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SYMMETRIC q-BERNOULLI NUMBERS AND POLYNOMIALS

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Abstract: In this work we are interested by giving a new q-analogue of Bernoulli numbers and polynomials which are symmetric under the interchange $q \leftrightarrow q^{-1}$ and deduce some important relations of them. Also, we deduce a q-analogue of the Euler-Maclaurin formulas

Keywords: q-Bernoulli, symmetric.

1. Introduction

In literature, q-analogue of some special functions like a q-exponential, q-Gamma, q-Betta and q-Bessel functions have been studied intensively for $0 < q < 1$.

In ([4]), G.Dattoli and A.Torre introduced a q-Bessel functions of index which are symmetric under the interchange $q \leftrightarrow q^{-1}$. The authors use a generating function obtained owing a product of symmetric q-exponential functions ([16],[17]).

Recently, Kamel Brahim and Yosr Sidomou ([2]) introduced a symmetric q-Gamma and q-Betta functions and extended the symmetric q-Bessel function of real index.

In the present paper, we introduce a symmetric q-Bernoulli polynomials and q-Bernoulli numbers and give some applications.

This paper is organized as follows: In Section 2, we present some results about quantum calculus and symmetric quantum calculus that will be useful in the sequel. In Section 3, we study the symmetric q-exponential function. In Section 4 and 5, we introduce and study symmetric q-Bernoulli polynomials and symmetric q-Bernoulli numbers. As an application we introduce in Section 6 a q-analogue of Euler-MacLaurin formulas.

2. Symmetric quantum calculus

We recall some usual notions and notations used in the q-theory (see [5] and [8]). Throughout this paper, we assume $q > 0$, $q \neq 1$.

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For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) = (1-a)(1-aq)....(1-aq^{n-1}), \quad n = 1,2,\ldots$$

We denote

$$[x]_q = \frac{1-q^x}{1-q}, \qquad x \in \mathbb{C},$$

and

$$\widetilde{[x]}_q = \frac{q^x - q^{-x}}{q-q^{-1}}, \qquad x \in \mathbb{C}.$$

We also denote

$$[n]_q! = \prod_{k=1}^{n}[k]_q = \frac{(q;q)_n}{(1-q)^n}, \qquad n \in \mathbb{N},$$

and

$$\widetilde{[n]}_q! = \prod_{k=1}^{n}\widetilde{[k]}_q, \qquad n \in \mathbb{N}.$$

The q-binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad k = 0,1,\ldots,n.$$

Similarly we can define the symmetric q-binomial coefficient by

$$\widetilde{\binom{n}{k}}_q = \frac{\widetilde{[n]}_q!}{\widetilde{[k]}_q!\widetilde{[n-k]}_q!}, \qquad k = 0,1,\ldots,n.$$

One can see that

1) $\widetilde{[x]}_q = \widetilde{[x]}_{q^{-1}}$.
2) $\widetilde{[x+y]}_q = q^y\widetilde{[x]}_q + q^{-x}\widetilde{[y]}_q$.
3) $\widetilde{\binom{n}{k}}_q = \widetilde{\binom{n}{k}}_{\frac{1}{q}}$.
4) $\widetilde{[x]}_q = q^{-(x-1)}[x]_{q^2}$.

The symmetric q-derivative $\widetilde{D}_q$ of a function f is given by

$$(\widetilde{D}_q f)(x) = \frac{f(qx) - f(q^{-1}x)}{(q-q^{-1})x}, \qquad \text{if } x \neq 0,$$

$(\widetilde{D}_q f)(0) = f'(0)$ provided $f'(0)$ exists.

We have the following relation

$$\widetilde{D}_q f(x) = D_{q^2} f(q^{-1}x)$$

where

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

The following properties hold ([8])

1) $\widetilde{D}_q(f(x) + g(x)) = \widetilde{D}_q f(x) + \widetilde{D}_q g(x)$,
2) $\widetilde{D}_q(f(x)g(x)) = g(q^{-1}x)\widetilde{D}_q f(x) + f(qx)\widetilde{D}_q g(x)$,
3) $\widetilde{D}_q x^n = \widetilde{[n]}_q x^{n-1}$,
4) $\widetilde{D}_q \widetilde{(x-a)}_q^n = \widetilde{[n]}_q \widetilde{(x-a)}_q^{n-1}$, where $\widetilde{(x-a)}_q^n = (x - q^{n-1}a)(x - q^{n-3}a)(x - q^{n-5}a)...(x - q^{-n+1}a)$ and $\widetilde{(x-a)}_q^0 = 1$.

In the particular case $a = 0$, we have $\widetilde{(x-0)}_q^n = \widetilde{(x)}_q^n = x^n$.

The following result is a q-analogue of the Gauss binomial formula

$$\widetilde{(x+a)}_q^n = \sum_{k=0}^{n} \widetilde{\binom{n}{k}}_q a^{n-k} x^k. \tag{1}$$

Provided that the series converges, the symmetric q-integral or $\widetilde{q}$-integral is given by ([8])

$$\int_0^a f(x) d_{\widetilde{q}} x = a(q^{-1} - q) \sum_{n=1,3,...} q^n f(q^n a),$$

$$\int_a^b f(x) d_{\widetilde{q}} x = \int_0^b f(x) d_{\widetilde{q}} x - \int_0^a f(x) d_{\widetilde{q}} x,$$

and

$$\int_0^\infty f(x) d_{\widetilde{q}} x = (q^{-1} - q) \sum_{n=\pm 1, \pm 3,...} q^n f(q^n a).$$

The $\widetilde{q}$-integral satisfy the following properties

Lemma 1.

a) *If F is any anti q-derivative of the function f, namely $\widetilde{D}_q F = f$, continuous at $x = 0$, then*

$$\int_0^a f(x) d_{\widetilde{q}} x = F(a) - F(0).$$

b) *For any function f we have*

$$\widetilde{D}_q \int_0^x f(t) d_{\widetilde{q}} t = f(x).$$

c) *We have*

$$\int_a^b f(q^{-1}x)\widetilde{D}_q fg(x)d_{\widetilde{q}}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)\widetilde{D}_q f(x)d_{\widetilde{q}}x.$$

Proof. 1) We have

$$\int_0^a f(x)d_{\widetilde{q}}x = a(q^{-1}-q)\sum_{n=1,3,\dots} q^n \widetilde{D}_q F(q^n a)$$
$$= -\sum_{n=1,3,\dots}\left[F(q^{n+1}a) - F(q^{n-1}a)\right] = F(a) - F(0).$$

2) We have

$$\widetilde{D}_q \int_0^x f(t)d_{\widetilde{q}}t = \frac{1}{(q-q^{-1})x}\left[\int_0^{qx} f(t)d_{\widetilde{q}}t - \int_0^{q^{-1}x} f(t)d_{\widetilde{q}}t\right]$$
$$= -\left(\sum_{n=1,3,\dots} q^{n+1}f(q^{n+1}x) - \sum_{n=1,3,\dots} q^{n-1}f(q^{n-1}x)\right) = f(x).$$

3) From the symmetric q-product derivative rule. ■

3. Symmetric q-exponential function

The classical exponential function e^z has two different natural q-extensions ([13]) one of them denoted by $e_q(z)$ and given by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n},$$

where $z \in \mathbb{C}$,$|z| < 1$ and $0 < q < 1$.

The function $e_q(z)$ can be considered as formal power series in the formal variable z and satisfies the relation $\lim_{q\to 1} e_q((1-q)z) = e^z$.

Let $C_q[[x, y]]$ be the complex associative algebra with 1 of formal power series

$$\sum_{k,l=0}^{\infty} c_{k,l}y^l x^k$$

with arbitrary complex coefficients $c_{k,l}$ and where x, y satisfy the relation $xy = qyx$.

In the algebra $C_q[[x, y]]$, the function $e_q(z)$ satisfy the following relation ([14])

$$e_q(x + y) = e_q(y)e_q(x).$$

A symmetric q-exponential (symmetric under the interchange $q \leftrightarrow q^{-1}$) is defined by D. S. McAnally in ([16],[17]):

$$\widetilde{e}_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{\widetilde{[n]}_q!},$$

for $z \in \mathbb{C}$ and $q \in]0, 1[\cup]1, +\infty[$.

The function $\widetilde{e}_q(z)$ can be considered as formal power series in the formal variable z and satisfies the relation $\lim_{q\to 1}\widetilde{e}_q(z)=e^z$.

The function $\widetilde{e}_q(z)$ can be extended in the following way:

$$\widetilde{e}_q(x+y)=\sum_{n=0}^{\infty}\frac{\widetilde{(x+y)}_q^{\,n}}{\widetilde{[n]}_q!},$$

in the particular case when y=0, we have

$$\widetilde{e}_q(x+0)=\sum_{n=0}^{\infty}\frac{\widetilde{(x)}_q^{\,n}}{\widetilde{[n]}_q!}=\sum_{n=0}^{\infty}\frac{x^n}{\widetilde{[n]}_q!}=\widetilde{e}_q(x).$$

Using (1), we have the following lemma

Lemma 2. *In the commutative algebra $C[[x,y]]$ we have the identity*

$$\widetilde{e}_q(x+y)=\widetilde{e}_q(y)\widetilde{e}_q(x).$$

Proof. We have

$$\begin{aligned}\widetilde{e}_q(x+y)&=\sum_{n=0}^{\infty}\frac{\widetilde{(x+y)}_q^{\,n}}{\widetilde{[n]}_q!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{\widetilde{[n]}_q!}\widetilde{\binom{n}{k}}_q y^{n-k}x^k\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{\widetilde{[k]}_q!\widetilde{[n-k]}_q!}y^{n-k}x^k\\&=\sum_{k,l=0}^{\infty}\frac{1}{\widetilde{[l]}_q!\widetilde{[k]}_q!}y^lx^k=\widetilde{e}_q(y)\widetilde{e}_q(x).\end{aligned}$$

■

4. Symmetric q-Bernoulli polynomials

The classical Bernoulli polynomials $B_n(x)$ are defined by the generating function

$$\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}=\frac{z}{e^z-1}e^{zx}.$$

The Bernoulli numbers are defined through the relation $B_n=B_n(0)$.

The q-Bernoulli polynomials $B_n(x,h\mid q)$ ([3], [10]) are defined by q-generating function

$$e^{\frac{t}{1-q}}\sum_{j=0}^{\infty}\frac{j+h}{[j+h]_q}q^{jx}(-1)^j\frac{1}{(1-q)^j}\frac{t^j}{j!}=\sum_{n=0}^{\infty}\frac{B_n(x,h\mid q)}{n!}t^n,\qquad h\in\mathbb{Z},\ x\in\mathbb{C}.$$

Note that

$$\lim_{q\to 1} B_n(x,h \mid q) = B_n(x).$$

The q-Bernoulli numbers are defined through the relation $B_n(h \mid q) = B_n(0,h \mid q)$.

In ([6]) the authors gave another approach to study the q-Bernoulli polynomials. They defined the q-Bernoulli polynomials $B_n(x,q)$ by q-generating function

$$\sum_{n=0}^{\infty} B_n(x,q)\frac{z^n}{[n]_q!} = \frac{z}{e^z-1}e_q((1-q)zx).$$

They proved that

$$B_n(x,q) = \sum_{k=0}^{n} \binom{n}{k}_q b_k(q)x^{n-k},$$

where $b_n(q) = \frac{b_n}{n!}[n]_q!$ is a q-analogue of the Bernoulli numbers.

In this paper we use the same approach in ([6]) to define and study a q-analogue of Bernoulli polynomials which is symmetric under the interchange $q \leftrightarrow q^{-1}$.

Let $\widehat{B}(t)$ be the generating function of the classical Bernoulli numbers ([15])

$$\widehat{B}(t) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z-1}.$$

Then we get

$$\widehat{B}\Big(\frac{\partial}{\partial x}\Big)x^k = \sum_{n=0}^{\infty} \frac{B_n}{n!}\Big(\frac{\partial}{\partial x}\Big)^n x^k = \sum_{n=0}^{k} \binom{k}{n} B_n x^{k-n}.$$

Also, on exponent

$$\widehat{B}\Big(\frac{\partial}{\partial x}\Big)e^{tx} = \widehat{B}(t)e^{tx} = B(x,t).$$

Now we will define a q-analogue of the generating function $\widehat{B}(t)$ as

$$\widetilde{B}_q(t) = \sum_{n=0}^{\infty} \frac{\widetilde{b}_n(q)}{\widetilde{[n]}_q!}t^n,$$

where $\widetilde{b}_n(q)$ is a q-analogue of the Bernoulli numbers. By using the q-difference operator $\widetilde{D}_q$ we get

$$\widetilde{B}_q(\widetilde{D}_q)x^k = \sum_{n=0}^{\infty} \frac{\widetilde{b}_n(q)}{\widetilde{[n]}_q!}\widetilde{D}_q^n x^k = \sum_{n=0}^{k} \frac{\widetilde{b}_n(q)}{\widetilde{[n]}_q!}\frac{\widetilde{[k]}_q!}{\widetilde{[k-n]}_q!}x^{k-n} = \sum_{n=0}^{k} \widetilde{\binom{k}{n}}_q \widetilde{b}_n(q)x^{k-n}.$$

This procedure will suggest the following q-analogue of Bernoulli polynomials

$$\widetilde{B}_k(x,q) = \sum_{n=0}^{k} \widetilde{\binom{k}{n}}_q \widetilde{b}_n(q)x^{k-n}.$$

Also,

$$\widetilde{B}_q(\widetilde{D}_q)\widetilde{e}_q(xt) = \sum_{n=0}^{\infty} \frac{\widetilde{b}_n(q)}{\widetilde{[n]_q!}} \widetilde{D}_q^n \left(\sum_{k=0}^{\infty} \frac{x^k}{\widetilde{[k]_q!}} t^k \right) = \sum_{k=0}^{\infty} \frac{t^k}{\widetilde{[k]_q!}} \sum_{n=0}^{\infty} \frac{\widetilde{b}_n(q)}{\widetilde{[n]_q!}} \widetilde{D}_q^n x^k$$
$$= \sum_{k=0}^{\infty} \frac{t^k}{\widetilde{[k]_q!}} \widetilde{B}_k(x,q) = B(x,t,q).$$

Using these notations we can define the symmetric q-Bernoulli polynomials

Definition 1. The symmetric q-Bernoulli polynomials $\widetilde{B}_n(x,q)$ are defined by

$$\sum_{n=0}^{\infty} \widetilde{B}_n(x,q) \frac{z^n}{\widetilde{[n]_q!}} = \frac{z}{e^z - 1} \widetilde{e}_q(zx), \tag{2}$$

where $\lim_{q\to 1} \widetilde{B}_n(x,q) = B_n(x)$, $B_n(x)$ are the ordinary Bernoulli polynomials.

Proposition 1.

$$\widetilde{D}_q \widetilde{B}_n(x,q) = \widetilde{[n]}_q \widetilde{B}_{n-1}(x,q).$$

Proof.

$$\sum_{n=1}^{\infty} \widetilde{D}_q \widetilde{B}_n(x,q) \frac{z^n}{\widetilde{[n]_q!}} = \frac{z^2}{e^z - 1} \widetilde{e}_q(zx) = \sum_{n=0}^{\infty} \widetilde{B}_n(x,q) \frac{z^{n+1}}{\widetilde{[n]_q!}}$$
$$= \sum_{n=1}^{\infty} \widetilde{B}_{n-1}(x,q) \frac{z^n}{\widetilde{[n-1]_q!}} = \sum_{n=1}^{\infty} \widetilde{[n]}_q \widetilde{B}_{n-1}(x,q) \frac{z^n}{\widetilde{[n]_q!}}. \quad \blacksquare$$

Proposition 2. *In the commutative algebra $C[[x,y]]$ we have the identity*

$$\widetilde{B}_n(x+y,q) = \sum_{k=0}^{n} \widetilde{\binom{n}{k}}_q y^{n-k} \widetilde{B}_k(x,q). \tag{3}$$

Proof. We have

$$\sum_{n=0}^{\infty} \widetilde{B}_n(x+y,q) \frac{z^n}{\widetilde{[n]_q!}} = \frac{z}{e^z - 1} \widetilde{e}_q(z(x+y)) = \frac{z}{e^z - 1} \widetilde{e}_q(zy) \widetilde{e}_q(zx)$$
$$= \widetilde{e}_q(zy) \left(\frac{z}{e^z - 1} \widetilde{e}_q(zx) \right) = \widetilde{e}_q(zy) \sum_{n=0}^{\infty} \widetilde{B}_n(x,q) \frac{z^n}{\widetilde{[n]_q!}}.$$

Also,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\widetilde{\binom{n}{k}}_q y^{n-k}\widetilde{B}_k(x,q)\frac{z^n}{\widetilde{[n]}_q!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{y^{n-k}\widetilde{B}_k(x,q)}{\widetilde{[k]}_q!\widetilde{[n-k]}_q!}z^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(zy)^{n-k}\widetilde{B}_k(x,q)}{\widetilde{[k]}_q!\widetilde{[n-k]}_q!}z^k$$
$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(zy)^l}{\widetilde{[l]}_q!}\frac{\widetilde{B}_k(x,q)}{\widetilde{[k]}_q!}z^k$$
$$= \widetilde{e}_q(zy)\sum_{n=0}^{\infty}\widetilde{B}_n(x,q)\frac{z^n}{\widetilde{[n]}_q!}.$$

Which achieves the proof. ■

The relation (3) is a q-analogue of the classical relation

$$B_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}y^{n-k}B_k(x),$$

where $B_n(x)$ are ordinary Bernoulli polynomials([1]).

5. Symmetric q-Bernoulli numbers

Definition 2. For $n \geqslant 0$, $\widetilde{b}_n(q) = \widetilde{B}_n(0,q)$ are called symmetric q-Bernoulli numbers.

We have the following result

Lemma 3. *We have*

$$\widetilde{b}_n(q) = \frac{b_n}{n!}\widetilde{[n]}_q!. \tag{4}$$

where $\lim_{q\to 1}\widetilde{b}_n(q) = b_n$*,* b_n *are the ordinary Bernoulli numbers.*

Proof. Putting $x = 0$ in equation (2), we get

$$\sum_{n=0}^{\infty}\widetilde{b}_n\frac{z^n}{\widetilde{[n]}_q!} = \frac{z}{e^z-1} = \sum_{n=0}^{\infty}b_n\frac{z^n}{n!}.$$

Then

$$\widetilde{b}_n(q) = \frac{b_n}{n!}\widetilde{[n]}_q!.$$

Also,

$$\lim_{q\to 1}\widetilde{b}_n(q) = \lim_{q\to 1}\frac{b_n}{n!}\widetilde{[n]}_q! = b_n.$$ ■

The knowledge of the Bernoulli numbers and the lemma (4) allows us to determinate the symmetric q-Bernoulli numbers. The first five of them are:

$$\widetilde{b}_0 = 1, \qquad \widetilde{b}_1 = -\frac{1}{2}, \qquad \widetilde{b}_2 = \frac{\widetilde{[2]}_q}{12}, \qquad \widetilde{b}_3 = 0, \qquad \widetilde{b}_4 = \frac{\widetilde{[2]}_q\widetilde{[3]}_q\widetilde{[4]}_q}{720}.$$

Using the properties of the ordinary Bernoulli numbers b_n ([8]), we can prove that

- $\widetilde{b}_n(q) = 0\ \forall n$ odd and $n \geqslant 3$,
- $\sum_{j=0}^{n-1} {}^nP_j \frac{\widetilde{b}_j(q)}{\widetilde{[j]}_q!} = 0$,
- $\sum_{j=1}^{n-1} (-1)^j\ {}^nP_j \frac{\widetilde{b}_{j+1}(q)}{\widetilde{[j+1]}_q!} = \frac{1-n}{2(1+n)}$.

Proposition 3. *For any $n \geqslant 1$*

$$\sum_{j=0}^{n-1} {}^nP_j \frac{\widetilde{B}_j(x,q)}{\widetilde{[j]}_q!} = \frac{n!}{\widetilde{[n-1]}_q!} x^{n-1}.$$

Proof. The case where $n = 1$ is obvious. If we assume that the relation is true for some $k \geqslant 1$, we have

$$\begin{aligned}
\widetilde{D}_q\left(\sum_{j=0}^{k} {}^{k+1}P_j \frac{\widetilde{B}_j(x,q)}{\widetilde{[j]}_q!}\right) &= \sum_{j=0}^{k} {}^{k+1}P_j \widetilde{[j]}_q \frac{\widetilde{B}_{j-1}(x,q)}{\widetilde{[j]}_q!} \\
&= (k+1)\sum_{j=0}^{k-1} {}^kP_j \frac{\widetilde{B}_j(x,q)}{\widetilde{[j]}_q!} \\
&= (k+1)\frac{k!}{\widetilde{[k-1]}_q!} x^{k-1} \\
&= \frac{(k+1)!}{\widetilde{[k-1]}_q!} x^{k-1} = \widetilde{D}_q\left(\frac{(k+1)!}{\widetilde{[k]}_q!} x^k\right).
\end{aligned}$$

Then

$$\sum_{j=0}^{k} {}^{k+1}P_j \frac{\widetilde{B}_j(x,q)}{\widetilde{[j]}_q!} = \frac{(k+1)!}{\widetilde{[k]}_q!} x^k + c.$$

Put $x = 0$, then

$$\sum_{j=0}^{k} {}^{k+1}P_j \frac{\widetilde{b}_j(q)}{\widetilde{[j]}_q!} = c.$$

Using the second property of $\widetilde{b}_j(q)$, we get $c = 0$. Hence, by induction, relation is true for any positive integer. ∎

Proposition 4.

$$\widetilde{B}_k(x,q)=\sum_{k=0}^{n}\widetilde{\binom{n}{k}}_q\widetilde{b}_k(q)x^{n-k}.$$

Proof. Let

$$F_n(x,q)=\sum_{k=0}^{n}\widetilde{\binom{n}{k}}_q\widetilde{b}_k(q)x^{n-k}.$$

It suffices to show that (i) $F_n(0,q)=\widetilde{b}_n(q)$ for $n\geqslant 0$ and (ii) $\widetilde{D}_qF_n(x,q)=\widetilde{[n]}_qF_{n-1}(x,q)$ for $n\geqslant 1$, since these two properties uniquely characterize $\widetilde{B}_k(x,q)$. The first property is obvious. As for the second property,

$$\begin{aligned}\widetilde{D}_qF_n(x,q)&=\sum_{k=0}^{n-1}\widetilde{\binom{n}{k}}_q\widetilde{b}_k(q)\widetilde{[n-k]}_qx^{n-k-1}\\&=\sum_{k=0}^{n-1}\frac{\widetilde{[n]}_q!}{\widetilde{[n-k-1]}_q!\widetilde{[k]}_q!}\widetilde{b}_k(q)x^{n-k-1}\\&=\widetilde{[n]}_q\sum_{k=0}^{n-1}\frac{\widetilde{[n-1]}_q!}{\widetilde{[n-k-1]}_q!\widetilde{[k]}_q!}\widetilde{b}_k(q)x^{n-k-1}\\&=\widetilde{[n]}_q\sum_{k=0}^{n-1}\widetilde{\binom{n-1}{k}}_q\widetilde{b}_k(q)x^{n-k-1}=\widetilde{[n]}_qF_{n-1}(x,q),\end{aligned}$$

and the proof follows. ■

The knowledge of q-Bernoulli numbers allow us to determine the q-Bernoulli polynomials. The five of them are listed below.

$$\begin{aligned}\widetilde{B}_0(x,q)&=1,\\\widetilde{B}_1(x,q)&=x-\frac{1}{2!},\\\widetilde{B}_2(x,q)&=x^2-\frac{\widetilde{[2]}_q}{2!}x+\frac{\widetilde{[2]}_q}{2(3!)},\\\widetilde{B}_3(x,q)&=x^3-\frac{\widetilde{[3]}_q}{2!}x^2+\frac{\widetilde{[2]}_q\widetilde{[3]}_q}{2(3!)}x,\\\widetilde{B}_4(x,q)&=x^4-\frac{\widetilde{[4]}_q}{2!}x^3+\frac{\widetilde{[3]}_q\widetilde{[4]}_q}{2(3!)}x^2+\frac{\widetilde{[2]}_q\widetilde{[3]}_q\widetilde{[4]}_q}{30(4!)}.\end{aligned}$$

Lemma 4. *The Symmetric q-Bernoulli polynomials have the following symmetry property*

$$(-1)^n\widetilde{B}_n(-x,q)=\widetilde{B}_n(x,q)+\widetilde{[n]}_qx^{n-1},\qquad\forall n\geqslant 1.$$

Proof. The case where $n = 1$ is obvious. If we assume that the relation is true for some $k \geqslant 1$, we get

$$\begin{aligned}\widetilde{D}_q\left((-1)^{k+1}\widetilde{B}_{k+1}(-x,q)\right) &= (-1)^k\widetilde{[k+1]}_q\widetilde{B}_k(-x,q)\\ &= \widetilde{[k+1]}_q\widetilde{B}_k(x,q) + \widetilde{[k+1]}_q\widetilde{[k]}_q x^{k-1}\\ &= \widetilde{D}_q\left(\widetilde{B}_{k+1}(x,q) + \widetilde{[k+1]}_q x^k\right),\end{aligned}$$

then

$$(-1)^{k+1}\widetilde{B}_{k+1}(-x,q) = \widetilde{B}_{k+1}(x,q) + \widetilde{[k+1]}_q x^k + c.$$

Put $x = 0$, then

$$((-1)^{k+1} - 1)\widetilde{b}_{k+1}(q) = c$$

but $((-1)^{k+1} - 1) = 0$ if k is an odd number and $\widetilde{b}_{k+1}(q) = 0$ if k is an even number. Then $c = 0$ and hence, by induction, the relation is true $\forall n \geqslant 1$. ∎

Lemma 5.

$$\int_a^x \widetilde{B}_n(t,q)d_{\widetilde{q}}t = \frac{\widetilde{B}_{n+1}(x,q) - \widetilde{B}_{n+1}(a,q)}{\widetilde{[n+1]}_q}.$$

Proof. By using $\widetilde{D}_q\widetilde{B}_n(t,q) = \widetilde{[n]}_q\widetilde{B}_{n-1}(t,q)$, then we get

$$\begin{aligned}\int_a^x \widetilde{B}_n(t,q)d_{\widetilde{q}}t &= \frac{1}{\widetilde{[n+1]}_q}\int_a^x \widetilde{B}_{n+1}(t,q)d_{\widetilde{q}}t\\ &= \frac{1}{\widetilde{[n+1]}_q}\widetilde{B}_{n+1}(t,q)\Big|_a^x = \frac{\widetilde{B}_{n+1}(x,q) - \widetilde{B}_{n+1}(a,q)}{\widetilde{[n+1]}_q}.\end{aligned}$$ ∎

6. A symmetric q-Euler Maclaurin formulas

Let the function $P(x) = \widetilde{B}_1(x - [x], q)$, in which $[x]$ means the greatest integer $\leqslant x$. The function $P(x)$ is periodic $P(x+1) = P(x)$. Also,

$$\int_0^1 P(x)d_{\widetilde{q}}t = \int_t^{t+1} P(x)d_{\widetilde{q}}t, \qquad \forall t \geqslant 0.$$

We employed $P(x)$ in obtaining a symmetric q-analogue of the Euler-Maclaurin formulas ([18]).

Theorem 1.

$$\sum_{k=0}^{n} f(k) = \frac{f(n)+f(0)}{2} + \int_0^n f(q^{-1}x)d_{\widetilde{q}}x + \int_0^n P(qx)\widetilde{D}_q f(x)d_{\widetilde{q}}x,$$

where $f(x)$ is differentiable.

Proof. First we write

$$\int_0^n P(x)\widetilde{D}_q f(x)d_{\widetilde{q}}x = \sum_{k=1}^{n}\int_{k-1}^{k} P(x)\widetilde{D}_q f(x)d_{\widetilde{q}}x.$$

Now

$$\int_{k-1}^{k} P(x)\widetilde{D}_q f(x)d_{\widetilde{q}}x = \int_{k-1}^{k}(x-k+\frac{1}{2})\widetilde{D}_q f(x)d_{\widetilde{q}}x$$

and we integrate by parts to obtain

$$\int_{k-1}^{k} P(qx)\widetilde{D}_q f(x)d_{\widetilde{q}}x = (x-k+\frac{1}{2})f(x)|_{k-1}^{k} - \int_{k-1}^{k} f(q^{-1}x)\widetilde{D}_q P(x)d_{\widetilde{q}}x$$

then

$$\int_{k-1}^{k} P(qx)\widetilde{D}_q f(x)d_{\widetilde{q}}x = \frac{f(k)+f(k-1)}{2} - \int_{k-1}^{k} f(q^{-1}x)d_{\widetilde{q}}x$$

$$\int_0^n P(qx)\widetilde{D}_q f(x)d_{\widetilde{q}}x = \sum_{k=0}^{n} f(k) - \frac{f(n)+f(0)}{2} - \int_0^n f(q^{-1}x)d_{\widetilde{q}}x$$

and the proof follows. ∎

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