

## REFINED SOLUTIONS OF SOME INTEGRAL EQUATIONS

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**Abstract:** The paper deals with linear and semi-linear integral equations in Sobolev spaces  $H_p^s(\mathbb{R}^n)$ . The main aim is to extend the resulting a.e. validity of the corresponding equations with respect to the Lebesgue measure to an  $\mu$ -a.e. validity for some Radon measures  $\mu$  in  $\mathbb{R}^n$ .

**Keywords:** integral equations, Sobolev spaces, fractals.

### 1. Introduction

Let  $1 < p < \infty$  and  $s \geq 0$ . Let  $H_p^s(\mathbb{R}^n)$  be the usual Sobolev space in  $\mathbb{R}^n$ . Let  $k \in L_1(\mathbb{R}^n)$  and  $h \in H_p^s(\mathbb{R}^n)$ . If  $\varepsilon > 0$  is small, then by Banach's contraction theorem,

$$u(x) = \varepsilon \int_{\mathbb{R}^n} k(y) u(x - y) dy + h(x), \quad (1)$$

has a (uniquely determined) solution  $u \in H_p^s(\mathbb{R}^n)$ . In particular, (1) holds a.e. (*almost everywhere*) with respect to the Lebesgue measure  $\mu_L$  in  $\mathbb{R}^n$ . If, in addition,  $s > \frac{n}{p}$ , then  $H_p^s(\mathbb{R}^n)$  is continuously embedded in the space  $C(\mathbb{R}^n)$ , consisting of all bounded uniformly continuous functions in  $\mathbb{R}^n$ . In particular, by the usual interpretation, there is a (uniquely determined) representative  $u \in C(\mathbb{R}^n)$  in the respective class  $[u] \in H_p^s(\mathbb{R}^n)$ . Choosing these continuous representatives both for  $u$  and  $h$  in (1), then, since the integral in (1) is also continuous, (1) holds for all  $x \in \mathbb{R}^n$ . In case of  $s = 0$ , which means  $H_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ , no improvement of the  $\mu_L$ -a.e. validity of (1) can be expected. The paper deals with the problem of improved validity in  $H_p^s(\mathbb{R}^n)$  if

$$1 < p < \infty \quad \text{and} \quad 0 < s \leq \frac{n}{p}. \quad (2)$$

We ask for Radon measures  $\mu$  in  $\mathbb{R}^n$  such that (1) with  $h \in H_p^s(\mathbb{R}^n)$  holds for suitable representatives not only  $\mu_L$ -a.e., but  $\mu$ -a.e. This will be the case, in rough

terms, if the trace

$$tr_\mu : H_p^s(\mathbb{R}^n) \mapsto L_1(\Gamma, \mu) \quad \text{with } \Gamma = \text{supp } \mu, \quad (3)$$

makes sense. Then it comes out that the validity of (1) can be extended  $\mu$ -a.e. to some fractal sets, boundaries of domains etc. having Lebesgue measure zero, in dependence on  $p$  and  $s$ . Our arguments are qualitative and (1) might be considered as a simple model case which can be generalized in many respects. This will not be done here with the following exception. If  $u(x)$  is real, then the truncation operator  $T^+$  is given by

$$T^+ : u(x) \mapsto u_+(x) = \max(u(x), 0). \quad (4)$$

Replacing in addition  $k(y)$  in (1) by some more general, but real, kernel  $k(x, y)$ , we ask for refined solutions of the semi-linear integral equation

$$u(x) = \varepsilon \int_{\mathbb{R}^n} k(x, y) u_+(x - y) dy + h(x) \quad (5)$$

in the real Sobolev space  $\mathbb{H}_p^s(\mathbb{R}^n)$ . This paper might be considered as a complement and continuation of some relevant parts of the recent book [9]. In section 2 we collect some notation, definitions, and prerequisites. Results, proofs, and examples are given in section 3.

## 2. Notation, definitions, prerequisites

**2.1. Basic notation.** Let  $\mathbb{N}$  be the collection of all natural numbers and let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . Let  $\mathbb{R}^n$  be euclidean  $n$ -space, where  $n \in \mathbb{N}$ ; put  $\mathbb{R} = \mathbb{R}^1$ . Let  $S(\mathbb{R}^n)$  be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on  $\mathbb{R}^n$ . By  $S'(\mathbb{R}^n)$  we denote its topological dual, the space of all tempered distributions on  $\mathbb{R}^n$ . As usual,  $\mathbb{Z}$  is the collection of all integers; and  $\mathbb{Z}^n$ , where  $n \in \mathbb{N}$ , denotes the lattice of all points  $m = (m_1, \dots, m_n) \in \mathbb{R}^n$  with  $m_j \in \mathbb{Z}$ .

We collect some more specific notation in connection with measures. Let  $Q_{jm}$  be a cube in  $\mathbb{R}^n$  with sides parallel to the axes of coordinates, centred at  $2^{-j}m$  and with side length  $2^{-j}$ , where  $m \in \mathbb{Z}^n$  and  $j \in \mathbb{N}_0$ . If  $Q$  is a cube in  $\mathbb{R}^n$  and  $r > 0$  then  $rQ$  is the cube in  $\mathbb{R}^n$  concentric with  $Q$  and with side length  $r$  times the side length of  $Q$ .

**2.2. Spaces.** As usual,  $L_p(\mathbb{R}^n)$  with  $1 < p < \infty$ , is the standard Banach space of complex-valued  $p$ -integrable functions  $f$  with respect to the Lebesgue measure  $\mu_L$ , normed by

$$\|f\|_{L_p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}. \quad (6)$$

Let  $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$  be the Laplacian, where  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . The (complex) Sobolev spaces

$$H_p^s(\mathbb{R}^n), \quad 1 < p < \infty, \quad s \in \mathbb{R}, \tag{7}$$

are defined (and normed) by lifting

$$H_p^s(\mathbb{R}^n) = (id - \Delta)^{-\frac{s}{2}} L_p(\mathbb{R}^n) \tag{8}$$

in  $S'(\mathbb{R}^n)$ . The corresponding real Sobolev spaces are denoted by  $H_p^s(\mathbb{R}^n)$  (only values  $s \geq 0$  are of interest).

**2.3. Radon measures.** Let  $\mu$  be a Radon measure in  $\mathbb{R}^n$  with

$$\Gamma = \text{supp } \mu \text{ compact; and } \mu(\mathbb{R}^n) = \mu(\Gamma) < \infty, \tag{9}$$

interpreted in the usual way as a tempered distribution  $\mu \in S'(\mathbb{R}^n)$ . We wish to classify these Radon measures with respect to function spaces. We follow essentially [9], 9.25, p. 145. Let

$$1 < v < \infty \quad \text{and} \quad t \geq 0. \tag{10}$$

Then

$$\mu_v^t = \left( \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{tjv} \mu(2Q_{j_m})^v \right)^{\frac{1}{v}}. \tag{11}$$

A discussion of these characteristic numbers of a Radon measure  $\mu$  with (9) may be found in [9], 9.26. Let, by definition,  $M_v^t$  be the collection of all those Radon measures  $\mu$  in  $\mathbb{R}^n$  with  $\mu_v^t < \infty$ . In 3.2 we introduce the local class  $M_v^{t,loc}$ .

**2.4. Representatives.** If  $s \geq 0$  then  $H_p^s(\mathbb{R}^n)$  is a subspace of  $L_1^{loc}(\mathbb{R}^n)$ . In particular, in any class  $[f] \in H_p^s(\mathbb{R}^n)$  one can select the (uniquely determined) representative  $f$  with

$$\lim_{r \rightarrow 0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y) - f(x)| dy = 0 \tag{12}$$

$\mu_L$ -a.e. in all Lebesgue points of  $f$ . Recall that  $\mu_L$  stands for the Lebesgue measure in  $\mathbb{R}^n$ . Furthermore  $B(x, r)$  is a ball in  $\mathbb{R}^n$  centred at  $x \in \mathbb{R}^n$  and of radius  $r$ . As usual,  $|B(x, r)| = \mu_L(B(x, r))$  is the volume of  $B(x, r)$ . As a consequence of (12) one has  $\mu_L$ -a.e.

$$f(x) = \lim_{r \rightarrow 0} |B(x, r)|^{-1} \int_{B(x, r)} f(y) dy. \tag{13}$$

Of course, (13) means that the right-hand side converges and that the respective limit equals  $f(x)$ . If  $s > \frac{n}{p}$  then  $f$  with (12) is the continuous representative mentioned in the Introduction. If  $s = 0$ , hence  $H_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ , then one can hardly say more than the  $\mu_L$ -a.e. convergence in (12) and (13). To clarify the situation if  $0 < s \leq \frac{n}{p}$  one needs the notion of capacity.

**2.5. Capacity and representatives.** Let  $\Gamma$  be a compact set in  $\mathbb{R}^n$  and let  $p$  and  $s$  be given by (2). By [1], Definition 2.2.6, p. 20, complemented by Corollary 2.6.8, p. 44,

$$C_{s,p}(\Gamma) = \inf \{ \|\varphi|H_p^s(\mathbb{R}^n)\|^p : \varphi \in S(\mathbb{R}^n), \varphi \geq 1 \text{ on } \Gamma \} \quad (14)$$

is called the  $(s,p)$ -capacity of  $\Gamma$ . Here the admitted functions  $\varphi$  are real. This notion can be extended to arbitrary sets  $E$  in  $\mathbb{R}^n$ , [1], p. 19. A property is said to hold  $(s,p)$ -quasi-everywhere,  $(s,p)$ -q.e. for short, if it is true for all  $x \in \mathbb{R}^n$  with exception of a set  $E$  with  $C_{s,p}(E) = 0$ . By [1], 6.1, 6.2, pp. 157-159, it follows that in each equivalence class  $[f] \in H_p^s(\mathbb{R}^n)$  there is a uniquely determined representative  $f$  such that (12), and, hence, (13), hold  $(s,p)$ -q.e. This representative can be described in terms of the lifting (8) as

$$f = (id - \Delta)^{-\frac{s}{2}} g \quad \text{with } g \in L_p(\mathbb{R}^n), \quad (15)$$

where  $g$  is uniquely determined. It coincides with the representative discussed in 2.4. As we shall see,  $(s,p)$ -q.e. in (12), (13) is much more than  $\mu_L$ -a.e.

**2.6. Traces.** Let  $\mu$  be a Radon measure in  $\mathbb{R}^n$  satisfying (9). Let  $H_p^s(\mathbb{R}^n)$  with  $s > 0$  and  $1 < p < \infty$  be the Sobolev spaces introduced in 2.2. If  $\varphi \in S(\mathbb{R}^n)$ , then, of course, the pointwise trace,

$$tr_\mu : \varphi \in S(\mathbb{R}^n) \mapsto \varphi|_\Gamma \in L_1(\Gamma, \mu) \quad (16)$$

makes sense. We ask whether there is a constant  $c > 0$  such that

$$\|tr_\mu \varphi|_{L_1(\Gamma, \mu)}\| \leq c \|\varphi|_{H_p^s(\mathbb{R}^n)}\| \quad (17)$$

for all  $\varphi \in S(\mathbb{R}^n)$ . If this is the case then one can extend (17) from  $S(\mathbb{R}^n)$  to  $H_p^s(\mathbb{R}^n)$  by completion, where we use that  $S(\mathbb{R}^n)$  is dense in  $H_p^s(\mathbb{R}^n) = F_{p,2}^s(\mathbb{R}^n)$ , [6], Theorem 2.3.3, p. 48. We refer to [9], section 9, for a more detailed discussion. In this way, any  $f \in H_p^s(\mathbb{R}^n)$  has a (uniquely determined) trace  $tr_\mu f \in L_1(\Gamma, \mu)$ , and

$$tr_\mu : H_p^s(\mathbb{R}^n) \mapsto L_1(\Gamma, \mu) \quad (18)$$

is denoted as *trace operator*. Let  $f \in H_p^s(\mathbb{R}^n)$  be the representative according to (15). In particular, (12) and (13) hold  $(s,p)$ -q.e. By the arguments in [9], 19.5, pp. 260-263, (which will be repeated and complemented below) it follows that we have (12) and (13)  $\mu$ -a.e. on  $\Gamma$  (or on  $\mathbb{R}^n$ , which is the same). In particular,  $tr_\mu f$  can be defined directly by (13). Further information and references may be found in [9], 19.5, 19.6, pp. 260-264.

**2.7. Pointwise multipliers and the spaces  $L_1(\mathcal{C}^\sigma)(\mathbb{R}^{2n})$ .** As said we are not only interested in kernels  $k(y)$  in (1) but also in the more general kernels  $k(x, y)$  in (5). This will be based on pointwise multipliers in  $H_p^s(\mathbb{R}^n)$ . We describe what we need. Recall that  $\mathcal{C}^\sigma(\mathbb{R}^n)$  with  $\sigma > 0$  are the classical Hölder-Zygmund spaces (Hölder spaces if  $0 < \sigma \notin \mathbb{N}$ , what is sufficient for us). Explicit descriptions may be found, for example, in [6], 2.2.2, p. 36. Let

$$\sigma > s > 0 \quad \text{and} \quad 1 < p < \infty. \tag{19}$$

Then there exists a positive constant  $c$  such that

$$\|gf | H_p^s(\mathbb{R}^n)\| \leq c \|g | \mathcal{C}^\sigma(\mathbb{R}^n)\| \cdot \|f | H_p^s(\mathbb{R}^n)\| \tag{20}$$

for all  $g \in \mathcal{C}^\sigma(\mathbb{R}^n)$  and all  $f \in H_p^s(\mathbb{R}^n)$ . This is a special case of the pointwise multiplier assertion in [7], Corollary on p. 205. We use this multiplier property in connection with the kernels  $k(x, y)$  belonging to some hybrid function spaces. Let  $\sigma > 0$ . Then  $L_1(\mathcal{C}^\sigma)(\mathbb{R}^{2n})$  is the collection of all (complex-valued) functions  $k(x, y)$  with  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^n$  such that

$$\|k | L_1(\mathcal{C}^\sigma)(\mathbb{R}^{2n})\| = \int_{\mathbb{R}^n} \|k(\cdot, y) | \mathcal{C}^\sigma(\mathbb{R}^n)\| dy < \infty. \tag{21}$$

We refer to [9], 27.2, 27.3, p. 391, for further details.

**2.8. Truncation.** Let  $s \geq 0$  and  $1 < p < \infty$ . Recall that  $\mathbb{H}_p^s(\mathbb{R}^n)$  is the real part of  $H_p^s(\mathbb{R}^n)$ . Then the truncation operator  $T^+$ , introduced in (4), makes sense. Let

$$1 < p < \infty, \quad 0 \leq s < 1 + \frac{1}{p}. \tag{22}$$

Then there is a positive number  $c$  such that

$$\|T^+ f | H_p^s(\mathbb{R}^n)\| \leq c \|f | H_p^s(\mathbb{R}^n)\| \quad \text{for all} \quad f \in \mathbb{H}_p^s(\mathbb{R}^n). \tag{23}$$

Hence,  $T^+$  is a bounded (non-linear) operator in  $\mathbb{H}_p^s(\mathbb{R}^n)$ . This assertion may be found in [4], p. 355. We refer also to [9], section 25, where we studied truncation problems in spaces  $B_{pq}^s(\mathbb{R}^n)$  and  $F_{pq}^s(\mathbb{R}^n)$  in detail. In particular, inequality (23) is a special case of [9], Corollary 25.11, pp. 378-379. We need this mapping property in connection with the semi-linear integral equation (5).

### 3. Results, proofs, and examples

**3.1. Proposition.** Let  $\mu$  be a Radon measure in  $\mathbb{R}^n$  with (9). Let  $M_\nu^t$  be the classes of measures introduced in 2.3. Let

$$p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 0 < s \leq \frac{n}{p}. \quad (24)$$

Let  $H_p^s(\mathbb{R}^n)$  be the Sobolev spaces and  $C_{s,p}$  be the related capacities according to 2.2 and 2.5, respectively.

(i) The trace operator  $tr_\mu$  in (18) exists according to 2.6 if, and only if,  $\mu \in M_{p'}^{\frac{n}{p'}-s}$ .

(ii) Let  $\mu \in M_{p'}^{\frac{n}{p'}-s}$ . There is a positive number  $c$  such that

$$\mu(K) \leq c C_{s,p}(K) \quad (25)$$

for all compact sets  $K$  in  $\mathbb{R}^n$ .

(iii) Let  $\mu \in M_{p'}^{\frac{n}{p'}-s}$ . For any representative  $f$  in  $H_p^s(\mathbb{R}^n)$  according to (15), both (12) and the equality (13) are valid  $\mu$ -a.e.

**Proof.** Part (i) is essentially a special case of [9], Theorem 9.9(ii), p. 131. We prove part (ii). Let  $\varphi \in S(\mathbb{R}^n)$  be real with  $\varphi \geq 1$  on  $K$ . By part (i) and (17) we have

$$\mu(K) \leq \int_{\mathbb{R}^n} |\varphi(x)|^p \mu(dx) \leq c \|\varphi\|_{H_p^s(\mathbb{R}^n)}^p. \quad (26)$$

Now (25) follows from (14). Finally we prove part (iii). For any  $\varepsilon > 0$  there is an open set  $E_\varepsilon$  in  $\mathbb{R}^n$  with  $C_{s,p}(E_\varepsilon) \leq \varepsilon$  such that

$$\lim_{r \rightarrow 0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y) - f(x)| dy = 0, \quad x \in \mathbb{R}^n \setminus E_\varepsilon. \quad (27)$$

This follows from [1], pp. 19 and 159. Let  $K_\varepsilon$  be a compact set with  $K_\varepsilon \subset E_\varepsilon$ . Then it follows from (25) that

$$\mu(K_\varepsilon) \leq c C_{s,p}(K_\varepsilon) \leq c C_{s,p}(E_\varepsilon) \leq c \varepsilon. \quad (28)$$

Since  $\mu$  is a Radon measure we have

$$\mu(E_\varepsilon) = \sup \{ \mu(K_\varepsilon) : K_\varepsilon \text{ compact, } K_\varepsilon \subset E_\varepsilon \} \quad (29)$$

and, consequently,

$$\mu(E_\varepsilon) \leq c \varepsilon, \quad (30)$$

where  $c$  is independent of  $\varepsilon$ . Now one may choose a monotonically decreasing sequence of these open sets  $E_{\varepsilon_j}$  with  $\varepsilon_j \rightarrow 0$ . Then

$$\mu(E) = 0 \quad \text{with} \quad E = \bigcap_{j=1}^{\infty} E_{\varepsilon_j}, \quad (31)$$

and we have (12) and (13) for all  $x \in \mathbb{R}^n \setminus E$ . ■

**3.2. Integral equations; the class  $M_v^{t,loc}$ .** As outlined in the Introduction we deal with the integral equations (1) and (5) in  $H_p^s(\mathbb{R}^n)$  and  $\mathbb{H}_p^s(\mathbb{R}^n)$ , respectively, and we wish to extend the a.e. (with respect to the Lebesgue measure) validity of these equations as much as possible. But there is a significant difference between these two equations. First we look at (1). Let  $1 < p < \infty$ ,  $s \geq 0$ , and

$$h \in H_p^s(\mathbb{R}^n), \quad k \in L_1(\mathbb{R}^n). \tag{32}$$

There is a number  $\varepsilon_0 > 0$  such that for any  $\varepsilon$  with  $0 < \varepsilon \leq \varepsilon_0$  the equation (1) has a uniquely determined solution  $u \in H_p^s(\mathbb{R}^n)$ . This is an immediate consequence of Banach's contraction theorem applied in  $H_p^s(\mathbb{R}^n)$ . If, in addition,  $s > \frac{n}{p}$ , then, as discussed in 2.4, one may choose both for  $u$  and  $h$  the respective continuous representatives. Since the integral in (1) is also continuous one can extend (1) from  $\mu_L$ -a.e. to all  $x \in \mathbb{R}^n$ . (Recall that  $\mu_L$  is the Lebesgue measure). As for classical and more recent sharp embeddings we refer to [5], 2.8; [6], 2.7; [4], 2.2; and [9], 11.4. In particular, if  $1 < p < \infty$  and  $0 < s \leq \frac{n}{p}$  then there is no continuous embedding of  $H_p^s(\mathbb{R}^n)$  in the space  $C(\mathbb{R}^n)$ , consisting of all bounded and uniformly continuous function in  $\mathbb{R}^n$ . In particular, the distinguished representatives for  $u$  and  $h$  according to 2.4 and 2.5 are no longer necessarily continuous. One may ask whether (1) is valid not only  $\mu_L$ -a.e. but also  $\mu$ -a.e. for some locally finite Radon measures in  $\mathbb{R}^n$ . For this purpose we extend the class  $M_v^t$  of finite Radon measures in  $\mathbb{R}^n$ , introduced in 2.3, to locally finite Radon measures in  $\mathbb{R}^n$ . A Radon measure  $\mu$  in  $\mathbb{R}^n$  is called locally finite if

$$\mu(B) < \infty \quad \text{for any ball } B \text{ in } \mathbb{R}^n. \tag{33}$$

Then the restriction  $\mu|B$  of  $\mu$  to  $B$  is a finite Radon measure. We collected the measure-theoretical background in [9], p. 2, with references to the literature, especially to [3]. Again let

$$1 < v < \infty \quad \text{and} \quad t \geq 0. \tag{34}$$

Then

$$M_v^{t,loc} = \{ \mu : \mu \text{ locally finite Radon measure in } \mathbb{R}^n \\ \text{with } \mu|B \in M_v^t \text{ for all balls } B \text{ in } \mathbb{R}^n \}. \tag{35}$$

Then one can apply Proposition 3.1(iii) to  $\mu \in M_{p'}^{\frac{n}{p}-s,loc}$  and identify  $u$  and  $h$  in (1) with their respective representatives. Afterwards it remains to check that the integral in (1) exists also  $\mu$ -a.e. This is the case as we shall see.

As for the semi-linear equation (5) the situation is different. Under the restriction (22) we have the truncation property (23). But with exception of  $s = 0$ , which means  $H_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ , the operator  $T^+$  has poor continuity properties. We discussed this problem in detail in [9], section 25. Then Banach's contraction theorem cannot be applied in  $\mathbb{H}_p^s(\mathbb{R}^n)$  with  $s > 0$ . We circumvented this difficulty in

[9], section 27, with the help of the so-called  $Q$ -method, which has a wider range of applications. It is one aim of this paper to show that for the comparatively simple equation (5) one has more direct arguments. But as in case of equation (1) we are mainly interested in extending the  $\mu_L$ -a.e. validity of (5) to a  $\mu$ -a.e. validity with  $\mu \in M_{p'}^{\frac{n}{p'}-s,loc}$ . As for (1), the restriction  $s \leq \frac{n}{p}$  is natural. But in case of (5) it seems to be reasonable to distinguish between the two regions in the  $(\frac{1}{p}, s)$ -diagram below the line  $s = 1 + \frac{1}{p}$  as indicated in Fig. 1.

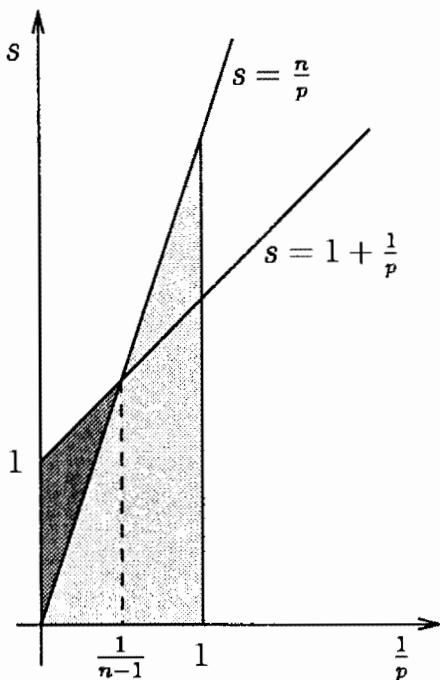


Fig. 1  $(n \geq 3)$

**3.3. Theorem. (i)** Let  $n \in \mathbb{N}$ ,

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 0 < s \leq \frac{n}{p} \quad \text{and} \quad k \in L_1(\mathbb{R}^n). \quad (36)$$

There is a positive number  $\varepsilon_0$  such that for any  $\varepsilon$  with  $0 < \varepsilon \leq \varepsilon_0$  and any  $h \in H_p^s(\mathbb{R}^n)$ ,

$$u(x) = \varepsilon \int_{\mathbb{R}^n} k(y) u(x - y) dy + h(x) \quad (37)$$

has a unique solution  $u \in H_p^s(\mathbb{R}^n)$ . Let  $\mu \in M_{p'}^{\frac{n}{p'}-s,loc}$  according to 2.3 and 3.2. Let both  $u$  and  $h$  be the distinguished representatives according to 2.5. Then (37) is valid  $\mu$ -a.e. (almost everywhere with respect to  $\mu$ ).



(ii) Let  $n \in \mathbb{N}$ ,

$$1 < p < \infty, \quad \frac{n}{p} < s < 1 + \frac{1}{p} \quad \text{and} \quad k \in L_1(C^\sigma)(\mathbb{R}^{2n}) \text{ real,} \quad (38)$$

according to 2.7 with  $\sigma > s$ . There is a positive number  $\varepsilon_0$  such that for any  $\varepsilon$  with  $0 < \varepsilon \leq \varepsilon_0$  and any  $h \in \mathbb{H}_p^s(\mathbb{R}^n)$ ,

$$u(x) = \varepsilon \int_{\mathbb{R}^n} k(x, y) u_+(x - y) dy + h(x) \quad (39)$$

has a unique solution  $u \in \mathbb{H}_p^s(\mathbb{R}^n)$ . Let both  $u$  and  $h$  be the continuous representatives. Then (39) is valid for all  $x \in \mathbb{R}^n$ .

(iii) Let  $n \in \mathbb{N}$ ,

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 0 < s \leq \frac{n}{p}, \quad s < 1 + \frac{1}{p}, \quad \text{and} \quad k \in L_1(C^\sigma)(\mathbb{R}^{2n}) \quad (40)$$

real, according to 2.7 with  $\sigma > s$ . There is a positive number  $\varepsilon_0$  such that for any  $h \in \mathbb{H}_p^s(\mathbb{R}^n)$ , (39) has a unique solution  $\mathbb{H}_p^s(\mathbb{R}^n)$ . Let  $\mu \in M_{p'}^{\frac{n}{p}-s, loc}$  according to 2.3 and 3.2. Let both  $u$  and  $h$  be the distinguished representatives according to 2.5. Then (39) is valid  $\mu$ -a.e.

**Proof.** Step 1. We prove (i). Let  $A$ ,

$$(Au)(x) = \varepsilon \int_{\mathbb{R}^n} k(y) u(x - y) dy + h(x), \quad u \in H_p^s(\mathbb{R}^n). \quad (41)$$

Then  $A$  is a bounded operator in  $H_p^s(\mathbb{R}^n)$  and

$$\|Au - Av\|_{H_p^s(\mathbb{R}^n)} \leq \varepsilon \|k\|_{L_1(\mathbb{R}^n)} \cdot \|u - v\|_{H_p^s(\mathbb{R}^n)}. \quad (42)$$

Hence, if  $\varepsilon > 0$  is small, then Banach's contraction theorem in the Banach space  $H_p^s(\mathbb{R}^n)$  can be applied. For given  $h$ , the operator  $A$  has a uniquely determined fixed point  $Au = u$ , which is the solution of (37) we are looking for. To prove the second part of (i) we may assume  $\mu \in M_{p'}^{\frac{n}{p}-s}$  with (9). We use the Fubini theorem for non-negative functions with respect to the product measure  $\mu \times \mu_L$  on  $\Gamma \times \mathbb{R}^n$ . Then we have that

$$\begin{aligned} \int_{\Gamma} \int_{\mathbb{R}^n} |k(y)| |u(\gamma - y)| dy \mu(d\gamma) &= \int_{\mathbb{R}^n} |k(y)| \|u(\cdot - y)\|_{L_1(\Gamma, \mu)} dy \\ &\leq c \|u\|_{H_p^s(\mathbb{R}^n)} \|k\|_{L_1(\mathbb{R}^n)} \\ &< \infty. \end{aligned} \quad (43)$$

Here we used Proposition 3.1(i), (18), and that the norm in  $H_p^s(\mathbb{R}^n)$  is translation invariant with respect to  $u(\cdot) \mapsto u(\cdot - y)$ . Hence,

$$(Ku)(x) = \int_{\mathbb{R}^n} k(y) u(x - y) dy \quad \text{exists } \mu\text{-a.e.} \quad (44)$$

Next we wish to prove that  $(Ku)(x)$  has  $\mu$ -a.e. the Lebesgue point property (13). Let  $Q_j$  with  $j \in \mathbb{N}_0$  be the cube in  $\mathbb{R}^n$  centred at the origin and with side-length  $2^{-j}$  (hence  $Q_j = Q_{j,0}$  with  $0 \in \mathbb{Z}^n$  in the notation introduced in 2.1). If  $f \in L_1^{loc}(\mathbb{R}^n)$  then we put

$$f^j(x) = \frac{1}{|Q_j|} \int_{Q_j} f(x + z) dz, \quad j \in \mathbb{N}_0. \quad (45)$$

For specified  $f$  we ask for which  $x$ ,

$$\lim_{j \rightarrow \infty} f^j(x) = f(x). \quad (46)$$

By the arguments given it will be clear that this is sufficient to prove (13). We apply the notation (45) to  $Ku$ , introduced in (44). We have

$$(Ku)^j(x) - (Ku)(x) = \int_{\mathbb{R}^n} k(y) \frac{1}{|Q_j|} \int_{Q_j} (u(x - y + z) - u(x - y)) dz dy, \quad (47)$$

and hence,

$$\begin{aligned} & \int_{\Gamma} |(Ku)^j(\gamma) - (Ku)(\gamma)| \mu(d\gamma) \\ & \leq \int_{\mathbb{R}^n} |k(y)| \frac{1}{|Q_j|} \int_{Q_j} \int_{\Gamma} |u(\gamma - y + z) - u(\gamma - y)| \mu(d\gamma) dz dy \\ & \leq c_1 \int_{\mathbb{R}^n} \int_{\Gamma} |k(y)| (|u(\gamma - y)| + (Mu)(\gamma - y)) \mu(d\gamma) dy, \\ & \leq c_2 \|u\|_{H_p^s(\mathbb{R}^n)} \cdot \|k\|_{L_1(\mathbb{R}^n)} < \infty, \end{aligned} \quad (48)$$

where  $Mu$  is the Hardy-Littlewood maximal function

$$(Mu)(x) = \sup_Q \frac{1}{|Q|} \int_Q |u(y)| dy. \quad (49)$$

The supremum in (49) is taken with respect to all cubes  $Q$  centred at  $x$ . To justify the last estimate in (48) we may assume that  $u$  is the distinguished representative according to 2.5, given by

$$u = (id - \Delta)^{-\frac{s}{2}} g \quad \text{with } g \in L_p(\mathbb{R}^n) \quad (50)$$

according to (15). The Bessel potential kernels related to  $(id - \Delta)^{-\frac{\alpha}{2}}$  are positive functions. Then it follows that

$$(Mu)(x) \leq ((id - \Delta)^{-\frac{\alpha}{2}} Mg)(x), \quad x \in \mathbb{R}^n. \tag{51}$$

Recall the classical Hardy-Littlewood maximal inequality

$$\|Mg\|_{L_p(\mathbb{R}^n)} \leq c \|g\|_{L_p(\mathbb{R}^n)}, \quad g \in L_p(\mathbb{R}^n), \tag{52}$$

in  $L_p(\mathbb{R}^n)$  with  $1 < p < \infty$ . The last estimate in (48) is now a consequence of (51), (52), (8), and (43). Furthermore by (12) we have for all  $y \in \mathbb{R}^n$ ,

$$\lim_{j \rightarrow \infty} \frac{1}{|Q_j|} \int_{Q_j} |u(\gamma - y + z) - u(\gamma - y)| dz = 0 \quad \mu\text{-a.e.} \tag{53}$$

Hence, using the Fubini theorem and then the Lebesgue dominated convergence theorem with respect to  $\mu_L \times \mu$ , [2], pp. 37, 44, we get

$$\lim_{j \rightarrow \infty} \int_{\Gamma} |(Ku)^j(\gamma) - (Ku)(\gamma)| \mu(d\gamma) = 0. \tag{54}$$

Finally by Fatou's lemma, [2], p. 38, we have that

$$\begin{aligned} & \int_{\Gamma} \left| \lim_{j \rightarrow \infty} (Ku)^j(\gamma) - (Ku)(\gamma) \right| \mu(d\gamma) \\ &= \int_{\Gamma} \lim_{j \rightarrow \infty} |(Ku)^j(\gamma) - (Ku)(\gamma)| \mu(d\gamma) \\ &\leq \lim_{j \rightarrow \infty} \int_{\Gamma} |(Ku)^j(\gamma) - (Ku)(\gamma)| d\mu = 0, \end{aligned} \tag{55}$$

and, consequently,

$$\lim_{j \rightarrow \infty} \frac{1}{|Q_j|} \int_{Q_j} (Ku)(x + y) dy = (Ku)(x), \quad \mu\text{-a.e.} \tag{56}$$

After this observation the proof of part (i) can be completed as follows. By (37) we have

$$u(x) = \varepsilon (Ku)(x) + h(x) \quad \mu_L\text{-a.e.} \tag{57}$$

and hence

$$u^j(x) = \varepsilon (Ku)^j(x) + h(x), \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N}_0. \tag{58}$$

Let  $u$  and  $h$  be the indicated distinguished representatives. Then we have counterparts of (56) with  $u$  and  $h$  in place of  $Ku$ . Now  $j \rightarrow \infty$  in (58) proves the validity of (37)  $\mu$ -a.e.

*Step 2.* Let

$$1 < p < \infty, \quad 0 < s < 1 + \frac{1}{p}, \quad k \in L_1(\mathcal{C}^\sigma)(\mathbb{R}^{2n}) \text{ real}, \quad h \in \mathbb{H}_p^s(\mathbb{R}^n), \quad (59)$$

where  $\sigma > s$ . We prove that (39) has a uniquely determined solution  $u \in \mathbb{H}_p^s(\mathbb{R}^n)$ . Let  $u_0 = 0$  and

$$u_{j+1}(x) = \varepsilon \int_{\mathbb{R}^n} k(x, y) T^+ u_j(x - y) dy + h(x), \quad j \in \mathbb{N}_0. \quad (60)$$

Assuming by mathematical induction  $u_j \in \mathbb{H}_p^s(\mathbb{R}^n)$  we get

$$\begin{aligned} & \|u_{j+1} | H_p^s(\mathbb{R}^n)\| \\ & \leq \varepsilon \int_{\mathbb{R}^n} \|k(\cdot, y) (T^+ u_j)(\cdot - y) | H_p^s(\mathbb{R}^n)\| dy + \|h | H_p^s(\mathbb{R}^n)\| \end{aligned} \quad (61)$$

and, using both the pointwise multiplier property 2.7 and the truncation property 2.8,

$$\begin{aligned} & \|u_{j+1} | H_p^s(\mathbb{R}^n)\| \\ & \leq c\varepsilon \int_{\mathbb{R}^n} \|k(\cdot, y) | \mathcal{C}^\sigma(\mathbb{R}^n)\| \|u_j | H_p^s(\mathbb{R}^n)\| dy + \|h | H_p^s(\mathbb{R}^n)\| \\ & \leq c\varepsilon \|k | L_1(\mathcal{C}^\sigma)(\mathbb{R}^{2n})\| \|u_j | H_p^s(\mathbb{R}^n)\| + \|h | H_p^s(\mathbb{R}^n)\|. \end{aligned} \quad (62)$$

Here  $c$  is independent of  $\varepsilon$  and  $j$ . In particular,  $u_{j+1} \in \mathbb{H}_p^s(\mathbb{R}^n)$  and there is a positive number  $\varepsilon_0$  such that for all positive  $\varepsilon$  with  $0 < \varepsilon \leq \varepsilon_0$ ,

$$\|u_{j+1} | H_p^s(\mathbb{R}^n)\| \leq \frac{1}{2} \|u_j | H_p^s(\mathbb{R}^n)\| + \|h | H_p^s(\mathbb{R}^n)\|. \quad (63)$$

Again by mathematical induction we obtain that

$$\|u_j | H_p^s(\mathbb{R}^n)\| \leq 2 \|h | H_p^s(\mathbb{R}^n)\|. \quad (64)$$

Hence, the sequence  $\{u_j\}$  is uniformly bounded in  $\mathbb{H}_p^s(\mathbb{R}^n)$ . However we have no counterpart of (42) since  $T^+$  is not Lipschitz continuous in  $\mathbb{H}_p^s(\mathbb{R}^n)$ . We refer to [9], section 25, for details. But  $T^+$  is Lipschitz continuous in  $L_p(\mathbb{R}^n)$ , which is the real part of  $L_p(\mathbb{R}^n)$ . Hence, there is a counterpart of (42) with  $L_p(\mathbb{R}^n)$  in place of  $H_p^s(\mathbb{R}^n)$ . Taking the above sequence  $\{u_j\}$  we get

$$u_j \rightarrow u \quad \text{if } j \rightarrow \infty \quad \text{in } L_p(\mathbb{R}^n). \quad (65)$$

The spaces  $H_p^s(\mathbb{R}^n)$  have the so-called Fatou property. A description may be found in [4], p. 15, or in [9], p. 360. In particular by (65) and (64) we obtain that  $u \in H_p^s(\mathbb{R}^n)$  and, hence,  $u \in \mathbb{H}_p^s(\mathbb{R}^n)$ . This proves that (39) has a solution in  $\mathbb{H}_p^s(\mathbb{R}^n)$ . The uniqueness in  $\mathbb{H}_p^s(\mathbb{R}^n)$  follows from the uniqueness in  $L_p(\mathbb{R}^n)$ .

*Step 3* It remains to prove the improved validity of (39) in the two cases considered. First we remark that there is a number  $\kappa$  with  $0 < \kappa < 1$  and a function  $\tilde{k} \in L_1(\mathbb{R}^n)$  such that for all  $x \in \mathbb{R}^n$ , all  $y \in \mathbb{R}^n$ , and all  $z \in \mathbb{R}^n$ ,

$$|k(x, y)| \leq \tilde{k}(y) \tag{66}$$

and

$$|k(x + z, y) - k(x, y)| \leq |z|^\kappa \tilde{k}(y). \tag{67}$$

This follows from  $k \in L_1(C^\sigma)(\mathbb{R}^{2n})$ . Hence,

$$\begin{aligned} &|k(x+z, y) u_+(x+z) - k(x, y) u_+(x-y)| \\ &\leq \tilde{k}(y) |u_+(x+z-y) - u_+(x, y)| + |z|^\kappa \tilde{k}(y) u_+(x-y). \end{aligned} \tag{68}$$

The counterpart of (44) is now denoted by

$$(K^+u)(x) = \int_{\mathbb{R}^n} k(x, y) u_+(x-y) dy. \tag{69}$$

First we assume that  $p$  and  $s$  are restricted by (38). We choose in (39) the respective continuous representatives  $u$  and  $h$ , which are even Hölder continuous with respect to the exponent  $\kappa$ , where  $0 < \kappa = s - \frac{n}{p} < 1$ . In (67) we can take the same  $\kappa$ . Then it follows by (68), (69),

$$|(K^+u)(x+z) - (K^+u)(x)| \leq c_1 |z|^\kappa \int_{\mathbb{R}^n} \tilde{k}(y) dy, \quad x \in \mathbb{R}^n, \quad |z| \leq c_2. \tag{70}$$

Hence also the integral in (39) is continuous. By construction, (39) is valid  $\mu_L$ -a.e. Since all three functions involved are continuous, it follows that (39) holds for all  $x \in \mathbb{R}^n$ . Next we assume that  $p$  and  $s$  are restricted by (40). For  $u$  and  $h$  in (39) we choose now the distinguished representatives according to 2.5. Now we combine the arguments from Step 1 with the above considerations. The counterpart of (47) is given by

$$\begin{aligned} &(K^+u)^j(x) - (K^+u)(x) \\ &= \int_{\mathbb{R}^n} \frac{1}{|Q_j|} \int_{Q_j} (k(x+z, y) u_+(x+z-y) - k(x, y) u_+(x-y)) dz dy. \end{aligned} \tag{71}$$

By (68) we get the following counterpart of (48),

$$\begin{aligned}
 & \int_{\Gamma} \left| (K^+u)^j(\gamma) - (K^+u)(\gamma) \right| \mu(d\gamma) \\
 & \leq \int_{\mathbb{R}^n} \tilde{k}(y) \frac{1}{|Q_j|} \int_{Q_j} \int_{\Gamma} |u_+(x+z-y) - u_+(x,y)| \mu(d\gamma) dz dy \\
 & \quad + c2^{-jx} \int_{\mathbb{R}^n} \tilde{k}(y) \int_{\Gamma} u_+(\gamma-y) \mu(d\gamma) dy \\
 & \leq c_1 \int_{\mathbb{R}^n} \int_{\Gamma} \tilde{k}(y) (|u(\gamma-y)| + (Mu)(\gamma-y)) \mu(d\gamma) dy \\
 & \quad + c_1 2^{-jx} \int_{\mathbb{R}^n} \tilde{k}(y) \int_{\Gamma} |u(\gamma-y)| \mu(d\gamma) dy \\
 & \leq c_2 \|u\|_{H_p^s(\mathbb{R}^n)} \cdot \|\tilde{k}\|_{L_1(\mathbb{R}^n)}. \tag{72}
 \end{aligned}$$

We have (53) with  $u_+$  in place of  $u$ . Now we are in the same position as in Step 1 after (48), (53). Using the theorems by Fubini, Lebesgue and Fatou it follows that (39) is valid  $\mu$ -a.e., where  $u$  and  $h$  are the distinguished representatives according to 2.5. ■

**3.4. Examples:  $d$ -sets.** We illustrate the theorem by looking at some examples. Again let  $n \in \mathbb{N}$ , and let  $0 \leq d \leq n$ . A compact set  $\Gamma$  in  $\mathbb{R}^n$  is called a  $d$ -set if there is a Radon measure  $\mu$  in  $\mathbb{R}^n$  and two positive numbers  $c_1$  and  $c_2$  such that

$$\text{supp } \mu = \Gamma \quad \text{and} \quad c_1 r^d \leq \mu(B(\gamma, r)) \leq c_2 r^d \tag{73}$$

for all  $\gamma \in \Gamma$ , all  $r$  with  $0 < r < 1$ , and all balls  $B(\gamma, r)$  centred at  $\gamma$  and of radius  $r$ . If  $\Gamma$  is a  $d$ -set with the measure  $\mu$ , then  $\mu$  is equivalent to the restriction  $\mathcal{H}^d|_{\Gamma}$  of the Hausdorff measure  $\mathcal{H}^d$  in  $\mathbb{R}^n$  to  $\Gamma$ . Further information and references to the literature may be found in [8], section 3, pp. 5-7. We check under which conditions a measure  $\mu$  with (73) belongs to the class  $M_v^t$  introduced in 2.3. Let  $N_j$  with  $j \in \mathbb{N}_0$  be the number of cubes  $2Q_{jm}$  in (11) having a non-empty intersection with  $\Gamma$ . There are two positive numbers  $c_3$  and  $c_4$  such that

$$c_3 \leq N_j 2^{-jd} \leq c_4 \quad \text{for } j \in \mathbb{N}_0. \tag{74}$$

Then

$$(\mu_v^t)^v \sim \sum_{j=0}^{\infty} 2^{jd} 2^{vj(t-d)} < \infty \tag{75}$$

if, and only if,

$$t - d + \frac{d}{v} < 0, \quad \text{hence } t < \frac{d}{v'} \quad \text{with } \frac{1}{v} + \frac{1}{v'} = 1. \tag{76}$$

In the above theorem we need the class  $M_p^{\frac{n}{p}-s}$ . Hence

$$\mu = \mathcal{H}^d|_{\Gamma} \in M_p^{\frac{n}{p}-s} \quad \text{if, and only if,} \quad \frac{n-d}{p} < s \leq \frac{n}{p}, \tag{77}$$

where again  $1 < p < \infty$ . In other words:

If  $\mu$  is a locally finite Radon measure in  $\mathbb{R}^n$  such that for any compact set  $K$  in  $\mathbb{R}^n$ ,

$$\Gamma_K = \text{supp } \mu|_K, \quad c_1^K r^d \leq \mu(B(\gamma, r)) \leq c_2^K r^d, \quad \gamma \in \Gamma_K, \quad 0 < r < 1, \tag{78}$$

is a  $d$ -set according to (73), then (37) with

$$h \in H_p^s(\mathbb{R}^n), \quad 1 < p < \infty, \quad \frac{n-d}{p} < s \leq \frac{n}{p}, \tag{79}$$

makes sense  $\mu$ -a.e.

If  $\mu = \mu_L$  is the Lebesgue measure, then we have  $d = n$  and, obviously, (37) makes sense  $\mu_L$ -a.e. in all spaces  $H_p^s(\mathbb{R}^n)$  with  $1 < p < \infty$  and  $0 \leq s \leq \frac{n}{p}$  (where  $s = 0$  refers to  $H_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ ). There are similar assertions with respect to (39) where one has the additional restriction  $s < 1 + \frac{1}{p}$ , now in the real spaces  $\mathbb{H}_p^s(\mathbb{R}^n)$ . Hence, it is quite clear that one has a refined validity of (37) and (39) in dependence on  $p$  and  $s$ , on some sets having Lebesgue measure zero. For example, let  $\Omega$  be a bounded  $C^\infty$  domain in  $\mathbb{R}^n$ . Then  $\Gamma = \partial\Omega$  is a  $(n-1)$ -set with respect to the surface measure  $\mu = \mathcal{H}^{n-1}|_{\partial\Omega}$ . Hence, if

$$\frac{1}{p} < s \leq \frac{n}{p} \quad \text{or} \quad \frac{1}{p} < s < 1 + \frac{1}{p}, \quad s \leq \frac{n}{p}, \tag{80}$$

then one has not only the  $\mu_L$ -a.e. validity of (37) or (39), respectively, but also  $\mu$ -a.e. validity on  $\partial\Omega$ .

### References

- [1] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory*, Springer, Berlin, 1996.
- [2] P. Malliavin, *Integration and probability*, Springer, New York, 1995.
- [3] P. Mattila, *Geometry of sets and measures in euclidean spaces*, Cambridge Univ. Press, 1995.
- [4] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, W. de Gruyter, Berlin, 1996.
- [5] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, 1978 (Sec. ed. Barth, Heidelberg, 1995).
- [6] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983.

- [7] H. Triebel, *Theory of function spaces*, II, Birkhäuser, Basel, 1992.
- [8] H. Triebel, *Fractals and spectra*, Birkhäuser, Basel, 1997.
- [9] H. Triebel, *The structure of functions*, Birkhäuser, Basel, 2001.

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