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MULTIPLICATIVE FUNCTION MEAN VALUES: ASYMPTOTIC ESTIMATES

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In celebration of the eighty fifth birthday
of Eduard Wirsing.

Abstract: Classical Mean-Value results of Wirsing type are established under weaker than classical constraints.

Keywords: multiplicative function, mean value, Maass form.

1. Statement of results

For many studies in analytic number theory a natural object against which to measure the mean-value of a complex-valued multiplicative arithmetic function $n \to g(n)$ is the mean-value of its attendant function $n \to |g(n)|$.

This reflects the decomposition $n \to |g(n)| \exp(i \arg g(n))$ of a non-vanishing completely multiplicative function into essentially a unitary character on the multiplicative group of the positive rationals, and a homomorphism $n \to \log |g(n)|$ of the positive rationals into the additive reals.

Some fifty years ago, papers of Delange [3] 1961, Wirsing [13] 1961, [14] 1967, Halász [9] 1968, catalysed the general study of multiplicative functions and moved the field seriously forward.

In the present paper I re-examine the theorems of Wirsing in the light of more recent developments and apply related ideas to the consideration of two open-ended questions.

The following four cumulative theorems will be established, all new. Several auxiliary propositions are also of independent interest.

Theorem 1. *Let g be a non-negative multiplicative function, uniformly bounded on the primes, for which the series $\sum q^{-1} g(q)$, taken over the prime-powers $q = p^k$ with $k \geqslant 2$, converges, and for which the sums $y^{-1} \sum_{q \leqslant y} g(q) \log q$, $y \geqslant 2$,*

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are uniformly bounded. Let $h(n)$ be a complex-valued multiplicative function that satisfies $|h(n)| \leqslant g(n)$. Set $G(x) = \sum_{n\leqslant x} n^{-1}g(n)$, $H(x) = \sum_{n\leqslant x} n^{-1}h(n)$, $x \geqslant 1$.

Then

$$H(x) = \left(\prod_{p\leqslant x} \left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots\right)\left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\right)^{-1} + o(1) \right) G(x)$$

as $x \to \infty$.

Remark. If the series $\sum p^{-1}(g(p) - \operatorname{Re} h(p))$ diverges or a sum $\sum_{k=1}^{\infty} p^{-k}h(p^k)$ has the value -1, then the product over the primes may be omitted. Otherwise, the product has the form $AL(\log x)$, where A is a non-zero constant and $L(y)$ a non-vanishing slowly oscillating function of y.

Theorem 2. *Let g be a non-negative multiplicative function that is uniformly bounded on the primes. Assume that for a positive c, and each b, $0 < b < 1$,*

$$\liminf_{x\to\infty} ((1-b)\log x)^{-1} \sum_{x^b < p \leqslant x} p^{-1} g(p) \log p \geqslant c.$$

Then for some positive c_0 and all $x \geqslant 2$,

$$\sum_{n\leqslant x} g(n) \geqslant \frac{c_0 x}{\log x} \prod_{p\leqslant x} \left(1 + \frac{g(p)}{p}\right).$$

Remark. Under the further assumptions on g in Theorem 1, there is a similar upper bound.

For each positive real τ, $\Delta(\tau)$ will denote a compact star-shaped region of the complex plane that contains the origin, has a representation

$$\{\rho e^{i\theta}, 0 \leqslant \theta < 2\pi, 0 \leqslant \rho \leqslant w(\theta)\},$$

with average radius

$$(2\pi)^{-1} \int_0^{2\pi} w(\theta)\, d\theta, \qquad w(2\pi) = w(0),$$

strictly less than τ.

Theorem 3. *Let the multiplicative function g satisfy the hypotheses of Theorems 1 and 2 and let h be a complex-valued multiplicative function with $|h(n)| \leqslant g(n)$ and values in $\Delta(c)$.*

Set

$$A(x) = \sum_{n\leqslant x} g(n), \qquad B(x) = \sum_{n\leqslant x} h(n).$$

Then

$$B(x) = \left(\prod_{p\leqslant x} \left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots\right)\left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\right)^{-1} + o(1) \right) A(x)$$

as $x \to \infty$.

Theorem 4. *Let the multiplicative function g satisfy the hypotheses of Theorems 1 and 2 and let h be a complex-valued multiplicative function with $|h(n)| \leqslant g(n)$.*

Then there are two possibilities.

(i) *For some real t the series $\sum p^{-1}(g(p) - \operatorname{Re} h(p)p^{it})$, taken over the primes, converges;*

$$B(x) = (1-it)^{-1}x^{-it} \prod_{p \leqslant x} \big(1 + h(p)p^{it-1} + \cdots\big)\big(1 + g(p)p^{-1} + \cdots\big)^{-1} A(x) + o(A(x)), \qquad x \to \infty.$$

(ii) *There is no such t, and*

$$B(x) = o(A(x)), \qquad x \to \infty.$$

Of particular interest in Theorems 1 and 4 is that beyond dominance by g, there is no non-structural constraint upon the complex values of the function h.

2. Background

Two central theorems of Wirsing's 1967 paper run as follows.

Satz 1.1. *Let $\lambda(n)$ be a non-negative multiplicative function, uniformly bounded on the primes, that for a positive τ satisfies*

$$\sum_{p \leqslant x} p^{-1} \log p \, \lambda(p) \sim \tau \log x, \qquad x \to \infty.$$

Assume further that the series $\sum q^{-1}\lambda(q)$, taken over the prime-powers $q = p^k$ with $k \geqslant 2$, converges, and that if $\tau \leqslant 1$ then $\sum_{q \leqslant x} \lambda(q) \ll x(\log x)^{-1}$ holds for $x \geqslant 2$.

Then

$$\sum_{n \leqslant x} \lambda(n) \sim \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p \leqslant x} \left(1 + \frac{\lambda(p)}{p} + \frac{\lambda(p^2)}{p^2} + \cdots\right), \qquad x \to \infty,$$

where γ is Euler's constant.

Satz 1.2. *Let $\lambda(n)$ be a multiplicative function that satisfies the conditions of Satz 1.1. Let $\lambda^*(n)$ be multiplicative, with values in $\Delta(\tau)$ and satisfy $|\lambda^*(n)| \leqslant \lambda(n)$. Then*

$$\sum_{n \leqslant x} \lambda^*(n) = \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p \leqslant x} \left(1 + \frac{\lambda^*(p)}{p} + \frac{\lambda^*(p^2)}{p^2} + \cdots\right) + o\left(\sum_{n \leqslant x} \lambda(n)\right)$$

as $x \to \infty$.

In what follows, a product of the form

$$\prod_{p\leqslant x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right),$$

when meaningful, may be denoted by $\prod_x(f)$.

The two theorems of Wirsing may be compared to the following result of Elliott and Kish [7], subsuming ideas from Wirsing and Halász, loc, cit.

Theorem 5. *Let $3/2 \leqslant Y \leqslant x$. Let g be a complex-valued multiplicative function that for positive constants β, c, c_1 satisfies $|g(p)| \leqslant \beta$,*

$$\sum_{w<p\leqslant x} p^{-1}(|g(p)|-c) \geqslant -c_1, \qquad Y \leqslant w \leqslant x,$$

on the primes. Suppose, further, that the series

$$\sum_q q^{-1}|g(q)|(\log q)^\kappa, \qquad \kappa = 1 + c\beta(c+\beta)^{-1},$$

taken over the prime-powers $q = p^k$ with $k \geqslant 2$, converges.

Then with

$$\lambda = \min_{|t|\leqslant T} \sum_{Y<p\leqslant x} p^{-1}(|g(p)| - \operatorname{Re} g(p)p^{it}),$$

$$\sum_{n\leqslant x} g(n) \ll x(\log x)^{-1} \prod_{p\leqslant x}(1+p^{-1}|g(p)|)\left(\exp(-\lambda c(c+\beta)^{-1}) + T^{-1/2}\right)$$

uniformly for $Y, x, T > 0$, the implied constant depending at most upon β, c, c_1 and a bound for the sum of the series over higher prime-powers.

An extension of Theorem 5, a proof of which will be given following that for Theorem 4, obviates the awkward condition involving the factor $(\log q)^\kappa$.

Theorem 6. *If the estimate in Theorem 5 is weakened to*

$$\sum_{n\leqslant x} g(n) \ll x(\log x)^{-1} \prod\nolimits_x(|g|)\left(\exp(-\lambda c(c+\beta)^{-1}) + T^{-1/2}\right)^{c/(3c+1)},$$

then the condition on the prime-power values $g(p^k)$, $k \geqslant 2$, may be relaxed to the convergence of the series $\sum_{p,k\geqslant 2} p^{-k}|g(p^k)|$ and a uniform bound for the sums $y^{-1}\sum_{p^k\leqslant y}|g(p^k)|\log p^k$, $y \geqslant 2$.

For the multiplicative function $\lambda_0(n)$ defined to be α, β with $0<\alpha<\beta$, on the primes in alternate intervals $(\exp(2^k), \exp(2^{k+1})]$, $k=-1,0,1,2,\ldots$, and to be zero on all other prime-powers,

$$\lim_{x\to\infty} (\log x)^{-1} \sum_{p\leqslant x} p^{-1}\lambda_0(p)\log p$$

does not exist, eliminating direct application of Sätze 1.1 and 1.2.

The lower bound of Theorem 2 is obtained in Elliott and Kish [7], Lemma 21, subject to the existence of a positive constant c_2 so that for all large x, $\sum_{p\leqslant x} g(p)\log p \geqslant c_2 x$. By modifying λ_0 to be zero on intervals $(y(\log y)^{-2}, y]$, $y = \exp(2^k)$, we obtain a multiplicative function λ_1 that will not satisfy such a criterion for any positive c_2.

Never-the-less, Theorems 1, 2 and 4 may be applied to λ_0, λ_1 with any dominated complex-valued multiplicative function, h.

3. Proof of Theorem 1

It is convenient to introduce several preliminary results.

Lemma 1. *The estimate*

$$\sum_{2\leqslant n\leqslant x} g(n) \leqslant \left(\frac{x}{\log x} + \frac{10x}{(\log x)^2}\right)\widetilde{\Delta}\sum_{n\leqslant x}\frac{g(n)}{n}$$

with

$$\widetilde{\Delta} = \sup_{1\leqslant y\leqslant x} y^{-1}\sum_{q\leqslant y} g(q)\log q,$$

where q denotes a prime-power, holds uniformly for all non-negative real multiplicative functions g, and all $x \geqslant 2$.

A proof of Lemma 1 may be found in Elliott [5], Chapter 2, Lemma 2.2. It is immediate that

$$\sum_{n\leqslant x} n^{-1}g(n) \leqslant \prod_{p\leqslant x}\left(1 + \sum_{k\leqslant \log x/\log p} p^{-k}g(p^k)\right)$$
$$\leqslant \exp\left(\sum_{q\leqslant x} q^{-1}g(q)\right).$$

A proof of the following qualitative corresponding lower bound, a result first obtained by Barban [1] using a different method, may be found in Lemma 20 of Elliott and Kish, [7].

Lemma 2. *To each positive β there is a further positive $c(\beta)$ so that a non-trivial non-negative multiplicative function, g, that satisfies $g(p) \leqslant \beta$ on the primes, also satisfies*

$$\sum_{n\leqslant x} g(n)n^{-1} \geqslant c(\beta)\prod_{p\leqslant x}(1 + p^{-1}g(p))$$

uniformly for $x \geqslant 1$.

Lemma 3. *Let g be a non-trivial non-negative multiplicative function uniformly bounded on the primes, for which the series $\sum q^{-1}g(q)$, taken over the prime-powers $q = p^k$ with $k \geqslant 2$, converges, and for which the sums $y^{-1}\sum_{q\leqslant y} g(q)\log q$, $y \geqslant 2$, are uniformly bounded.*

Then

$$\sum_{u<n\leqslant v} \frac{g(n)}{n} \ll \left(\log\left(\frac{\log v}{\log u}\right) + \frac{1}{\log x}\right) \sum_{n\leqslant x} \frac{g(n)}{n}$$

uniformly for $x^{1/2} \leqslant u \leqslant v \leqslant x^{3/2}$, $x \geqslant 2$.

Proof of Lemma 3. In view of the hypothesis on g, Lemma 1 delivers the uniform estimate

$$\sum_{n\leqslant y} g(n) \ll \frac{y}{\log y} \prod_{p\leqslant x^{3/2}} \left(1 + \frac{g(p)}{p}\right), \qquad 2 \leqslant y \leqslant x^{3/2},$$

which Lemma 2 shows to be $\ll y(\log y)^{-1}G(x)$. The asserted result then follows from an integration by parts. ■

For better appreciation the following theorem is given in both its abelian and tauberian aspects. A proof may be found, together with a history of the result from Feller [8] to Stadtmüller and Trautner [12], in Bingham, Goldie and Teugels [2], Chapter 2, Theorem 2.10.1, pp. 116–118, and Korevaar [11], Chapter IV, Theorem 10.1, pp. 197–199.

Let $C(y)$, $D(y)$ be non-negative real-valued functions on the non-negative reals, non-decreasing and right continuous. To each corresponds a Laplace transform, typically

$$s \to \widehat{C}(s) = \int_0^\infty e^{-sy}\, dC(y),$$

here assumed to be defined for $s > 0$.

Lemma 4. *Assume that for each $y > 1$*

$$D^*(y) = \limsup_{u\to\infty} D(u)^{-1}D(uy)$$

is finite, D implicitly assumed not to be identically zero.

If, for some constant A and slowly-oscillating function $L(y)$,

$$C(y) = (AL(y) + o(1))D(y), \qquad y \to \infty,$$

then

$$\widehat{C}(s) = (AL(s^{-1}) + o(1))\widehat{D}(s), \qquad s \to 0+\,.$$

Further, if $D^*(y) \to 1$ as $y \to 1+$, then the converse is valid.

Remark. The non-decreasing nature of D ensures that $\lim D^*(y)$, $y \to 1$, exists.

Completion of the proof of Theorem 1. We apply Lemma 4 to the pair $2G(e^x) + \mathrm{Re}(H(e^x))$, $G(e^x)$; to the pair with $\mathrm{Im}(H(e^x))$ in place of $\mathrm{Re}(H(e^x))$; and to the pair $G(e^x)$, $G(e^x)$.

Computation with Euler products shows $\widehat{C}(s)$, $\widehat{D}(s)$, the Laplace transforms of the first pair, to exist for all positive s and satisfy $\widehat{C}(s) = f(s)\widehat{D}(s)$, where

$$f(s) - 2 = \mathrm{Re}\left(\prod_p \left(1 + \sum_{k=1}^{\infty} p^{-k(1+s)} h(p^k)\right)\left(1 + \sum_{m=1}^{\infty} p^{-m(1+s)} g(p^m)\right)^{-1}\right).$$

In particular,

$$|f(s) - 2| \ll \exp\left(-\sum_p p^{-1-s}(g(p) - \mathrm{Re}\, h(p))\right),$$

so that if the series in the exponent diverges for $s = 0$, then $f(s) \to 2$ as $s \to 0+$, and we may apply Lemma 4 with $A = 2$, L identically 1.

We may therefore assume the series $\sum p^{-1}(g(p) - \mathrm{Re}\, h(p))$ to converge.

From the Chebyshev bound $\pi(y) \ll y(\log y)^{-1}$, integration by parts shows the series $\sum_{p>x^\varepsilon} p^{-1} \exp(-\log p/\log x)$ to be bounded in terms of ε alone. Since

$$|g(p) - h(p)|^2 \leqslant 2g(p)(g(p) - \mathrm{Re}\, h(p)),$$

an application of the Cauchy-Schwarz inequality, confined to the primes on which g does not vanish, shows that

$$\sum_{p>x^\varepsilon} p^{-1-1/\log x}|g(p) - h(p)| \ll \left(\sum_{p>x^\varepsilon} g(p)p^{-1-1/\log x}\right)^{1/2} \times \left(\sum_{p>x^\varepsilon} p^{-1}(g(p) - \mathrm{Re}\, h(p))\right)^{1/2}$$

and $o(1)$ as $x \to \infty$.

Moreover,

$$\sum_{p\leqslant x^\varepsilon} (p^{-1} - p^{-1-1/\log x}) \ll \sum_{p\leqslant x^\varepsilon} p^{-1} \log p/\log x \ll \varepsilon,$$

the implied constant absolute for all values of x sufficiently large in terms of ε.

Letting $x \to \infty$, $\varepsilon \to 0+$, we see that as $x \to \infty$

$$f\left((\log x)^{-1}\right) - 2 = \mathrm{Re}\left(B \exp\left(\sum_{p\leqslant x} p^{-1}\, \mathrm{Im}(h(p))\right)\right) + o(1),$$

with B the product of

$$\prod_p \left(1+\sum_{k=1}^{\infty} p^{-k}h(p^k)\right) \exp\left(-p^{-1}h(p)\right) \prod_p \left(1+\sum_{m=1}^{\infty} p^{-m}g(p^m)\right)^{-1} \exp\left(p^{-1}g(p)\right)$$

and $\exp(-\sum_p p^{-1}(g(p)-\operatorname{Re} h(p)))$. Its genesis in terms of Euler products ensures that $|B| \leqslant 1$; moreover, B will vanish only if for some prime p the sum $1+\sum_{k=1}^{\infty} p^{-k}h(p^k)$ vanishes.

Note that for any $\beta \geqslant 1$, the above argument shows that

$$\begin{aligned}\sum_{x<p\leqslant x^\beta} p^{-1}\operatorname{Im}(h(p)) &= -\sum_{x<p\leqslant x^\beta} p^{-1}\operatorname{Im}(g(p)-h(p)) \\ &\ll \left(\sum_{x<p\leqslant x^\beta} p^{-1}\right)^{1/2} \left(\sum_{p>x} p^{-1}|g(p)-h(p)|^2\right)^{1/2} = o(1)\end{aligned}$$

as $x\to\infty$, so that $\exp(\sum_{p\leqslant e^s} p^{-1}\operatorname{Im}(h(p)))$ is a slowly oscillating function of s.

In view of Lemma 3,

$$\lim_{y\to 1+} \limsup_{u\to\infty} G(e^x)^{-1}G(e^{xy}) = 1.$$

Three applications of Lemma 4 in its Tauberian aspect, typically with $A=1$,

$$L(s) = 2+\operatorname{Re}\left(B\exp\left(\sum_{p\leqslant e^s} p^{-1}\operatorname{Im}(h(p))\right)\right),$$

delivers the asymptotic estimate

$$H(e^x) = \left(f(x^{-1})+o(1)\right)G(e^x), \qquad x\to\infty,$$

from which Theorem 1 follows rapidly. ■

4. Proof of Theorem 2

Again a preliminary result is advantageous.

Let $0\leqslant g(p)\leqslant \beta$ for each prime, p.

If, for some $\tau>0$,

$$\sum_{p\leqslant y} p^{-1}g(p)\log p \sim \tau\log y, \qquad y\to\infty,$$

then for each ε, $0<\varepsilon<1$,

$$\liminf_{x\to\infty} (\varepsilon\log x)^{-1} \sum_{x^{1-\varepsilon}<p\leqslant x} p^{-1}\log p \geqslant \tau.$$

The converse need not be true, as may be seen from the example λ_0 in Section 2. However, the following converse is valid.

Lemma 5. *Assume that for $c > 0$ and each ε, $0 < \varepsilon < 1$, the function $g(p)$, uniformly bounded on the primes, satisfies*

$$\liminf_{x\to\infty}(\varepsilon \log x)^{-1} \sum_{x^{1-\varepsilon}<p\leqslant x} p^{-1}g(p)\log p \geqslant c.$$

Then for each α, $0 < \alpha < c$, there is a subsequence of primes, r, such that

$$\lim_{x\to\infty}(\log x)^{-1}\sum_{r\leqslant x} r^{-1}g(r)\log r = \alpha.$$

Proof of Lemma 5. We begin with an outline of the argument. Fix a prime t for which $\sum_{p\leqslant t} p^{-1}g(p)\log p \geqslant \alpha \log t$.

We define a function $\overline{g}(p)$ by choosing, for each prime p, to retain $g(p)$ or to replace it by zero. For ease of notation $\sum_{p\leqslant y} p^{-1}\overline{g}(p)\log p$ will be denoted by $S(y)$.

We choose $\overline{g}(p) = g(p)$ for $p \leqslant t$.

The primes $y_1 < y_2 < \cdots$ are defined successively as follows. We replace $g(p)$ by zero on the primes following t until, for the first time, $S(y)/\log y$ falls strictly below α. The corresponding value of y is y_1.

We choose $\overline{g}(p) = g(p)$ on the primes $p > y_1$ until, for the first time with $y > y_1$, the ratio $S(y)/\log y$ climbs above α. The corresponding value of y is y_2; and so on.

Our initial aim is to show the turning values y_j not to be logarithmically far apart.

A few preliminary remarks are helpful.

Let $0 < \theta < 1$, $x \geqslant 2$, $3/2 \leqslant y \leqslant x^\theta$. With $0 < \varepsilon < 1-\theta$ determine the integer k by $x^{(1-\varepsilon)^k} < y \leqslant x^{(1-\varepsilon)^{k-1}} = \psi$, so that $k \geqslant 2$. Assume that for all sufficiently large values of w

$$\sum_{w^{1-\varepsilon}<p\leqslant w} p^{-1}g(p)\log p \geqslant \varepsilon c \log w.$$

By partitioning the interval $(x^{(1-\varepsilon)^k}, x]$ into adjoining subintervals $(x^{(1-\varepsilon)^m}, x^{(1-\varepsilon)^{m-1}}]$, $m = 1, 2, \ldots, k$, we see that provided $x^{(1-\varepsilon)^k}$ is sufficiently large in terms of ε,

$$\sum_{y<p\leqslant x} p^{-1}g(p)\log p \geqslant c\log(x/\psi) \geqslant c(\log(x/y) - \log(\psi/y))$$
$$\geqslant c\left(1-\varepsilon(1-\theta)^{-1}\right)\log(x/y),$$

since $\log(\psi/y) \leqslant \log(\psi/\psi^{1-\varepsilon}) = \varepsilon\log\psi \leqslant \varepsilon\log x \leqslant \varepsilon(1-\theta)^{-1}\log(x/y)$.

For the purposes of proving Lemma 5 we may therefore replace its lower-bound hypothesis by:

For each ε, $0 < \varepsilon < 1$,

$$\sum_{y<p\leqslant x} p^{-1}g(p)\log p \geqslant c\log(x/y)$$

uniformly for $1 \leqslant y \leqslant x^{1-\varepsilon}$ and all x sufficiently large in terms of ε.

It is clear that the initial prime t exists.

As a second preliminary remark, if $2 \leqslant y \leqslant w$, then

$$(\log w)^{-1} S(w) - (\log y)^{-1} S(y) = \left((\log w)^{-1} - (\log y)^{-1}\right) S(y) + (\log w)^{-1} \sum_{y<p\leqslant w} p^{-1}\overline{g}(p) \log p.$$

Hence

$$\begin{aligned} \left|(\log w)^{-1} S(w) - (\log y)^{-1} S(y)\right| &\leqslant (\log w \log y)^{-1} S(y) \log(w/y) \\ &\quad + c_0 (\log w)^{-1} \sum_{y<p\leqslant w} p^{-1} \log p \\ &\leqslant c_1 (\log(w/y) + 1)(\log w)^{-1} \end{aligned}$$

with a positive constant c_1 dependent at most upon the upper bound for the $g(p)$. Here we have employed the elementary estimate $\sum_{p\leqslant y} p^{-1} \log p = \log y + O(1)$, $y \geqslant 2$.

In particular, if y is a prime adjacent to a turning value y_k, then

$$S(y)/\log y - S(y_k)/\log y_k \ll (|\log(y_k/y)| + 1)/\log y_k \ll 1/\log y_k,$$

since the ratio of successive increasing primes approaches 1.

We now show the y_j not to increase too rapidly.

Suppose that $S(y_k)/\log y_k < \alpha$, so that for the next prime $p > y_k$, $g(p)$ is kept. In particular $S(y_k) \geqslant \alpha \log y_k + O(1)$. If $y_k < (\frac{1}{2}y_{k+1})^{1-\varepsilon} < \frac{1}{2}y_{k+1}$ and y_k is sufficiently large, then $\frac{1}{2}y_{k+1}y_k^{-1} > y_k^{\varepsilon}$,

$$\begin{aligned} S(\tfrac{1}{2}y_{k+1}) &= S(\tfrac{1}{2}y_{k+1}) - S(y_k) + S(y_k) \\ &\geqslant c \log(\tfrac{1}{2}y_{k+1}y_k^{-1}) + \alpha \log y_k + O(1) \\ &= \alpha \log(\tfrac{1}{2}y_{k+1}) + (c-\alpha)\log(\tfrac{1}{2}y_{k+1}y_k^{-1}) + O(1). \end{aligned}$$

With w a nearest prime to $\frac{1}{2}y_{k+1}$, $S(w)/\log w > \alpha$ before the next change point, y_{k+1}.

Thus $y_k \geqslant (\frac{1}{2}y_{k+1})^{1-\varepsilon}$.

If $S(y_k) \geqslant \alpha \log y_k$, then again $S(y_k) = \alpha \log y_k + O(1)$, and $\overline{g}(p) = 0$ on the primes in the interval $(y_k, \frac{1}{2}y_{k+1}]$. Hence

$$\begin{aligned} S(\tfrac{1}{2}y_{k+1})(\log(\tfrac{1}{2}y_{k+1}))^{-1} &= S(y_k)(\log(\tfrac{1}{2}y_{k+1}))^{-1} \\ &= \alpha \log y_k (\log y_{k+1})^{-1} + O((\log y_k)^{-1}). \end{aligned}$$

If, now, $y_k < y_{k+1}^{1-\varepsilon}$ and y_k is sufficiently large then

$$S(\tfrac{1}{2}y_{k+1})(\log(\tfrac{1}{2}y_{k+1}))^{-1} \leqslant \alpha(1-\varepsilon) + O((\log y_k)^{-1}),$$

again leading to a premature change point.

In this case $y_k \geqslant y_{k+1}^{1-\varepsilon}$.

For all large values of y_k, $\frac{1}{2}y_{k+1}^{1-\varepsilon} \leqslant y_k \leqslant y_{k+1}$. As a consequence

$$S(y_{k+1})/\log y_{k+1} - S(y_k)/\log y_k \ll \log(y_{k+1}/y_k)/\log y_{k+1} \ll \varepsilon,$$

the implied constant independent of ε. Since $S(y_k)/\log y_k = \alpha + O(1/\log y_k)$, $S(y)/\log y - \alpha \ll \varepsilon$ for all sufficiently large values of y, first for prime values then for otherwise arbitrary real values.

The construction of the function $\overline{g}$ does not depend upon the value of ε and we may apply the argument with $\varepsilon = 2^{-m}$, $m = 1, 2, 3, \ldots$, in turn.

Lemma 5 is established. ■

Completion of the proof of Theorem 2. Let $0 < \alpha < c$ and let r run through a sequence of primes for which $\sum_{r \leqslant y} r^{-1}g(r)\log r \sim \alpha \log y$, $y \to \infty$.

Define multiplicative functions g_j, $j = 1, 2$, by

$$g_1(p) = \begin{cases} g(p) & \text{if } p \neq r, \\ 0 & \text{if } p = r, \end{cases} \qquad g_2(p) = \begin{cases} 0 & \text{if } p \neq r, \\ g(p) & \text{if } p = r, \end{cases}$$

and $g_j(p^k) = 0$ on all other prime powers.

On squarefree integers g coincides with $g_1 * g_2$, the Dirichlet convolution of g_1 and g_2; hence

$$\sum_{n \leqslant x} g(n) \geqslant \sum_{u \leqslant \sqrt{x}} g_1(u) \sum_{v \leqslant x/u} g_2(v).$$

Satz 1.1 of Wirsing (c.f. §2) gives for a typical innersum the asymptotic estimate

$$(1 + o(1))\frac{e^{-\gamma\alpha}}{\Gamma(\alpha)}\frac{x}{u\log(x/u)} \prod_{p \leqslant x/u}\left(1 + \frac{g_2(p)}{p}\right), \quad x/u \to \infty.$$

The doublesum thus exceeds a constant multiple of

$$\frac{x}{\log x}\prod_{p \leqslant x}\left(1 + \frac{g_2(p)}{p}\right)\sum_{u \leqslant \sqrt{x}}\frac{g_1(u)}{u}.$$

An appeal to Lemma 2 completes the proof. ■

5. Proof of Theorem 3

Choose a real α to lie strictly between the average radius of $\Delta(c)$, and c.

Choose a subsequence of primes r for which

$$\sum_{r \leqslant y} r^{-1}g(r)\log r \sim \alpha \log y, \qquad y \to \infty.$$

We define multiplicative functions g_j, $j = 1, 2$, by

$$g_1(p^k) = \begin{cases} g(p^k) & p \neq r, \\ 0 & \text{otherwise} \end{cases}, \qquad g_2(p^k) = \begin{cases} g(p^k) & p = r, \\ 0 & \text{otherwise.} \end{cases}$$

The function g has a Dirichlet convolution representation $g_1 * g_2$.

We likewise define multiplicative functions h_j, $j = 1, 2$, so that $h = h_1 * h_2$, $|h_j| \leqslant g_j$, $j = 1, 2$. There is a representation

$$M = \sum_{n \leqslant x} h(n) = \sum_{u \leqslant x} h_1(u) \sum_{v \leqslant x/u} h_2(v).$$

Let $0 < \varepsilon < 1/2$. We remove the contribution from the terms with $u \leqslant x^\varepsilon$ and $x^{1-\varepsilon} < u \leqslant x$. Typically, by Lemma 1,

$$\sum_{u \leqslant x^\varepsilon} g_1(u) \sum_{v \leqslant x/u} g_2(v) \ll \sum_{u \leqslant x^\varepsilon} g_1(u) \frac{x}{u \log(x/u)} \prod_{p \leqslant x/u} \left(1 + \frac{g_2(p)}{p} + \cdots\right)$$
$$\ll \frac{x}{\log x} \prod\nolimits_x (g_2) \sum_{u \leqslant x^\varepsilon} \frac{g_1(u)}{u}.$$

Moreover,

$$\sum_{u \leqslant x^\varepsilon} \frac{g_1(u)}{u} \ll \prod_{p \leqslant x^\varepsilon} \left(1 + \frac{g_1(p)}{p}\right) \ll \prod\nolimits_x (g_1) \prod_{x^\varepsilon < p \leqslant x} \left(1 + \frac{g_1(p)}{p}\right)^{-1}.$$

From the lower bound hypothesis on g and the construction of the sequence r, an integration by parts shows that

$$\sum_{x^\varepsilon < p \leqslant x} \frac{1}{p} g_1(p) \geqslant \frac{1}{2}(c - \alpha) \log \frac{1}{\varepsilon} + O(1).$$

The contribution to M from the terms with $u \leqslant x^\varepsilon$ is

$$\ll \varepsilon^{(c-\alpha)/2} x (\log x)^{-1} \prod\nolimits_x (g), \qquad x \to \infty.$$

For the range $x^{1-\varepsilon} < u \leqslant x$, $v \leqslant x^\varepsilon$ and we may invert summations, replacing $(c - \alpha)/2$, as the exponent of ε, by $\alpha/2$.

We are reduced to the estimation of

$$M_\varepsilon = \sum_{x^\varepsilon < u \leqslant x^{1-\varepsilon}} h_1(u) \sum_{v \leqslant x/u} h_2(v).$$

Since h_2 inherits its properties relative to g_2 from h, applied to the innersum in M_ε, Satz 1.2 delivers the asymptotic estimate

$$\frac{e^{-\gamma\alpha}}{\Gamma(\alpha)} \frac{x/u}{\log(x/u)} \left(\prod\nolimits_{x/u} (h_2) + o\left(\prod\nolimits_{x/u} (g_2) \right) \right), \qquad x \to \infty,$$

uniformly for $x^\varepsilon \leqslant u \leqslant x^{1-\varepsilon}$.

Introducing factors $\exp(-p^{-1}h_2(p))$, $\exp(-p^{-1}g_2(p))$, respectively, the ratio $\prod_y(h_2)\prod_y(g_2)^{-1}$ has an estimate

$$(B+o(1))\exp\left(-\sum_{p\leqslant y}p^{-1}(g_2(p)-h_2(p))\right), \qquad y\to\infty,$$

with

$$B=\prod_p\left(\sum_{k=0}^{\infty}p^{-k}h_2(p^k)\exp(-p^{-1}h_2(p))\right)\prod_p\left(\sum_{m=0}^{\infty}p^{-m}g_2(p^m)\right)^{-1}\exp(p^{-1}g_2(p)).$$

If the series $\sum p^{-1}(g_2(p)-\mathrm{Re}(h_2(p)))$ diverges, then uniformly for $x^{\varepsilon}\leqslant u\leqslant x^{1-\varepsilon}$,

$$\prod\nolimits_{x/u}(h_2)\prod\nolimits_{x/u}(g_2)^{-1}=\prod\nolimits_x(h_2)\prod\nolimits_x(g_2)^{-1}+o(1), \qquad x\to\infty,$$

since both product ratios asymptotically vanish.

If the series $\sum p^{-1}(g(p)-\mathrm{Re}(h_2(p)))$ converges, then we may argue as in the proof of Theorem 1. For each positive real τ, $0<\tau\leqslant 1$,

$$\sum_{x^\tau<p\leqslant x}p^{-1}(g_2(p)-h_2(p))\to 0, \qquad x\to\infty,$$

and we formally obtain the same asymptotic equality of ratios.

Likewise, there is a representation

$$(\log y)^{-\alpha}\prod\nolimits_y(g_2)=(C+o(1))\exp\left(\sum_{p\leqslant y}p^{-1}g_2(p)-\alpha\log\log y\right), \qquad y\to\infty,$$

with

$$C=\prod_p\left(\sum_{m=1}^{\infty}p^{-m}g_2(p^m)\right)\exp(-p^{-1}g_2(p)).$$

An integration by parts shows that for each τ, $0<\tau<1$,

$$\sum_{x^\tau<p\leqslant x}p^{-1}g_2(p)+\alpha\log\tau\to 0, \qquad x\to\infty,$$

so that

$$(\log(x/u))^{-\alpha}\prod\nolimits_{x/u}(g_2)=(\log x)^{-\alpha}\prod\nolimits_x(g_2)+o(1), \qquad x\to\infty.$$

Altogether, the innersum of M_ε has the estimate

$$\frac{e^{-\gamma\alpha}}{\Gamma(\alpha)}\frac{x}{u(\log x)^{\alpha}}\cdot\frac{1}{(\log(x/u))^{1-\alpha}}\left(\prod\nolimits_x(h_2)+o\left(\prod\nolimits_x(g_2)\right)\right), \qquad x\to\infty,$$

uniformly for $x^{\varepsilon}\leqslant u\leqslant x^{1-\varepsilon}$.

The error terms contribute towards M_ε

$$o\left(\frac{x}{\log x}\prod\nolimits_x(g)\sum_{x^\varepsilon<u\leqslant x^{1-\varepsilon}}\frac{g_1(u)}{u}\right)=o\left(\frac{x}{\log x}\prod\nolimits_x(g)\right),\qquad x\to\infty,$$

within which M_ε has the estimate

$$\frac{e^{-\gamma}}{\Gamma(\alpha)}\frac{x}{(\log x)^\alpha}\prod\nolimits_x(h_2)\sum_{x^\varepsilon<u\leqslant x^{1-\varepsilon}}\frac{h_1(u)}{u(\log(x/u))^{1-\alpha}}.$$

Setting

$$H_1(y)=\sum_{n\leqslant y}h_1(n)n^{-1},\qquad G_1(y)=\sum_{n\leqslant y}g_1(n)n^{-1},$$

an integration by parts gives a representation

$$\sum_{x^\varepsilon<u\leqslant x^{1-\varepsilon}}\frac{h_1(u)}{u(\log(x/u))^{1-\alpha}}=\frac{H_1(x^{1-\varepsilon})}{(\varepsilon\log x)^{1-\alpha}}-\frac{H_1(x^\varepsilon)}{((1-\varepsilon)\log x)^{1-\alpha}}$$
$$-(1-\alpha)\int_{x^\varepsilon}^{x^{1-\varepsilon}}\frac{H_1(y)}{y(\log(x/u))^{2-\alpha}}\,dy,$$

provided x^ε, $x^{1-\varepsilon}$ are not positive integers, a situation that we may avoid by choosing a slightly larger value of x.

According to Theorem 1,

$$H_1(y)=\left(\prod\nolimits_y(h_1)\prod\nolimits_y(g_1)^{-1}+o(1)\right)G_1(y),\qquad y\to\infty,$$

where, as above, we may replace the products $\prod_y$ by $\prod_x$, uniformly for $x^\varepsilon\leqslant y\leqslant x^{1-\varepsilon}$, $x\to\infty$.

As a consequence,

$$\sum_{x^\varepsilon<u\leqslant x^{1-\varepsilon}}\frac{h_1(u)}{u(\log(x/u))^{1-\alpha}}=\left(\prod\nolimits_x(h_1)\prod\nolimits_x(g_1)^{-1}+o(1)\right)$$
$$\times\sum_{x^\varepsilon<u\leqslant x^{1-\varepsilon}}\frac{g_1(u)}{u(\log(x/u))^{1-\alpha}},\qquad x\to\infty.$$

Once again, the argument is expedited by considering $2G_1(x)+\mathrm{Re}(H_1(x))$, $2G_1(x)+\mathrm{Im}(H_1(x))$.

Rewinding,

$$\begin{aligned} M_\varepsilon &= \prod_x(h_1)\prod_x(g_1)^{-1}\frac{e^{-\gamma\alpha}}{\Gamma(\alpha)}\frac{x}{(\log x)^\alpha}\prod_x(h_2)\sum_{x^\varepsilon<u\leqslant x^{1-\varepsilon}}\frac{g_1(u)}{u(\log(x/u))^{1-\alpha}} \\ &\quad + o\left(\frac{x}{\log x}\prod_x(g)\right) \\ &= \prod_x(h)\prod_x(g)^{-1}\sum_{x^\varepsilon<u\leqslant x^{1-\varepsilon}}\frac{e^{-\gamma\alpha}}{\Gamma(\alpha)}\frac{x}{\log(x/u)}\prod_{x/u}(g_2)\frac{g_1(u)}{u} \\ &\quad + o\left(\frac{x}{\log x}\prod_x(g)\right) \\ &= \prod_x(h)\prod_x(g)^{-1}\sum_{x^\varepsilon<u\leqslant x^{1-\varepsilon}}g_1(u)\sum_{v\leqslant x/u}g_2(v) + o\left(\frac{x}{\log x}\prod_x(g)\right) \\ &= \prod_x(h)\prod_x(g)^{-1}\sum_{n\leqslant x}g(n) + O\left(\varepsilon^\nu\sum_{n\leqslant x}g(n)\right) \end{aligned}$$

with $\nu = \min((c-\alpha)/2, \alpha/2)$ and, for all sufficiently large values of x, an implied constant independent of ε.

A similar estimate holds for M.

Letting $x\to\infty$, $\varepsilon\to 0+$ completes the proof. ■

6. Proof of Theorem 4

Case (i). From the assumption that the series $\sum p^{-1}(g(p)-\mathrm{Re}(h(p)p^{it}))$ converges, for each positive δ the series taken over the primes p for which $g(p)-\mathrm{Re}(h(p)p^{it}) > \delta$ also converges.

On the remaining primes

$$|g(p)-h(p)p^{it}|^2 \leqslant 2g(p)(g(p)-\mathrm{Re}(h(p)p^{it})) \leqslant 2\beta\delta.$$

The values of $h(p)p^{it}$ lie in a box about the real axis, with corners at $(-(2\beta\delta)^{1/2}, \pm(2\beta\delta)^{1/2})$, $(\beta+(2\beta\delta)^{1/2}, \pm(2\beta\delta)^{1/2})$, and area $2(2\beta\delta)^{1/2}(\beta+2(2\beta\delta)^{1/2})$.

Assuming that δ is sufficiently small and, in particular, that $2(2\beta\delta)^{1/2}\leqslant\beta$, this is a region of the type $\Delta(\tau)$ with an average radius

$$\frac{1}{2\pi}\int_0^{2\pi} w(\theta)\,d\theta \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} w(\theta)^2\,d\theta\right)^{1/2} \leqslant \left(4\pi^{-1}(2\beta^3\delta)^{1/2}\right)^{1/2}$$

that can be fixed at a value as small as desired.

We may follow the proof of Theorem 3, first selecting a subsequence of primes r for which $(\log x)^{-1}\sum_{r\leqslant x} r^{-1}g(r)\log r\to\alpha$, $x\to\infty$, then removing from that subsequence those primes for which $h(p)p^{it}$ does not belong to a region $\Delta(\alpha)$ defined by a value of δ that satisfies $4\pi^{-1}(2\beta^3\delta)^{1/2} < \alpha^2$.

The removal of these exceptional primes does not affect the existence or the value of the asymptotic limit for $(\log x)^{-1}\sum_{r\leqslant x} r^{-1}g(r)\log r$.

The upshot is an asymptotic estimate

$$\sum_{n\leqslant x} h(n)n^{it} = \prod_{p\leqslant x}\left(1+h(p)p^{it-1}+\cdots\right)\left(\prod\nolimits_x(g)\right)^{-1}\sum_{n\leqslant x} g(n) + o\left(\sum_{n\leqslant x} g(n)\right), \qquad x\to\infty.$$

We would like to integrate by parts and remove the weight n^{it} from $h(n)n^{it}$, but have insufficient control over the values of the function h. Since, in some sense, we are considering the ratio $h(n)n^{it}(g(n))^{-1}$, at an appropriate moment we switch the weight n^{it} from h to g and consider the ratio $h(n)(g(n)n^{-it})^{-1}$.

Following the argument for Theorem 3, the study of the sum $\sum_{n\leqslant x} h(n)$ is reduced to that of

$$\widetilde{M_\varepsilon} = \sum_{x^\varepsilon < u\leqslant x^{1-\varepsilon}} h_1(u)\sum_{v\leqslant x/u} h_2(v),$$

where Theorem 3 is applicable to the pair $h_2(n)n^{it}$, $g_2(n)$. There is a corresponding estimate

$$\sum_{n\leqslant y} h_2(n)n^{it} = L(\log y)\sum_{n\leqslant y} g_2(n) + o\left(\sum_{n\leqslant y} g_2(n)\right), \qquad y\to\infty,$$

with

$$L(\log y) = \prod_{p\leqslant y}\left(1+h_2(p)p^{it-1}+\cdots\right)\left(\prod\nolimits_y(g_2)\right)^{-1}, \qquad y\geqslant 2.$$

Set

$$H_2(y) = \sum_{n\leqslant y} h_2(n)n^{it}, \qquad G_2(y) = \sum_{n\leqslant y} g_2(n), \qquad y\geqslant 1/2.$$

An integration by parts gives a representation

$$\sum_{n\leqslant y} h_2(n) = y^{-it}H_2(y) + it\int_{1/2}^{y} w^{-it-1}H_2(w)\,dw,$$

provided y is not an integer. Since $G_2(w)\ll w(\log w)^{-1}\prod_w(g_2)$, $w\geqslant 2$,

$$\int_2^x w^{-1}G_2(w)\,dw \ll \prod\nolimits_x(g_2)\int_2^x(\log w)^{-1}\,dw \ll x(\log x)^{-1}\prod\nolimits_x(g_2) \ll G_2(x), \qquad x\geqslant 2.$$

Hence

$$\sum_{n\leqslant y} h_2(n) = y^{-it}L(\log y)G_2(y) + it\int_2^y w^{-it-1}L(\log w)G_2(w)\,dw + o(G_2(y)),$$

as $y\to\infty$.

As in the proof of Theorem 3, within an acceptable error $L(\log w)$, for $y^\varepsilon \leqslant w \leqslant y$, may be replaced by $L(\log y)$ and factored out of the representation:

$$\sum_{n\leqslant y} h_2(n) = L(\log y)\left(y^{-it}G_2(y) + it\int_2^y w^{-it-1}G_2(w)\,dw\right) + o(G_2(y)), \quad y\to\infty.$$

We appeal to the asymptotic estimate

$$G_2(x) = (1+o(1))\frac{e^{-\gamma\alpha}}{\Gamma(\alpha)}\frac{x}{\log x}\prod\nolimits_x(g_2), \qquad x\to\infty,$$

vouchsafed by Satz 1.1. Once again, as for Theorem 3, we employ the slow oscillation of the function $\prod_x(g_2)(\log x)^{-\alpha}$ to obtain a representation

$$\begin{aligned}\sum_{n\leqslant y} h_2(n) &= \frac{e^{-\gamma\alpha}}{\Gamma(\alpha)}L(\log x)\frac{\prod_x(g_2)}{(\log x)^\alpha}\left(\frac{y^{1-it}}{(\log y)^{1-\alpha}} + it\int_2^y \frac{w^{-it}}{(\log w)^{1-\alpha}}\,dw\right)\\ &\quad + o(G_2(y))\\ &= \frac{e^{-\gamma\alpha}}{\Gamma(\alpha)}L(\log x)\frac{\prod_x(g_2)}{(\log x)^\alpha}\frac{y^{1-it}}{(1-it)(\log y)^{1-\alpha}} + o(G_2(y)),\end{aligned}$$

uniformly for $x^\varepsilon \leqslant y \leqslant x$, as $x\to\infty$; stepping from w to y to x.

Accordingly,

$$\widetilde{M_\varepsilon} = \frac{e^{-\gamma\alpha}}{\Gamma(\alpha)}\frac{x^{1-it}}{1-it}L(\log x)\frac{\prod_x(g_2)}{(\log x)^\alpha}\sum_{x^\varepsilon<u\leqslant x^{1-\varepsilon}}\frac{h_1(u)u^{it}}{u(\log x/u)^{1-\alpha}} + o(G(x)), \quad x\to\infty.$$

We may now formally follow the argument for Theorem 3, the rôle of $h_1(n)$ there here played by $h_1(n)n^{it}$, although on a slightly different set of primes. Eventually only the extra factor $x^{-it}(1-it)^{-1}$ remains.

Case (ii). The series $\sum p^{-1}(g(p) - \mathrm{Re}(h(p)p^{it}))$ diverges for every real t. The partial sums of this series are non-decreasing in x and continuous in t. Divergence of the series is uniform on every compact interval $|t|\leqslant T$ and Theorem 3 follows from an application of Theorem 6, depending upon whether the series $\sum p^{-1}(g(p) - |h(p)|)$ converges or not. ■

Remark. Under the hypothesis of Case (i) the series $\sum p^{-1}|g(p) - h(p)p^{it}|^2$ converges. The series $\sum p^{-1}|g(p) - |h(p)||^2$ and $\sum p^{-1}g(p)|1 - e^{i\theta_p}p^{it}|^2$, where $h(p) = |h(p)|e^{i\theta_p}$, then also converge.

Suppose further that, for some positive integer k, $h(p)^k$ is real. The inequality $|1-z^k| \leqslant k|1-z|$, valid for every z in the complex unit disc, guarantees the series $\sum p^{-1}g(p)|1-p^{2ikt}|^2$ to converge.

In the present circumstances $\sum_{p\leqslant x} p^{-1}g(p) \geqslant (c+o(1))\log\log x$ as $x\to\infty$ and an application of Lemma 15 from Elliott and Kish [6] shows that $t=0$.

A simple example is given by $h(n) = g(n)\chi(n)$, where χ is a Dirichlet character.

The argument of this remark may be given a topological aspect by defining a metric $\sigma(f, g) = (\sum p^{-1}|f(p) - g(p)|^2)^{1/2}$ on equivalence classes of multiplicative functions that coincide of the primes, and restricting study to those functions g whose distance $\sigma(g, g_0)$ to a fixed multiplicative function g_0 is defined, i.e. finite. The topological space of complex-valued multiplicative functions is in this manner locally metrised and correspondingly disconnected.

7. Proof of Theorem 6

We assume the new, weaker restraints upon g. If g is exponentially multiplicative, i.e. $g(p^k) = g(p)^k/k!$, and $|g(p)| \leqslant \beta$, then for any γ the series

$$\sum_{p,k\geqslant 2} p^{-k}|g(p^k)|(\log p^k)^\gamma$$

converges, so that Theorem 4 is applicable. Indeed, for such functions the original exposition of Elliott and Kish, [7] Theorem 2, already contains a proof.

In general, we define an exponentially multiplicative function g_1 by $g_1(p) = g(p)$, and a complementary multiplicative function g_2 by Dirichlet convolution: $g = g_1 * g_2$.

Calculation with Euler products shows that $g_2(p) = 0$ and for $k \geqslant 2$,

$$g_2(p^k) = \sum_{r=0}^{k} (r!)^{-1}(-g(p))^r g(p^{k-r}).$$

In particular,

$$|g_2(p^k)| \leqslant \sum_{r=0}^{k} (r!)^{-1}\beta^r |g(p^{k-r})|, \qquad k \geqslant 2.$$

As a consequence

$$\begin{aligned}\sum_{p,k\geqslant 2} p^{-k}|g_2(p^k)| &\leqslant \sum_{r=0}^{\infty}(r!)^{-1}\beta^r \sum_{p,k\geqslant 2} p^{-k}|g(p^{k-r})| \\ &\leqslant \left(\tfrac{3}{2}\beta^2 + \tfrac{1}{4}\beta^3\right)\sum p^{-2} + \left(1 + \tfrac{1}{2}\beta^2\right)\sum_{p,k\geqslant 2} p^{-k}|g(p^k)|,\end{aligned}$$

and converges.

Moreover,

$$\begin{aligned}\sum_{p^k\leqslant y} |g_2(p^k)| &\leqslant \sum_{r=0}^{\infty}(r!)^{-1}\beta^r \sum_{p^k\leqslant y, k\geqslant 2} |g(p^{k-r})| \\ &\ll \sum_{r=0}^{\infty}(r!)^{-1}\beta^r y(\log y)^{-1} \ll y(\log y)^{-1}\end{aligned}$$

uniformly for $y \geqslant 2$.

We may apply Lemma 1 and obtain for $|g_2|$ the uniform estimate

$$\sum_{n\leqslant y}|g_2(n)| \ll y(\log y)^{-1}, \qquad y \geqslant 2.$$

With δ a real number to be chosen presently in the range $0<\delta<1$,

$$\rho = \exp\left(-\frac{c}{c+\beta}\lambda\right) + T^{-1/2},$$

as in the statement of Theorem 5, we define $w = \exp(\rho^\delta \log x)$, so that w is effectively a function of x for $x \geqslant 2$.

It is convenient to note that we may assume $\rho^\delta \leqslant 1/2$, otherwise Theorem 6 follows directly from Lemma 1.

Moreover, provided $2\delta\beta c < c+\beta$ and Y does not exceed a certain fixed power of x, which we may likewise assume, $Y \leqslant w$. For otherwise

$$\begin{aligned}\log x/\log Y \leqslant \rho^{-\delta} &\leqslant \exp\left(\frac{\delta c}{c+\beta}\lambda\right)\\ &\ll \exp\left(\frac{\delta c}{c+\beta}\sum_{Y<p\leqslant x}2p^{-1}|g(p)|\right) \ll (\log x/\log Y)^{2\delta\beta c/(c+\beta)}.\end{aligned}$$

In particular, uniformly for $w<y\leqslant x$,

$$\min_{|t|\leqslant T}\sum_{Y<p\leqslant y}p^{-1}(|g(p)|-\mathrm{Re}(g(p)p^{it})) \geqslant \lambda - 2\sum_{w<p\leqslant x}p^{-1}|g(p)| \geqslant \lambda + 2\delta\beta\log\rho + O(1).$$

Applied to g_1 over the same range of y-values, Theorem 5 delivers an estimate

$$\begin{aligned}\sum_{n\leqslant y}g_1(n) &\ll \frac{y}{\log y}\prod\nolimits_y(|g_1|)\left(\exp\left(-\frac{c\lambda}{c+\beta}\right)\rho^{-2\delta\beta c/(c+\beta)} + T^{-1/2}\right)\\ &\ll \frac{y}{\log y}\prod\nolimits_y(|g_1|)\rho^{1-2\delta\beta c/(c+\beta)},\end{aligned}$$

this last step somewhat wasteful.

We decompose the mean-value of g into two sums:

$$\sum_{n\leqslant x}g(n) = \sum_{b\leqslant x/w}g_2(b)\sum_{a\leqslant x/b}g_1(a) + \sum_{a<w}g_1(a)\sum_{x/w<b\leqslant x/a}g_2(b).$$

The first doublesum is

$$\begin{aligned}&\ll \sum_{b\leqslant x/w}|g_2(b)|xb^{-1}(\log(x/b))^{-1}\prod\nolimits_{x/b}(|g_1|)\rho^{1-2\delta\beta c/(c+\beta)}\\ &\ll x(\log x)^{-1}\prod\nolimits_x(|g|)\rho^{1-2\delta\beta c/(c+\beta)-\delta}.\end{aligned}$$

The second doublesum is

$$\ll \sum_{a<w} |g_1(a)| x a^{-1} (\log(x/a))^{-1}$$

and $w \leqslant x^{1/2}$, so that the bound does not exceed a constant multiple of

$$x(\log x)^{-1} \prod_{p \leqslant w} (1 + p^{-1}|g(p)|) \ll x(\log x)^{-1} \prod\nolimits_x (|g|) \exp\left(- \sum_{w<p\leqslant x} p^{-1}|g(p)| \right).$$

According to the lower bound hypothesis on $|g(p)|$ in Theorem 5, still in force in Theorem 6, noting that $w \geqslant Y$,

$$\sum_{w<p\leqslant x} p^{-1}|g(p)| \geqslant c \sum_{w<p\leqslant x} p^{-1} + O(1) \geqslant -\delta c \log \rho + O(1).$$

Altogether,

$$\sum_{n \leqslant x} g(n) \ll \frac{x}{\log x} \prod\nolimits_x (|g|)(\rho^{1-\delta c_0} + \rho^{\delta c})$$

with $c_0 = 2\beta c(c+\beta)^{-1} + 1$.

We choose δ to satisfy $1 - \delta c_0 = \delta c$. The earlier condition $2\delta\beta c < \beta + c$ is amply satisfied, c_0 increases with β and δc descends to a limiting value $c(3c+1)^{-1}$. ■

8. Concluding remarks

The present Theorem 4, with quite different argument, improves the formally similar 2001 Theorem of Indlekofer, Kátai and Wagner [10] by appreciably weakening its main hypothesis.

Note that since its lower bound hypothesis remains valid with $\max(g(p), 0)$ in place of $g(p)$, the function g in Lemma 5 may be assumed non-negative. Moreover, the argument for that lemma also allows the choice $\alpha = c$.

The hypothesis on $|g|$ in Theorem 6 remains essentially weaker than that on g in Theorem 4. What might a best-possible condition on g be in order to guarantee the validity of Theorem 4?

Likewise, what might the weakest hypothesis on g be in order to guarantee the validity of the lower bound in Theorem 2?

In response to a request of the referee the author adds the following remarks concerning the possibility of giving the present results a quantitative aspect:

The present Theorem 4 (ii) is a direct application of Theorem 6, a gloss on Theorem 5, for which the complete argument given in Elliott and Kish, [7], is already localised.

Although employing new ideas, the argument for Theorem 4 (i) rests ultimately upon the pioneering work of Wirsing, loc. cit. Its thorough overhaul to effect a localisation would be an enterprise of considerable interest in itself.

An effective estimate for modestly perturbed multiplicative functions is provided by combining the argument of Elliott and Kish [7], Theorem 2 with that of the taxonomy section of Elliott and Kish [6]. The following serves:

Example. If g is a non-negative exponentially multiplicative function, uniformly bounded by β on the primes and, for some positive constants c, c_1, satisfying

$$\sum_{w<p\leqslant x} p^{-1}(g(p)-c) \geqslant -c_1, \quad 3/2 \leqslant w \leqslant x,$$

then for any positive integer $D \geqslant 2$,

$$\sum_{n\leqslant x,(n,D)=1} g(n) = \prod_{p|D}\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\cdots\right)^{-1}\sum_{n\leqslant x} g(n) + O\left(\frac{(\log\log 2D)^{\beta+1}}{(\log x)^{1+\eta}}\prod_{p\leqslant x}\left(1+\frac{g(p)}{p}\right)\right)$$

with η a complicated expression that simplifies to $c(1+3456(\beta/c)^2)^{-1}$ if $c \leqslant 12(2\beta)^{1/2}$.

The implied constant depends at most upon c, c_1 and β.

With adequate control over g on the higher prime-powers, g may be assumed only multiplicative rather than exponentially multiplicative. In particular, for g with values in the unit interval $[0,1]$, this widens the uniformity of the corresponding Theorem 2 in the author's 1989 paper, [4].

Moreover, the example may be combined with the present Theorem 2 to provide an effective important particular case of the present Theorem 4.

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