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ON A GENERAL DIOPHANTINE INEQUALITY

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Abstract: In [Ru15], the author introduced the notion of Nevanlinna constant (denoted by Nev(D)) for any effective Cartier divisor D on a normal projective variety X, and established a defect relation for Zariski-dense holomorphic mappings $f : \mathbb{C} \to X$ in terms of Nev(D). In this paper, we prove its counterpart result in Diophantine approximation, according to Vojta's correspondence (or Vojta's dictionary [Voj87]). The results obtained gave the quantitative extension of the earlier results of Corvaja-Zannier [CZ04a][CZ04b], Evertse-Ferretti [EF02] [EF08], A. Levin [Lev09], P. Autissier [Aut1], and others.

Keywords: Schmidt's subspace theorem, integral points, Diophantine approximation.

1. Introduction and the statement of the main results

The celebrated Vojta's Conjecture (discussed e.g. in [Vojcm]) predicts a lower bound for the rational approximation to a configuration of hypersurfaces on a projective variety. The conjectured inequality vastly generalizes theorems of Roth and Schmidt-Schlickewei. This paper makes a step in this direction, using techniques introduced by Corvaja-Zannier [CZ02], Evertse-Ferretti and developed by Levin, Autissier, Min Ru and others. The Main Theorem proved in this paper provides a Diophantine inequality (inequality (3)) which bounds the proximity function to an effective divisor in term of the corresponding height. The dependence on the height is linear and the involved 'constant' is the Nevanlinna constant introduced by the author in [Ru15], depends on the geometric data. As it is customary in this kind of results, this Nevanlinna constant (denoted by Nev(D)) becomes smaller (so that the result is better) whenever the divisor is highly reducible; in particular, it cannot be made dependent only on the divisor class in the Picard group, as should be the case following Vojta's conjecture. As we shall see in this paper, the Nevanlinna constant clarifies, quantifies and unifies in some sense the applicability of the method of Corvaja-Zannier ([CZ04a], [CZ04b]), and Evertse-Ferretti ([EF08]) based on the Subspace Theorem (see also Levin [Lev09], Heier-Ru [HR12], and

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P. Autissier [Aut1]). In the last part of the paper, an explicit computation of this constant recovers the important result of Evertse-Ferretti ([EF08]). It also derives new results, by explicit computation of $\mathrm{Nev}(D)$ for the divisor $D = D_1 + \cdots + D_q$, where (multiples of) $D_1, \ldots, D_q$ are not necessarily linearly equivalent (see Section 4). We note that it is the first time that Diophantine inequalities are obtained for divisors which are not necessarily linearly equivalent on X.

To state the results, we recall the notion of Weil functions. Let k be a number field and let $\mathcal{O}_k$ denote the ring of integers of k. As usual, we have a set M_k of places of k consisting of one place for each nonzero prime ideal in $\mathcal{O}_k$, one place for each real embedding $\sigma : k \to \mathbb{R}$, and one place for each pair of conjugate embeddings $\sigma, \bar{\sigma} : k \to \mathbb{C}$. Denote by k_v the completion of k with respect to v. We normalize our absolute values so that $\|p\|_v = p^{-[k_v:\mathbb{Q}_p]/[k:\mathbb{Q}]}$ if v corresponds to the prime ideal above the prime $p \in \mathbb{Q}$, $\|x\|_v = |\sigma(x)|^{1/[k:\mathbb{Q}]}$ if v corresponds to the real embedding σ, and $\|x\|_v = |\sigma(x)|^{2/[k:\mathbb{Q}]}$ if v corresponds to the pair of conjugate embeddings $\sigma, \bar{\sigma} : k \to \mathbb{C}$. Let X be a projective variety over a number field k. To every Cartier divisor D on X and every place $v \in M_k$, we can associate a local Weil function $\lambda_{D,v} : X(k) \setminus \operatorname{supp} D \to \mathbb{R}$ (see, for example, [Lan87] or [Vojcm]), where $\operatorname{supp} D$ is the support of the divisor D. When D is effective, the Weil function $\lambda_{D,v}$ gives a measurement of the v-adic distance of a point to D. If $X = \mathbb{P}^n$ and $D \subset \mathbb{P}^n$ is a hypersurface defined by a homogeneous polynomial Q of degree d, then

$$\lambda_{D,v}([x_0 : \cdots : x_n]) := \log \frac{\max\{\|x_0\|_v^d, \ldots, \|x_n\|_v^d\}}{\|Q(x_0, \ldots, x_n)\|_v}.$$

The height $h_D(x)$ for points $x \in X(k)$ is defined as

$$h_D(x) = \sum_{v \in M_k} \lambda_{D,v}(x).$$

It is independent of, up to $O(1)$, the choice of Weil functions. Let $S \subset M_k$ be a finite set of places containing all archimedean ones. We define, for $x \in X(k) \setminus \operatorname{supp} D$,

$$m_S(x, D) = \sum_{v \in S} \lambda_{D,v}(x).$$

Definition 1.1. Let X be a projective variety X of dimension $n \geqslant 1$. Divisors $D_1, \ldots, D_q$ on a projective variety X with $q > n$ are said to be *in general position on* X if any $\{i_1, \ldots, i_{n+1}\} \subset \{1, \ldots, q\}$, $\operatorname{supp} D_{i_1} \cap \cdots \cap \operatorname{supp} D_{i_{n+1}} = \emptyset$..

For two divisors D_1, D_2 on X, we use the notation $D_1 \sim D_2$ to denote that D_1, D_2 are linearly equivalent on X. The slightly re-formulated recent result of Evertse-Ferretti [EF08] can be stated as follows (see Theorem 3.1 in Levin [Lev14]).

Theorem A (Evertse–Ferretti [EF08], reformulated). *Let X be a projective variety and $D_1, \ldots, D_q$ be Cartier divisors in general position on X, both defined over a number field k. Let $S \subset M_k$ be a finite set of places. Assume that there* *exist an ample divisor A on X, defined over k, and positive integers d_i such that $D_i \sim d_iA$ for all i. Then, for every $\varepsilon > 0$,*

$$\sum_{j=1}^{q} \frac{1}{d_j} m_S(x, D_j) \leqslant (\dim X + 1 + \epsilon) h_A(x),$$

holds for all k-rational points outside a proper Zariski closed subset of X.

The special case when $X = \mathbb{P}^n$ was due to Corvaja-Zannier [CZ04a]. Note that in the theorem above, the multiples of $D_1, \ldots, D_q$ are assumed to be linearly equivalent on X.

In this paper, we establish a Diophantine inequality of the above type for a general divisor D on a normal variety X, in terms of $\mathrm{Nev}(D)$. We recall here the notion of the Nevanlinna constant $\mathrm{Nev}(D)$ (see [Ru15]). Let X be a normal projective variety over k and D be an effective Cartier divisor on X defined over k. Note that the condition of normality of X is assumed so that $\mathrm{ord}_E D$ (called the *coefficient of D in E*) is defined for any prime divisor E and any effective Cartier divisor D on X (See [Laz04], Remark 1.1.4). For any section $s \in H^0(X, \mathcal{O}(D))$, we use $\mathrm{ord}_E\, s$, or $\mathrm{ord}_E(s)$, to denote the coefficients of (s) in E where (s) is the divisor on X associated to s.

Definition 1.2. Let X be a normal projective variety, and D be an effective Cartier divisor on X, both defined over k. The *Nevanlinna constant* of D, denoted by $\mathrm{Nev}(D)$, is defined by

$$\mathrm{Nev}(D) := \inf_N \left(\inf_{\{\mu_N, V_N\}} \frac{\dim V_N}{\mu_N} \right), \tag{1}$$

where the infimum "$\inf\limits_N$" is taken over all positive integers N and the infimum "$\inf\limits_{\{\mu_N, V_N\}}$" is taken over all pairs $\{\mu_N, V_N\}$ where μ_N is a positive real number and $V_N \subset H^0(X, \mathcal{O}(ND))$ is a linear subspace with $\dim V_N \geqslant 2$ such that, for all $P \in \operatorname{supp} D$, there exists a basis B of V_N (may depend on P) with

$$\frac{1}{\mathrm{ord}_E(ND)} \sum_{s \in B} \mathrm{ord}_E(s) \geqslant \mu_N \tag{2}$$

for all irreducible component E of D passing through P. If $\dim H^0(X, \mathcal{O}(ND)) \leqslant 1$ for all positive integers N, we define $\mathrm{Nev}(D) = +\infty$.

Remark 1.3. The Nevanlinna constant $\mathrm{Nev}(D)$ depends on the ground field k which is denoted, more precisely, by $\mathrm{Nev}_k(D)$. However, From Proposition 2.3 below, if $L \supset k$ are two fields, then $\mathrm{Nev}_L(D) \leqslant \mathrm{Nev}_k(D)$. Therefore the Nevanlinna constant stabilizes after enlarging sufficiently the number field (so that each irreducible component of each divisor is defined over such a larger field). Hence, we can always assume that the ground field k is large enough that the Nevanlinna constant becomes a 'geometric' datum only.

Example 1.4. Let $X = \mathbb{P}^n$ and $D = H_1 + \cdots + H_q$ where $H_1, \cdots, H_q$ are hyperplanes in $\mathbb{P}^n$ in general position. We take $N = 1$ and consider $V_1 := H^0(\mathbb{P}^n, \mathcal{O}(D)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$. Then $\dim V_1 = \binom{q+n}{n}$. For each $P \in \mathrm{Supp}D$, since $H_1, \cdots, H_q$ are in general position, $P \in H_{i_1} \cap \cdots \cap H_{i_l}$ with $\{i_1, \ldots, i_l\} \subset \{1, \ldots, q\}$ and $l \leqslant n$ and $P \notin H_j$ for $j \neq i_1, \ldots, i_l$. Without loss of generality, we can just assume $H_{i_1} = \{z_1 = 0\}, \cdots, H_{i_l} = \{z_l = 0\}$ by taking proper coordinates for $\mathbb{P}^n$. Now we take the basis $B = \{z_0^{i_0} \cdots z_n^{i_n} \mid i_0 + \cdots + i_n = q\}$ for $V_1 = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$. Then, for each irreducible component E of D containing P, say $E = \{z_{j_0} = 0\}$, for some $1 \leqslant j_0 \leqslant l$, we have $ord_E\{z_j = 0\} = 0$ for $j \neq j_0$, $ord_E\{z_{j_0} = 0\} = 1$ and thus $ord_E D = 1$. On the other hand,

$$\sum_{s \in B} ord_E s = \sum_{\vec{i}} i_{j_0} = \frac{1}{n+1} \sum_{\vec{i}} (i_0 + \cdots + i_n) = \frac{q}{n+1} \binom{q+n}{n} = \frac{q}{n+1} \dim V_1,$$

where, in above, the sum is taken for all $\vec{i} = (i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = q$, and we used the fact that the number of choices of $\vec{i} = (i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = q$ is $\binom{q+n}{n}$. Thus we can take $\mu_1 = \frac{q}{n+1} \dim V_1$, and hence,

$$\mathrm{Nev}(D) \leqslant \frac{\dim V_1}{\mu_1} = \frac{n+1}{q}.$$

A similar but more sophisticated argument (see [Ru04] and [Ru15]) shows that $\mathrm{Nev}(D) \leqslant \frac{n+1}{q}$ is still valid for $X = \mathbb{P}^n$ and $D = D_1 + \cdots + D_q$, where $D_1, \cdots, D_q$ are hypersurfaces of same degree, located in general position.

The Main Result of this paper is as follows.

Main Theorem.

(a) *Let k be a number field and M_k be the set of places on k. Let $S \subset M_k$ be a finite set of places containing all archimedean ones. Let X be a normal projective variety and D be an effective Cartier divisor on X, both defined over k (we further assume that all irreducible components of D are Cartier divisors). Then, for every $\varepsilon > 0$, the inequality*

$$m_S(x, D) \leqslant (Nev(D) + \epsilon)\, h_D(x), \tag{3}$$

holds for all $x \in X(k)$ outside a Zariski closed subset Z of X.

(b) *If X is projective but not normal. Let $\pi : \tilde{X} \to X$ be the normalization of X. Then, for every $\varepsilon > 0$, the inequality*

$$m_S(x, D) \leqslant (Nev(\pi^* D) + \epsilon)\, h_D(x), \tag{4}$$

holds for all $x \in X(k)$ outside a Zariski closed subset Z of X.

Remark 1.5. From the definition, we have $m_S(x, D) + N_S(x, D) = h(x) + O(1)$ where $N_S(x, D) = \sum_{v \notin S} \lambda_{S,v}(x)$. Therefore (3) automatically holds if $\mathrm{Nev}(D) \geqslant 1$. On the other hand, if $\mathrm{Nev}(D) < 1$, then the set of (D, S)-integral points of $X(\bar{k}) \backslash D$ (See Definition 4.3 below or Definition 2.7 in [HR12]) is always degenerate if D is big (similar to the argument in Corollary 2.10 in [HR12]).

Consequences of the Main Theorem can be derived by computing the Nevanlinna constant Nev(D) in various situations. For instance, combining Example 1.4 and the Main Theorem will recover Schmidt's subspace theorem. The fact that Nev$(D) \leqslant \frac{n+1}{q}$ still holds for $D = D_1 + \cdots + D_q$, where $D_1, \cdots, D_q$ are hypersurfaces of same degree in general position in $\mathbb{P}^n$, implies the result of Corvaja-Zannier [CZ04a]. Further computation (see Proposition 3.1) will yield the Theorem A as well. In addition, the Main Theorem will derive various results of this type for the divisor $D = D_1 + \cdots + D_q$, where (multiples of) $D_1, \ldots, D_q$ are not necessarily linearly equivalent (see Section 4).

2. The Proof of the Main Theorem

Throught the rest of the paper, we always assume, unless otherwise indicated, that X is a normal projective variety and D is an effective Cartier divisor on X, both defined over the given number field k. The proof the Main Theorem in our paper is based on the following proposition.

Proposition 2.1. *Let $S \subset M_k$ be a finite set of places containing all archimedean ones. Assume that there exists a positive number $\mu > 0$ and a linear subspace $V \subset H^0(X, \mathcal{O}(D))$ with $\dim V \geqslant 2$, such that for all $P \in \operatorname{supp} D$, there exists a basis B of V with*

$$\frac{1}{\operatorname{ord}_E(D)} \sum_{s \in B} \operatorname{ord}_E(s) \geqslant \mu$$

for all irreducible component E of D passing through P. Then for every $\varepsilon > 0$, the inequality

$$m_S(x, D) \leqslant \left(\frac{\dim V}{\mu} + \epsilon \right) h_D(x),$$

holds for all $x \in X(k)$ outside a Zariski closed subset Z of X.

To prove the Proposition, we introduce the following definition.

Definition 2.2. Let $\mu > 0$. The divisor D is said to have *μ-growth with respect to V*, where $V \subset H^0(X, \mathcal{O}(D))$ is a subspace with $\dim V \geqslant 2$, if for all $P \in \operatorname{supp} D$ there exists a basis B of V such that

$$\frac{1}{\operatorname{ord}_E(D)} \sum_{s \in B} \operatorname{ord}_E(s) \geqslant \mu \tag{5}$$

for all irreducible component E of D passing through P.

Following [Vojcm], we derive the following functoriality property of μ-growth divisors with respect to a subspace $V \subset H^0(X, \mathcal{O}(D))$ (compare with [Vojcm], Proposition 20.2).

Proposition 2.3. *Let X' and X be two normal projective varieties over fields L and k, respectively, with $L \supset k$, and let $\phi : X' \to X$ be a morphism of schemes such that the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ Spec\ L & \longrightarrow & Spec\ k \end{array}$$

commutes. Let D be an effective Cartier divisor on X whose support doesn't contain $\phi(X')$, and let $D' = \phi^* D$ be the corresponding divisor on X'. Assume that the natural map

$$\alpha : H^0(X, \mathcal{O}(D)) \otimes_k L \to H^0(X', \mathcal{O}(D')) \tag{6}$$

is an isomorphism. If D has μ-growth with respect to a subspace $V \subset H^0(X, \mathcal{O}(D))$, then D' also has μ-growth with respect to the subspace $V' \subset H^0(X', \mathcal{O}(D'))$ with $V' = \alpha(V \otimes_k L)$.

Proof. Let P' be a point on X' with $\phi(P') \in \operatorname{supp} D$. Since D has μ-growth with respect to a subspace $V \subset H^0(X, \mathcal{O}(D))$, for the point $\phi(P') \in \operatorname{supp} D$, let B a basis of V such that

$$\sum_{s \in B} \operatorname{ord}_E s \geqslant \mu \operatorname{ord}_E D \tag{7}$$

for all irreducible component E of D passing through $\phi(P')$. Let $B' = \{\alpha(s \otimes 1) \mid s \in B\}$; it is a basis for V'. Let E' be an irreducible component of D' passing through P'. For each irreducible component E of D passing through $\phi(P')$, let n_E be the multiplicity of E' in $\phi^* E$. Then, for $s \in H^0(X, \mathcal{O}(D))$,

$$\operatorname{ord}_{E'} \alpha(s \otimes 1) \geqslant \sum_E n_E \operatorname{ord}_E s$$

and

$$\operatorname{ord}_{E'}(D') = \operatorname{ord}_{E'} \alpha(s_D \otimes 1) = \sum_E n_E \operatorname{ord}_E s_D = \sum_E n_E \operatorname{ord}_E D,$$

where s_D is the canonical section, i.e. $(s_D) = D$. Note that the strictness in the first inequality may arise if E' is exceptional for ϕ and s vanishes along prime divisors containing $\phi(E')$ that do not occur in D. Thus, using (7),

$$\sum_{s' \in B'} \operatorname{ord}_{E'} s' \geqslant \sum_E n_E \sum_{s \in B} \operatorname{ord}_E s \geqslant \mu \sum_E n_E \operatorname{ord}_E D = \mu \operatorname{ord}_{E'} D'. \qquad \blacksquare$$

Corollary 2.4 (see [Vojcm], Corollary 20.3). *Let X be a normal projective variety over a field k, let D be a Cartier divisor on X, let L be a field containing k, let $X_L = X \times_k L$ with projection $\phi : X_L \to X$, and let $D_L = \phi^* D$. If D has μ-growth with respect to a subspace $V \subset H^0(X, \mathcal{O}(D))$, then D_L also has μ-growth with respect to the corresponding subspace of the same dimension.*

Proof. Note that (6) is an isomorphism because L is flat over k (see [Har77] III Prop. 9.3). Thus the corollary follows from Proposition 2.3. ■

Corollary 2.5 (see [Vojcm], Corollary 20.4). *Let $\phi : X' \to X$ be a proper birational morphism of normal projective varieties over a field, and let D be a Cartier divisor on X whose support doesn't contain $\phi(X')$. If D has μ-growth with respect to a subspace $V \subset H^0(X, \mathcal{O}(D))$, then $\phi^* D$ also has μ-growth with respect to the corresponding subspace of the same dimension.*

Proof. Note that (6) is an isomorphism because $\phi_* \mathcal{O}_{X'} = \mathcal{O}_X$ (see [Har77], proof of III Cor. 11.4 and III Remark 8.8.1). Thus the corollary follows from the Proposition 2.3. ■

We also recall some properties of Weil functions.

Lemma 2.6 (See Theorem 8.8 on page 140 in [Vojcm]). *Let X be a projective variety over a number field k. Then the following properties hold.*

(a) Additivity. *If λ_1 and λ_2 are Weil functions for Cartier divisors D_1 and D_2 on X, respectively, then $\lambda_1 + \lambda_2$ extends uniquely to a Weil function for $D_1 + D_2$.*
(b) Functoriality. *If λ is a Weil function for a Cartier divisor D on X, and if $f : X' \to X$ is a morphism of k-varieties such that $f(X') \not\subset SuppD$, then $x \mapsto \lambda(f(x))$ is a Weil function for the Cartier divisor f^*D on X'.*
(c) Normalization. *If $X = \mathbb{P}^n_k$, and if $D = \{x_0 = 0\} \subset X$ is the hyperplane at infinity, then the function*

$$\lambda_{D,v}([x_0 : \cdots : x_n]) := \log \frac{\max\{\|x_0\|_v, \ldots, \|x_n\|_v\}}{\|x_0\|_v}$$

is a Weil function for D.
(d) Uniqueness. *If both λ_1 and λ_2 are Weil functions for a Cartier divisor D on X, then $\lambda_1 = \lambda_2 + O_{M_k}(1)$ (for the definition of the M_k-boundedness, see [Lan87]).*
(e) Boundedness from below. *If D is an effective Cartier divisor and λ is a Weil function for D, then λ is bounded from below by an M_k-constant.*
(f) Principal divisors. *If D is a principal divisor (f), then $-\log \|f\|_v$ is a Weil function for D.*

Lemma 2.7 ([Lan87], Ch. 10, Prop. 3.2). *Let $\lambda_1, \ldots \lambda_n$ be Weil functions for Cartier divisors $D_1, \ldots, D_n$, respectively, on a projective variety X over a number field k. Assume that the divisors D_i are of the form $D_i = D_0 + E_i$, where D_0 is a fixed Cartier divisor and E_i are effective for all i. Assume also that*

$$\operatorname{supp} E_1 \cap \cdots \cap \operatorname{supp} E_n = \emptyset.$$

Then the function

$$\lambda(x) = \min\{\lambda_i(x) : x \notin \operatorname{supp} E_i\}$$

is defined everywhere on $(X \setminus \operatorname{supp} D_0)(M_k)$, and is a Weil function for D_0.

Finally we recall the following (generalized) version of Schmidt's Subspace Theorem from [Voj97].

Theorem 2.8. *Let k be a number field and $S \subset M_k$ be a finite set containing all archimedean places. Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n$ defined over $\bar{k}$ with corresponding Weil functions $\lambda_{H_1}, \dots, \lambda_{H_q}$. Then there exists a finite union of hyperplanes Z, depending only on $H_1, \dots, H_q$ (and not k, S), such that for any $\varepsilon > 0$,*

$$\sum_{v \in S} \max_I \sum_{i \in I} \lambda_{H_i,v}(P) \leqslant (n+1+\varepsilon)h(P)$$

holds for all but finitely many $P \in \mathbb{P}^n(k) \setminus Z$, where the maximum is taken over subsets $I \subset \{1, \dots, q\}$ such that the linear forms defining $H_i, i \in I$, are linearly independent.

Proof of Proposition 2.1. From the assumption, D has μ-growth with respect to $V \subset H^0(X, \mathcal{O}(D))$. Let $\Phi : X \to \mathbb{P}^{m-1}$ be the canonical rational map associated to V where $m = \dim V$.

We may assume that Φ is a morphism. Indeed, let X' be a desingularization of the closure of the graph of Φ. Replace X with X' and D with its pull-back. By Corollary 2.5 the pull-back still has μ-growth with respect to the corresponding vector space of the same dimension. Moreover, by functoriality of Weil functions, the corresponding Weil function and height function all keep.

Denote by σ_0 the set of all prime divisors occurring in D, so we can write

$$D = \sum_{E \in \sigma_0} \mathrm{ord}_E(D)E.$$

Let

$$\Sigma := \{\sigma \subset \sigma_0 \mid \cap_{E \in \sigma} E \neq \emptyset\}.$$

For each subset $\sigma \in \Sigma$, write

$$D = D_{\sigma,1} + D_{\sigma,2}$$

where

$$D_{\sigma,1} := \sum_{E \in \sigma} \mathrm{ord}_E(D)E, \qquad D_{\sigma,2} := \sum_{E \notin \sigma} \mathrm{ord}_E(D)E.$$

Pick a Weil function for each divisors D, $D_{\sigma,1}$ and $D_{\sigma,2}$. We first claim that there exists a M_k-constant $(C_v)_{v \in M_k}$, depending only on X and D, such that

$$\min_{\sigma \in \Sigma} \lambda_{D_{\sigma,2},v}(x) \leqslant C_v.$$

for all $x \in X(\mathbb{C}_v)$ and all $v \in M_k$ (for the definition of the M_k-constant, see [Lan87]). Indeed, the definition of the set Σ implies that

$$\cap_{\sigma \in \Sigma} \operatorname{supp} D_{\sigma,2} = \emptyset,$$

since for all $x \in X$ the set $\sigma := \{E \in \sigma_0 \mid x \in E\}$ is an element of Σ, and then $x \notin \operatorname{supp} D_{\sigma,2}$. Our claim then follows from Lemma 2.7, since Σ is a finite set.

Now for each $\sigma \in \Sigma$, since D has μ-growth with respect to V, let B_σ be a basis of V that satisfies

$$\sum_{s \in B_\sigma} \operatorname{ord}_E(s) \geqslant \mu \operatorname{ord}_E D \tag{8}$$

at some (and hence) all points $P \in \cap_{E \in \sigma} E$. Since Σ is finite, $\{B_\sigma \mid \sigma \in \Sigma\}$ is a finite collection of bases of V. Thus, the distinct hyperplanes in $\mathbb{P}^{m-1}$ corresponding to elements of the union $\cup_{\sigma \in \Sigma} B_\sigma$ is finite, say they are $H_1, \ldots, H_q$ in $\mathbb{P}^{m-1}$. Choose a Weil function $\lambda_{H_j,v}$ for each H_j, $1 \leqslant j \leqslant q$ and $v \in M_k$.

For an arbitrary $x \in X$, from the claim above, pick $\sigma \in \Sigma$ for which

$$\lambda_{D_{\sigma,2},v}(x) \leqslant C_v. \tag{9}$$

Let $J \subset \{1, \ldots, q\}$ be the subset for which $\{H_j, j \in J\}$ are the hyperplanes corresponding to the elements of B_σ. Then (8), when applying to B_σ, implies that,

$$\sum_{j \in J} \operatorname{ord}_E \Phi^* H_j \geqslant \mu \operatorname{ord}_E D$$

for all $E \in \sigma$; and therefore, by the "boundedness from below" property of the Weil functions for effective divisors,

$$\sum_{j \in J} (\operatorname{ord}_E \Phi^* H_j) \lambda_{E,v}(x) \geqslant \mu (\operatorname{ord}_E D) \lambda_{E,v}(x) + O_{M_k}(1)$$

for all $E \in \sigma$. Now, since

$$D = \sum_{E \in \sigma} (\operatorname{ord}_E D) \cdot E + D_{\sigma,2},$$

(9) gives

$$\lambda_{D,v}(x) = \sum_{E \in \sigma} (\operatorname{ord}_E D) \lambda_{E,v}(x) + O_{M_k}(1).$$

Hence

$$\begin{aligned}
\sum_{j \in J} \lambda_{H_j,v}(\Phi(x)) &\geqslant \sum_{j \in J} \sum_{E \in \sigma} (\operatorname{ord}_E \Phi^* H_j) \lambda_{E,v}(x) + O_{M_k}(1) \\
&\geqslant \mu \sum_{E \in \sigma} (\operatorname{ord}_E D) \lambda_{E,v}(x) + O_{M_k}(1) \\
&= \mu \lambda_{D,v}(x) + O_{M_k}(1).
\end{aligned}$$

Note that, since $\{H_j, j \in J\}$ are the hyperplanes corresponding to the elements of B_σ, we see that $\{H_j, j \in J\}$ are in general position. Thus, for any $x \in X$,

$$\lambda_{D,v}(x) \leqslant \frac{1}{\mu} \left(\max_J \sum_{j \in J} \lambda_{H_j,v}(\Phi(x)) + O_{M_k}(1) \right).$$

Summing over the places $v \in S$, noting that $h(\Phi(x)) = h_D(x)$, applying Theorem 2.8, we obtain Proposition 2.1. ■

Proof of the Main Theorem. (a) For every $\epsilon > 0$, from the definition of $\mathrm{Nev}(D)$ there exists N such that

$$\frac{\dim V_N}{\mu_N} < \mathrm{Nev}(D) + (\epsilon/2)$$

with some pair $\{\mu_N, V_N\}$, where $V_N \subset H^0(X, \mathcal{O}(ND))$ is a subspace with $\dim V_N \geqslant 2$ such that, for all $P \in \operatorname{supp} D$, there exists a basis B of V_N with

$$\sum_{s \in B} \mathrm{ord}_E(s) \geqslant \mu_N \, \mathrm{ord}_E(ND)$$

for all irreducible component E of D passing through P. Applying Proposition 2.1 to the divisor ND, we get that

$$m_S(r, ND) \leqslant \left(\frac{\dim V_N}{\mu_N} + (\epsilon/2)\right) h_{ND}(x) < (\mathrm{Nev}(D) + \epsilon) h_{ND}(x)$$

holds for all $x \in X(k)$ outside a Zariski closed subset Z of X. Since $m_S(x, ND) = Nm_S(x, D) + O(1), h_{ND}(x) = Nh_D(x) + O(1)$, this proves (a).

For (b), for any $x \in X$, take a normal open subscheme U of X such that π is an isomorphism over U. Then we apply (a) to $\tilde{X}$ (which is normal) and π^*D (or more precisely on $\pi^{-1}(U)$), we get, for every $\epsilon > 0$,

$$m_S(\pi^*x, \pi^*D) \leqslant (\mathrm{Nev}(\pi^*D) + \epsilon) \, h_{\pi^*D}(\pi^*x).$$

Now, since $\pi : \tilde{X} \to X$ is a morphism, the "functoriality property" for the Weil function and height function implies that $m_S(\pi^*, \pi^*D) = m_S(x, D) + O(1)$, and $h_{\pi^*D}(\pi^*x) = h_D(x) + O(1)$. Thus (b) holds. ■

3. Computation of the Nevanlinna constants

In this section, we compute the Nevanlinna constant $\mathrm{Nev}(D)$ in various cases in order to apply the Main Theorem. We note that all results in this section have been contained in [Ru15] (by replacing the field of complex numbers $\mathbb{C}$ with the number field k). We include the proofs here for sake of completeness.

Proposition 3.1. *Let X be a normal projective variety of dimension n and $D = D_1 + \cdots + D_q$ be a sum of very ample effective Cartier divisors in general position on X, both defined over a number field k. We further assume that $D_i \sim D_j$ for $1 \leqslant i, j \leqslant q$. Then*

$$Nev(D) \leqslant \frac{\dim X + 1}{q}.$$

Proof. From assume that $D_i \sim D_j$ for $1 \leqslant i, j \leqslant q$ and D_j is very ample for all $j = 1, \ldots, q$, we write $D_i \sim A$ for $i = 1, \ldots, q$ where A is a (fix) very ample divisor. Then $\phi_A : X \to \mathbb{P}^u$, the canonical map associated to A, is an embedding. Let $Q_1, \ldots, Q_q$ be the linear forms in $(u+1)$-variables such that $D_i = \phi_A^*\{Q_i = 0\}$. Let

$$\psi : X \to \mathbb{P}^{q-1}, \quad x \mapsto [Q_1(\phi_A(x)), \ldots, Q_q(\phi_A(x))].$$

Let $Y := \psi(X) \subset \mathbb{P}^{q-1}$. By the general position assumption for $D_1, \ldots, D_q$, ψ is a finite morphism from X to Y.

On $\mathbb{P}^{q-1}$, we have for all $N \in \mathbb{N}$ a short exact sequence

$$0 \to \mathcal{I}_Y(N) \to \mathcal{O}_{\mathbb{P}^{q-1}}(N) \to \mathcal{O}_Y(N) \to 0.$$

The beginning of the corresponding long exact sequence reads

$$\begin{aligned} 0 &\to H^0(\mathbb{P}^{q-1}, \mathcal{I}_Y(N)) \to H^0(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(N)) \xrightarrow{\tau} H^0(Y, \mathcal{O}_Y(N)) \\ &\to H^1(\mathbb{P}^{q-1}, \mathcal{I}_Y(N)) \end{aligned}$$

where τ denotes the restriction map. Since $H^1(\mathbb{P}^{q-1}, \mathcal{I}_Y(N)) = 0$ for N big enough, we have, for N big enough,

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(N)) &\cong H^0(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(N))/\ker(\tau) \\ &\cong H^0(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(N))/H^0(\mathbb{P}^{q-1}, \mathcal{I}_Y(N)) \\ &\cong k[Y_0, \ldots, Y_{q-1}]_N/(I_Y)_N, \end{aligned} \tag{10}$$

where $k[Y_0, \ldots, Y_{q-1}]_N$ denotes the set of those homogeneous polynomials of degree N and $(I_Y)_N$ denotes the set of those homogeneous polynomials of degree N vanishing on Y. We now estimate the Nevanlinna constant by letting, for $\tilde{N} = \frac{N}{q}$ where N is a multiple of q and big enough,

$$V_{\tilde{N}} := \psi^* H^0(Y, \mathcal{O}_Y(N)) \subset H^0(X, \mathcal{O}(\frac{N}{q}D)) = H^0(X, \mathcal{O}(\tilde{N}D)).$$

Since $\psi : X \to Y$ is a finite surjective morphism, by using (10)

$$\dim(V_{\tilde{N}}) = \dim H^0(Y, \mathcal{O}_Y(N)) = \dim\left(k[Y_0, \ldots, Y_{q-1}]_N/(I_Y)_N\right) = H_Y(N),$$

where $H_Y(N)$ is the Hilbert function of Y.

To continue, let $P \in \operatorname{supp} D$. The condition that $D_1, \ldots, D_q$ are in general position implies that $P \in \cap_{t=1}^{l} \left(\phi_{N_0 A}^*\{Q_{i_t} = 0\}\right)$ for some distinct $Q_{i_1}, \ldots, Q_{i_l} \in \{Q_1, \ldots, Q_q\}$ with $l \leqslant n$. Without loss of generality, we can assume that $l = n$ (otherwise we just add more polynomials). Let $\vec{c} = (c_1, \ldots, c_q)$ be the q-vector whose i_j-th entry ($1 \leqslant j \leqslant n$) is 1, with all other entries being 0. Let $\vec{y}^{\vec{a}^{(1)}}, \ldots, \vec{y}^{\vec{a}^{(H_Y(N))}}$ be monomials such that their equivalence classes in $k[Y_0, \ldots, Y_{q-1}]_N/(I_Y)_N$ give a basis and such that

$$S_Y(N, \vec{c}) = \sum_{i=1}^{H_Y(N)} \vec{a}^{(i)} \bullet \vec{c},$$

where $S_Y(N,\vec{c})$ is the N-th Hilbert weight and the bullet denotes the usual dot product. Recall that the N-th Hilbert weight of Y with respect to the weight $\vec{c}$ is given by

$$S_Y(N,\vec{c}) = \max \sum_{i=1}^{H_Y(N)} \vec{a}^{(i)} \bullet \vec{c},$$

where the maximum is taken over all sets of monomials $\vec{y}^{\vec{a}^{(1)}}, \dots, \vec{y}^{\vec{a}^{(H_Y(N))}}$ whose residue class modulo I_Y form a basis of $k[Y_0,\dots,Y_{q-1}]_N/(I_Y)_N$. For $\nu = 1,\dots,H_Y(N)$, and N a positive multiple of q, let

$$s_\nu = (Q_1^{a_1^{(\nu)}} \dots Q_q^{a_q^{(\nu)}})|_{\phi_{N_0A}(X)}.$$

These functions form a basis for $V_{\tilde{N}}$ understood as a subspace of $H^0(X,\mathcal{O}(\tilde{N}D))$.

We recall the following key lemma which is due to J.-H. Evertse and R. Ferretti (see [EF08], also see the combination of Theorem 2.1 and Lemma 3.2 in [Ru09]).

Lemma 3.2. *Let $Y \subset \mathbb{P}^l$ be an algebraic variety of dimension n and degree $\triangle$. Let $m > \triangle$ be an integer and let $\vec{c} = (c_0,\dots,c_l) \in \mathbb{R}_{\geqslant 0}^{l+1}$. Let $\{i_0,\dots,i_n\}$ be a subset of $\{0,\dots,l\}$ such that*

$$Y \cap \{y_{i_0} = 0,\dots,y_{i_n} = 0\} = \emptyset.$$

Then

$$\frac{1}{mH_Y(m)} S_Y(m,\vec{c}) \geqslant \frac{1}{(n+1)}(c_{i_0} + \dots + c_{i_n}) - \frac{(2n+1)\triangle}{m} \cdot \left(\max_{i=0,\dots,l} c_i\right).$$

We now continue our proof. For any irreducible component E in D with $P \in \operatorname{supp} E$. We assume that E is contained in $\phi_{N_0A}^*\{Q_{j_0} = 0\}$. With our chosen $\vec{c}$ and $\vec{a}^{(i)}$, using Lemmas 3.2 (notice the condition that $D_1,\dots,D_q$ are in general position on X), and the symmetry property of the $\vec{a}^{(1)},\dots,\vec{a}^{(H_Y(N))}$,

$$\begin{aligned}\frac{1}{\operatorname{ord}_E D}\sum_\nu \operatorname{ord}_E s_\nu &= \sum_{\nu=1}^{H_Y(N)} a_{j_0}^{(\nu)} = \frac{1}{n}\sum_{\nu=1}^{H_Y(N)} \vec{a}^{(\nu)} \bullet \vec{c} \\ &= \frac{1}{n} S_Y(N,\vec{c}) \geqslant \frac{1}{n}\frac{1}{n+1} N H_Y(N)(\sum_{j=1}^{n} c_{i_j}) + O(H_Y(N)) \\ &= \frac{1}{n+1} N(H_Y(N) + o(H_Y(N))).\end{aligned}$$

Thus

$$\begin{aligned}\sum \operatorname{ord}_E s_\nu &\geqslant \frac{q}{n+1}(H_Y(N) + o(H_Y(N)) \operatorname{ord}_E\left(\frac{N}{q}D\right) \\ &= \frac{q}{n+1}(H_Y(N) + o(H_Y(N)) \operatorname{ord}_E(\tilde{N}D).\end{aligned}$$

Therefore, from the definition of $\mathrm{Nev}(D)$ (see (1)), we have

$$\begin{aligned}\mathrm{Nev}(D) &\leqslant \liminf_{\tilde{N}\to+\infty} \frac{\dim V_{\tilde{N}}}{\frac{q}{n+1}(H_Y(N)+o(H_Y(N))} \\ &= \liminf_{N\to+\infty} \frac{H_Y(N)}{\frac{q}{n+1}(H_Y(N)+o(H_Y(N))} = \frac{n+1}{q}.\end{aligned}$$

Proposition 3.1 is thus proved. ■

Under the assumptions in Theorem A, there exist an ample divisor A on X and positive integers d_j such that $D_j \sim d_j A$. Thus Theorem A can be obtained by applying Proposition 3.1 together with the Main Theorem to the divisors $\frac{N_0}{d_j} D_j$ with N_0 being a positive integer divisible by $d_i, 1 \leqslant i \leqslant q$, and such that $N_0 A$ is very ample.

Next, we consider the case that the given divisors (more precisely the multiple of the divisors) are not necessarily linearly equivalent on X. Denote by $h^0(D) := \dim H^0(X, \mathcal{O}(D))$.

Definition 3.3. Divisors $D_1, \ldots, D_q$ with $q > l$ on a projective variety X are said to be *in l-subgeneral position* if any $\{i_1, \ldots, i_{l+1}\} \subset \{1, \ldots, q\}$, $\operatorname{supp} D_{i_1} \cap \cdots \cap \operatorname{supp} D_{i_{l+1}} = \emptyset$.

Following Levin [Lev09], we introduce the concept of "equidegree" for the divisor $D = D_1 + \cdots + D_q$, where $D_1, \ldots, D_q$ are Cartier divisors.

Definition 3.4. Let X be a projective variety of dimension n. Let $D = \sum_{j=1}^q D_j$ be a sum of effective Cartier divisors on X. We say that D *has equidegree with respect to* $D_1, \ldots, D_q$ if

$$D_i.D^{n-1} = \frac{1}{q} D^n, \qquad \text{for } \ i = 1, \ldots, q.$$

We say that D *is equidegreelizable with respect to* $D_1, \ldots, D_q$ if there are some real numbers $r_i > 0$ such that $D' := \sum_{j=1}^q r_j D_j$ has equidegree with respect to $r_1 D_1, \ldots, r_q D_q$ (where we extend intersections to $Div X \otimes \mathbb{R}$ in the canonical way).

Obviously, if $D_1, \ldots, D_q$ are linearly equivalent on X, then $D = \sum_{j=1}^q D_j$ has equidegree with respect to $D_1, \ldots, D_q$.

Lemma 3.5 ([Lev09], Lemma 9.7). *Let X be a projective variety and $D_1, \ldots, D_q$ be big and nef Cartier divisors on X. Then $D := \sum_{i=1}^q D_i$ is equidegreelizable with respect to $D_1, \ldots, D_q$.*

Lemma 3.6 (See [Laz04], Corollary1.4.41). *Suppose D is a nef Cartier divisor on a projective variety X with* $\dim X = n$. *Then*

$$h^0(ND) = \frac{D^n}{n!} N^n + O(N^{n-1}). \tag{11}$$

If particular, $D^n > 0$ if and only if D is big.

We recall a lemma from Autissier [Aut1].

Lemma 3.7 (See [Aut1], Lemma 2.1). *Suppose E is a big and base-point free Cartier divisor on a projective variety X with $\dim X = n \geqslant 2$, and F be a nef Cartier divisor on X such that $F - E$ is also nef. Let $\beta > 0$ be a positive real number. Then for any positive integers N, m with $1 \leqslant m \leqslant \beta N$, we have*

$$h^0(NF - mE) \geqslant \frac{F^n}{n!}N^n - \frac{F^{n-1}.E}{(n-1)!}N^{n-1}m + \frac{(n-1)F^{n-2}.E^2}{n!}N^{n-2}\min\{m^2, N^2\} + O(N^{n-1})$$

where O depends on β.

Proposition 3.8. *Let X be a normal projective variety with $\dim X = n \geqslant 2$. Let $D = D_1 + \cdots + D_q$ be a sum of big, nef and base-point free Cartier divisors on X in l-subgeneral position. Assume that, for any $\epsilon > 0$, there are positive integers $a_i > 0$ such that*

$$(a_iD_i).D'^{n-1} \leqslant \frac{1}{q}D'^n + \epsilon \qquad \text{for all } \ i = 1, \dots, q$$

where $D' := \sum_{i=1}^q a_iD_i$. Then, for

$$\epsilon < \min\left\{D'^n, (n-1)\min\left\{\frac{2nl}{3q}, \frac{q^2}{4n^2l^2}\right\}\left(\min_{1\leqslant j\leqslant q} a_j^2(D'^{n-2}.D_j^2)\right)\right\},$$

we have

$$Nev(D') < \frac{2l \dim X}{q}.$$

Proof. For $P \in \operatorname{supp} D'$, denote by $D'_P = \sum_{i:P\in \operatorname{supp} D_i} a_iD_i$, and consider the filtration, for N big enough,

$$V_N := H^0(X, \mathcal{O}(ND')) \supset H^0(X, \mathcal{O}(ND' - D'_P)) \supset H^0(X, \mathcal{O}(ND' - 2D'_P)) \supset \cdots \supset H^0(X, \mathcal{O}(ND' - MD'_P)) \supset H^0(X, \mathcal{O}(ND' - (M+1)D'_P)) = \{0\}.$$

Choose a basis for V_N according to the above filtration. With this basis B, we compute μ appeared in (2) of the definition of the μ-growth. Let E be an irreducible component of D' such that $P \in E$. Notice that $\operatorname{ord}_E s \geqslant m \operatorname{ord}_E D'$ for any $s \in H^0(X, \mathcal{O}(ND' - mD'_P))$, therefore we have

$$\frac{1}{\operatorname{ord}_E D'}\sum_{s\in B}\operatorname{ord}_E s \geqslant \sum_{m=1}^{\infty} m\left(h^0(ND' - mD'_P) - h^0(ND' - (m+1)D'_P\right) \tag{12}$$
$$= \sum_{m=1}^{\infty} h^0(ND' - mD'_P).$$

Applying Lemma 3.7 with $F = D'$, $E = D'_P$ and $\beta = \frac{D'^n}{nD'^{n-1}.D'_P}$, and denote $A := (n-1)D'^{n-2}.D'^2_P$, yields

$$\begin{aligned}
&\sum_{m=1}^{\infty} h^0(ND' - mD'_P) \qquad (13)\\
&\geqslant \sum_{m=1}^{[\beta N]} \left(\frac{D'^n}{n!} N^n - \frac{D'^{n-1}.D'_P}{(n-1)!} N^{n-1} m + \frac{A}{n!} N^{n-2} \min\{m^2, N^2\} \right) + O(N^n)\\
&= \left(\frac{D'^n}{n!}\beta - \frac{D'^{n-1}.D'_P}{(n-1)!} \frac{\beta^2}{2} + \frac{A}{n!} g(\beta) \right) N^{n+1} + O(N^n)\\
&\geqslant \left(\frac{\beta}{2} + \frac{A}{D'^n} g(\beta) \right) D'^n \frac{N^{n+1}}{n!} + O(N^n)\\
&= \left(\frac{\beta}{2} + \alpha \right) N h^0(ND') + O(N^n)
\end{aligned}$$

where $\alpha := \frac{A}{D'^n} g(\beta)$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the function given by $g(x) = \frac{x^3}{3}$ if $x \leqslant 1$ and $g(x) = x - \frac{2}{3}$ for $x \geqslant 1$. Now from the assumptions that

$$(a_i D_i).D'^{n-1} \leqslant \frac{1}{q} \left(D'^n + \epsilon \right),$$

and that the intersection of any $l+1$ distinct D_j is empty, we have

$$D'^{n-1}.D'_P = D'_P.D'^{n-1} = \sum_{i, P \in \operatorname{supp} D_i} (a_i D_i).D'^{n-1} \leqslant \frac{l}{q} \left(D'^n + \epsilon \right)$$

Hence

$$\beta = \frac{D'^n}{nD'^{n-1}.D'_P} \geqslant \frac{q}{nl} \frac{D'^n}{(D'^n + \epsilon)}. \qquad (14)$$

Thus, from (12) and (14),

$$\begin{aligned}
\sum_{s \in B} \operatorname{ord}_E s &\geqslant \left(\left(\frac{q}{2nl} \left(\frac{D'^n}{D'^n + \epsilon} \right) + \alpha \right) N h^0(ND') + O(N^n) \right) \operatorname{ord}_E D'\\
&\geqslant \left(\left(\frac{q}{2nl} \left(\frac{D'^n}{D'^n + \epsilon} \right) + \alpha \right) h^0(ND') + O(N^{n-1}) \right) \operatorname{ord}_E (ND')
\end{aligned}$$

From the definition of $\operatorname{Nev}(D')$, we have

$$\begin{aligned}
\operatorname{Nev}(D') &\leqslant \lim \inf_{N \to +\infty} \frac{h^0(ND')}{\left(\frac{q}{2nl} \left(\frac{D'^n}{D'^n + \epsilon} \right) + \alpha \right) h^0(ND') + O(N^{n-1})}\\
&= \frac{2nl}{q} \left(\frac{D'^n + \epsilon}{D'^n + \frac{2nl\alpha}{q} (D'^n + \epsilon)} \right) < \frac{2nl}{q} \left(\frac{D'^n + \epsilon}{D'^n + \frac{2nl\alpha}{q} D'^n} \right),
\end{aligned}$$

so if

$$\epsilon < \frac{2nl\alpha}{q} D'^n, \tag{15}$$

then

$$\mathrm{Nev}(D') < \frac{2nl}{q}.$$

We now find the lower bound of $\alpha D'^n$. First notice that for

$$\epsilon < D'^n, \tag{16}$$

we have, from (14) $\beta \geqslant \frac{q}{2nl}$ and thus

$$g(\beta) \geqslant \min\left\{\frac{1}{3}, \frac{q^3}{8n^3l^3}\right\},$$

so

$$\begin{aligned}\frac{2nl\alpha}{q} D'^n &= \frac{2nl}{q}(n-1)D'^{n-2}.D_P'^2 g(\beta) \\ &\geqslant (n-1)\min\left\{\frac{2nl}{3q}, \frac{q^2}{4n^2l^2}\right\}\left(\min_{1\leqslant j\leqslant q} a_j^2(D'^{n-2}.D_j^2)\right).\end{aligned} \tag{17}$$

Thus if we let, from (16) and (17),

$$\epsilon < \min\left\{D'^n, (n-1)\min\left\{\frac{2nl}{3q}, \frac{q^2}{4n^2l^2}\right\}\left(\min_{1\leqslant j\leqslant q} a_j^2(D'^{n-2}.D_j^2)\right)\right\},$$

then (15) is satisfied. This finishes the proof. ■

4. Important consequences of the Main Theorem

In this section, as an important consequences of the Main Theorem, we establish a Diophantine inequality for divisors which are not necessarily linear equivalent.

Theorem 4.1. *Let k be a number field and $S \subset M_k$ be a finite set containing all archimedean places. Let X be a normal projective variety with $\dim X \geqslant 2$ and $D = D_1 + \cdots + D_q$ be a sum of big and nef Cartier divisors in l-subgeneral position on X, both defined over k. Let $r_i > 0$ be real numbers such that $D' := \sum_{i=1}^q r_i D_i$ is equidegree (such numbers exist due to Lemma 3.5). We further assume that the linear system $|ND|$ is base-point free for $N \geqslant N_0$ where $D = D_1 + \cdots + D_q$. Then, for $\varepsilon_0 > 0$ small enough (which depends explicitly only on the divisors $r_1D_1, \ldots, r_qD_q$),*

$$\sum_{j=1}^q r_j m_S(x, D_j) < \left(\frac{2l \dim X}{q} - \epsilon_0\right)\left(\sum_{j=1}^q r_j h_{D_j}(x)\right),$$

holds for all $x \in X(k)$ outside a Zariski closed subset Z of X.

Remark: The condition of nefness of D_j for $i = j, \dots, q$ in the above theorems is only used to guarantee that $D = \sum_{i=1}^q D_i$ is equidegreelizable with respect to $D_1, \dots, D_q$ by Lemma 3.5. In the case when we study the degeneracy of (S, D)-integral points of $X \backslash D$, by doing a blowing up, the smoothness condition (or the normal condition) of X, as well as the nefness condition of $D_j, 1 \leqslant j \leqslant q$, can all be removed. The following is the exact statement.

Lemma 4.2 (Lemma 9.10 in [Lev09]). *Let X be a projective variety. Let $D = \sum_{j=1}^q D_j$ be a sum of effective Cartier divisors on X. Then there exists a nonsingular projective variety X', a birational morphism $\pi : X' \to X$, and a divisor $D' = \sum_{j=1}^q D'_j$ on X' such that $SuppD'_j \subset Supp\pi^* D_j$ for all j, every irreducible component of D' is nonsingular, $|D'_j|$ is base-point free for all j (in particular D'_j is nef), and $\kappa(D'_j) = \kappa(D_j) = \dim \Phi_{D'_j}(X')$ for all j (where $\kappa(D_j)$ is the Kodaira dimension of D_j). Also, if X and D are defined over a number field, then X', D', and π are defined over some number field.*

Definition 4.3. Let $S \subset M_k$ be a finite set containing the archimedean places. Let $R \subset X(k) \backslash \operatorname{supp}(D)$. The set R is defined to be *a (D, S)–integral set of points* if there exists a global Weil function $\lambda_{D,v}$ such that

$$\lambda_{D,v}(x) \leqslant 0 \qquad \text{for } \ \forall x \in R \ \text{ and } \ \forall v \notin S.$$

Note that, if set R is a (D, S)–integral set of points, then

$$m_S(x, D) = h_D(x) + O(1) \qquad \text{for } \ \forall x \in R.$$

Recall that divisors $D_1, \dots, D_q$ (with $q > n$) are said to be in general position on a variety X with $\dim X = n$ if the intersection of the support of any $n+1$ distinct D_i on X is empty. Let $Y \subset X$. Then it is obvious that, if $D_1, \dots, D_q$ are in general position on X and D_i $(i = 1, \dots, q)$ doesn't contain any component of Y, then the intersection any $n+1$ distinct D_i on Y is still empty, so $D_1, \dots, D_q$ are in n-subgeneral position on Y. Thus, from the above discussions, we have the following Corollary.

Corollary 4.4. *Let X be a projective variety with $\dim X = n \geqslant 2$, and $D = \sum_{j=1}^q D_j$ be a sum of big Cartier divisors on X, located in general position on X, both defined over k. If $q \geqslant 2n^2$, then any (D, S)–integral set of points of $X(k) \backslash SuppD$ is finite.*

For the related results, see [RuW91].

Although in general the positive real numbers $r_1, \dots r_q$ appeared in Theorem 4.1 may be hard to compute explicitly, in the special case that $D_i \sim d_i A$ for $i = 1, \dots, q$, where A is a fixed big divisor, it is obvious that

$$D' = \frac{1}{d_1} D_1 + \cdots + \frac{1}{d_q} D_q$$

is equi-degree. So we can indeed take $r_1 = (1/d_1), \dots, r_q = (1/d_q)$. Also, from the definition, $h_{D'}(x) = q h_A(x)$, thus Theorem 4.1 applying to above D' (i.e. taking $r_1 = (1/d_1), \dots, r_q = (1/d_q)$) implies the following Corollary.

Corollary 4.5. *Let X be a projective variety with* $\dim X = n \geqslant 2$ *and let* $D_1, \ldots, D_q$ *be effective ample Cartier divisors on* X*, in* l*-subgeneral position on* X*, both defined over* k*. We further assume that there are positive integers* $d_i > 0$ *such that* $D_i \sim d_i A$*, where* A *is an effective Cartier divisor on* X*. Then, for* $\epsilon_0 > 0$ *small enough, the inequality*

$$\sum_{j=1}^{q} \frac{1}{d_j} m_S(x, D_j) \leqslant (2ln - \epsilon_0) h_A(x)$$

holds for all x *outside a Zariski closed subset* Z *of* $X(k)$*.*

In particular, if we assume that $D_1, \ldots, D_q$ *are in general position on* X *(instead of* l*-subgeneral position), then, for* $\epsilon > 0$ *small enough, the inequality*

$$\sum_{j=1}^{q} \frac{1}{d_j} m_S(x, D_j) \leqslant (2n^2 - \epsilon_0) h_A(x)$$

holds for all $x \in X(k)$ *except for a finite number of points.*

Proof. When X is normal, it is derived from Theorem 4.1 above. If X is not normal then we consider the normalization $\pi : \tilde{X} \to X$ and the divisors $\pi^* A$ and $\pi^* D_i$ for all i. Notice that $\pi^* D_i, 1 \leqslant i \leqslant q$, are still in l-subgeneral position, and, since A is ample, $\pi^* A$ is big and the linear system $|\pi^* A|$ is base-point free for N large enough, so again Theorem 4.1 implies our result. ∎

Proof of Theorem 4.1. From the functoriality and additivity of Weil functions and height functions and by replacing D with $N_0 D$ if necessary, we can assume that $N_0 = 1$. From the assumption, we have, for $j = 1, \ldots, q$,

$$(r_j D_j).(D')^{n-1} = \frac{1}{q} D'^n,$$

where $D' = r_1 D_1 + \cdots + r_q D_q$. For small $0 < \delta_1 < 1$ which will be chosen later (see (19)), choose rational numbers $a_j, 1 \leqslant j \leqslant q$, such that

$$|a_j - r_j| \leqslant \min \left\{ \frac{\delta_1}{4} \left(\min_{1 \leqslant j \leqslant q} r_j \right), \ \frac{q\delta_1}{8nl} \left(\min_{1 \leqslant j \leqslant q} r_j \right) \right\},$$

and, for $j = 1, \ldots, q$,

$$a_j D_j . \left(\sum_{i=1}^{q} a_i D_i \right)^{n-1} < \frac{1}{q} \left(\sum_{i=1}^{q} a_i D_i \right)^n + \delta_2,$$

where δ_2 will be chosen below (see (18)). Let $\tilde{D} = \sum_{j=1}^{q} da_j D_j$, where d is the product of the denominators of $a_1 \ldots, a_q$ so da_j is an integer for $1 \leqslant j \leqslant q$. Notice that

$$\tilde{D}^n \geqslant \left(\sum_{j=1}^{q} da_j D_j \right)^n \geqslant \frac{d^n}{2^n} \left(\sum_{j=1}^{q} r_j D_j \right)^n \geqslant \frac{1}{2^n} D'^n,$$

and, similarly, for any $P \in \mathrm{Supp} D$,

$$\min_{1\leqslant j\leqslant q} a_j^2(\tilde{D}^{n-2}.D_j^2) \geqslant \frac{1}{2^n}\min_{1\leqslant j\leqslant q} r_j^2({D'}^{n-2}.D_j^2).$$

So if we choose

$$\delta_2 < \min\left\{\frac{1}{2^n}D'^n, (n-1)\min\left\{\frac{2nl}{3q}, \frac{q^2}{4n^2l^2}\right\}\frac{1}{2^n}\left(\min_{1\leqslant j\leqslant q} r_j^2(D'^{n-2}D_j^2)\right)\right\}, \quad (18)$$

then, by applying Proposition 3.8 to $\tilde{D}$, we have

$$\mathrm{Nev}(\tilde{D}) < \frac{2nl}{q}.$$

Choose δ_1 small enough, so that

$$\mathrm{Nev}(\tilde{D}) < \frac{2nl}{q} - \frac{3\delta_1}{2}. \quad (19)$$

Thus, applying the Main Theorem to the divisor $\tilde{D}$ (with ϵ is taken as $\epsilon < \frac{2nl}{q} - \frac{3\delta_1}{2} - \mathrm{Nev}(\tilde{D})$), we get

$$m_S(x,\tilde{D}) \leqslant \left(\frac{2nl}{q} - \frac{3\delta_1}{2}\right) h_{\tilde{D}}(x) \quad \|,$$

i.e.

$$\sum_{j=1}^q a_j m_S(x, D_j) \leqslant \left(\frac{2nl}{q} - \frac{3\delta_1}{2}\right)\left(\sum_{j=1}^q a_j h_{D_j}(x)\right) \quad \|,$$

here we use $\|$ to denote that the inequality holds for all $x \in X(k)$ outside a Zariski closed subset Z of X. Therefore

$$\begin{aligned}
\sum_{j=1}^q r_j m_S(x,D_j) &\leqslant \sum_{j=1}^q a_j m_S(x,D_j) + \left(\frac{\delta_1}{4}\min_{1\leqslant j\leqslant q} r_j\right)\left(\sum_{j=1}^q m_S(x,D_j)\right)\\
&\leqslant \left(\frac{2nl}{q} - (3\delta_1/2)\right)\left(\sum_{j=1}^q a_j h_{D_j}(x)\right) + \left(\frac{\delta_1}{4}\min_{1\leqslant j\leqslant q} r_j\right) h_D(x) \quad \|\\
&\leqslant \left(\frac{2nl}{q} - (3\delta_1/2)\right)\left(\sum_{j=1}^q r_j h_{D_j}(x)\right)\\
&\quad + (\delta_1/4)(\min_{1\leqslant j\leqslant q} r_j) h_D(x) + \left(\frac{\delta_1}{4}\min_{1\leqslant j\leqslant q} r_j\right) h_D(x) \quad \|\\
&= \left(\frac{2nl}{q} - (3\delta_1/2)\right)\left(\sum_{j=1}^q r_j h_{D_j}(x)\right) + (\delta_1/2)(\min_{1\leqslant j\leqslant q} r_j) h_D(x) \quad \|.
\end{aligned}$$

Now

$$\sum_{j=1}^{q} r_j h_{D_j}(x) \geqslant (\min_{1\leqslant j\leqslant q} r_j) \sum_{j=1}^{q} h_{D_j}(x) = (\min_{1\leqslant j\leqslant q} r_j) h_D(x).$$

So we have

$$\begin{aligned}\sum_{j=1}^{q} r_j m_S(x, D_j) &\leqslant \left(\frac{2nl}{q} - (3\delta_1/2)\right)\left(\sum_{j=1}^{q} r_j h_{D_j}(x)\right) \\ &\quad + (\delta_1/2)\left(\sum_{j=1}^{q} r_j h_{D_j}(x)\right) \quad \| \\ &= \left(\frac{2nl}{q} - \delta_1\right)\left(\sum_{j=1}^{q} r_j h_{D_j}(x)\right) \quad \|.\end{aligned}$$

This proves Theorem 4.1. ■

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