{\log \delta_{\mathbb{K}}}$. However (1.1) is not very good anyway because in several steps we dropped terms in $-\frac{1}{T^2}$, including a term in $-\frac{w}{T^2}$, which now do not tend to 0.

Remark 2. One should keep in mind that (1.2) is thought for fixed x and diverging $\delta_{\mathbb{K}}$. To prove it, we chose $T = 10$. We could have used a technique similar to the one we used for (1.1) optimizing T in terms of $\delta_{\mathbb{K}}$. However this is not worth it, because we would obtain a bound which differs from (1.2) by $(-c_1 + o(1))\frac{\sqrt{x}}{\log \delta_{\mathbb{K}}}$. Meanwhile, all the approximations and choices we made affect the coefficient of $\sqrt{x}\log \delta_{\mathbb{K}}$ in (1.2), thus to improve the bound it is more efficient, for instance, to increase the minimal value above which Theorem 2.5 is valid or to refine the bounds in Lemma 2.3.

Lemma 3.1. *In the settings of the proof of Theorem 1.1,*

$$R := \frac{18.7781 + \frac{27.5673}{n_{\mathbb{K}}} - 5.0593w}{T_0^2} \leqslant \frac{0.5167\pi}{n_{\mathbb{K}}\sqrt{x}}.$$

Proof. Letting $L := \frac{2}{\kappa} + \frac{\kappa}{2} + \log \delta_{\mathbb{K}}$ and $S := \frac{\kappa\pi\sqrt{x}}{2n_{\mathbb{K}}} + 21.3270 + \frac{33.3542}{n_{\mathbb{K}}}$, we rewrite (3.6) as

$$f(T_0, L) = S.$$

Using the implicit function theorem, it is then easy to see that

$$\frac{\partial w}{\partial L} = \frac{\big(T_0 + \frac{10.1186}{T_0}\big)\big(\log\big(\frac{T_0}{2\pi}\big) + L\big)}{\big(T_0 + \frac{10.1186}{T_0}\big)\big(\log\big(\frac{T_0}{2\pi}\big) + L\big) + \big(T_0 - 7.0604 - \frac{10.1186}{T_0}\big)} \geqslant 0$$

and that $\frac{\partial T_0}{\partial x} \geqslant 0$ and thus w is increasing with both L and x. It is thus obvious that R is decreasing with x. We now prove that, if $n_{\mathbb{K}} \geqslant 2$ and x is fixed, the maximum value of R is obtained for the minimal discriminant. Indeed we observe that $\frac{\partial R}{\partial L}$ is

$$\frac{2\big(T_0 - 7.0604 - \frac{10.1186}{T_0}\big)\big(18.7781 + \frac{27.5673}{n_{\mathbb{K}}}\big) - 5.0593w(3T_0 - 2\cdot 7.0604 - \frac{10.1186}{T_0})}{T_0^2\Big(\big(T_0 + \frac{10.1186}{T_0}\big)\big(\log\big(\frac{T_0}{2\pi}\big) + L\big) + \big(T_0 - 7.0604 - \frac{10.1186}{T_0}\big)\Big)}$$

which is negative if

$$\log \delta_{\mathbb{K}} \geqslant \frac{2\big(T_0 - 7.0604 - \frac{10.1186}{T_0}\big)\big(18.7781 + \frac{27.5673}{n_{\mathbb{K}}}\big)}{5.0593\big(3T_0 - 2\cdot 7.0604 - \frac{10.1186}{T_0}\big)} - \log\Big(\frac{T_0}{2\pi}\Big) - \sqrt{5}.$$

This is true because as a function of T_0 the right hand side has a maximum value equal to $0.1366\dots$ (attained for $n_{\mathbb{K}} = 2$ and $T_0 \approx 21.2153$) while $\log \delta_{\mathbb{K}} \geqslant \frac{1}{2}\log 3$. For each degree, we thus just need to bound R for the field with minimal absolute discriminant. For the first few $n_{\mathbb{K}}$ we determine the lowest possible value for $\Delta_{\mathbb{K}}$ using the "megrez" number field tables [5] and for $n_{\mathbb{K}} \geqslant 8$ we use Odlyzko's Table 3 in [4].

For any increasing sequence (x_n) let $c_n := \frac{n_{\mathbb{K}}\sqrt{x_{n+1}}}{\pi} R(x_n)$. Since R is decreasing in x and (x_n) is increasing, if $c_{\max} := \max c_n$, we have

$$\forall x, \quad R(x) \leqslant \frac{c_{\max}\pi}{n_{\mathbb{K}}\sqrt{x}}.$$

We use the sequence (x_n) defined as follows:

$$x_1 := 3, \qquad x_{n+1} := x_n + \begin{cases} 1 & \text{if } x_n \in [3, 5000) \\ 10 & \text{if } x_n \in [5000, 10^4) \\ 100 & \text{if } x_n \in [10^4, 10^5) \\ x_n & \text{otherwise.} \end{cases}$$

The table below shows the values of $c_{\max}$ for each degree, the point $x_{n_{\max}}$ where it is reached and the total number of points we compute (we stop as soon as $R(x_n) < 0$). For $n_{\mathbb{K}} \geqslant 9$, we used the general formula for $n_{\mathbb{K}} = 9$ with $S = 21.3270$ and $T_0 = T_F$, which ensures that the result is valid for all $n_{\mathbb{K}} \geqslant 9$.

$n_{\mathbb{K}}$	$c_{\max}$	$x_{n_{\max}}$	n_{points}
1	0.2110	2810	6411
2	0.4644	4350	6402
3	0.5167	3986	6398
4	0.4443	2927	5809
5	0.1325	694	4177
6	0.0144	63	280
7	< 0	3	1
8	< 0	3	1
$\geqslant 9$	< 0	3	1

■

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