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ON SOME EXTENSIONS OF BERNSTEIN'S INEQUALITY FOR TRIGONOMETRIC POLYNOMIALS

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Abstract: We give an approach to treating inequalities of Bernstein type for trigonometric polynomials. Some necessary and sufficient conditions of their validity are established.

1. Introduction

In the present paper we study some inequalities for trigonometric polynomials of d variables in L_p with $0 < p \leq +\infty$. Some of these inequalities like Bernstein's inequality for derivatives of trigonometric polynomials T_n of order at most n (see, for instance, [4], pp. 97-98, 104-109)

$$\|T_n'\|_p \leq n\|T_n\|_p \ , \quad n \in \mathbb{N}_0 \ , \quad 0 < p \leq +\infty \tag{1.1}$$

or its counterpart for the Riesz derivative (see [3], p. 427)

$$\|T_n^{(')}\|_p \leq c(p) \cdot n\|T_n\|_p \ , \quad n \in \mathbb{N}_0 \ , \quad 1 \leq p \leq +\infty \tag{1.2}$$

are well-known. Some of them like inequalities containing the fractional Laplacian seem to be new.

We have worked out an unified approach for treating such inequalities based on methods of Fourier analysis. For $0 < p \leq 1$ and $p = +\infty$ we are able to give necessary and sufficient conditions providing the validity of a wide class of multipliers in the L_p-spaces. This enables us to give complete solutions of some problems. As a rule, the spaces L_p with $1 < p < +\infty$ have better properties and they are more convenient to studying in difference to the cases $0 < p \leq 1$ and $p = +\infty$. However, this is not the case if inequalities of multiplier type are considered. We mention, for example, the problem of Bochner-Riesz multipliers that does not have a complete solution for $1 < p < +\infty$ (see, for references,

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[12], p. 388-394). The results of the present paper give another confirmation of this remark.

The paper is organized as follows. In section 2 we describe the general problem and give some examples. In section 3 we prove some general assertions and, in particular, the criteria for $0 < p \le 1$, $p = +\infty$ mentioned above. Section 4 is devoted to the study of the behaviour at infinity of the Fourier transform of some functions. In section 5 we deal with applications.

Henceforth, we will use the following notations: $\|\cdot\|_p$, $0 < p \le +\infty$, is the p-norm (the quasi-norm, if $0 < p < 1$) on the d-dimensional torus $\mathbb{T}^d$, $d \in \mathbb{N}$;

$$xy = x_1y_1 + \ldots + x_dy_d; \quad |x|_q = (x_1^q + \ldots + x_d^q)^{1/q}; \quad |x| = |x|_2;$$

$$D_r = \{x \in \mathbb{R}^d : |x| < r\}; \quad \overline{D}_r = \{x \in \mathbb{R}^d : \ |x| \le r\};$$

$\mathcal{T}$ is the space of all complex or real valued trigonometric polynomials; $\mathcal{T}_\lambda = \operatorname{span}\{e^{ikx} : |k| \le \lambda\}$.

2. General problem and its special cases

Let $\mu(\xi)$ be a complex valued function defined on $\mathbb{R}^d$. It generates a family of operators $\{A_\lambda\}_{\lambda \ge 1}$ given by

$$A_\lambda t(x) = \sum_{k \in \mathbb{Z}^d} \mu\left(\frac{k}{\lambda}\right) c_k e^{ikx}, \quad \left(t(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{ikx} \in \mathcal{T}\right). \tag{2.1}$$

Clearly, A_λ maps $\mathcal{T}$ into itself. We will say that $\{A_\lambda\}_{\lambda \ge 1}$ is of multiplier type and is generated by μ, so that $A_\lambda \equiv A_\lambda(\mu)$. We deal with the inequality

$$\|A_\lambda(\mu)t\|_p \le c(d;p;\mu) \cdot \|t\|_p \,, \quad t \in \mathcal{T}_\lambda \,, \quad \lambda \ge 1 \,, \tag{$*$}$$

where $\{A_\lambda(\mu)\}$ is generated by μ. Inequality $(*)$ is said to be valid for $p \in (0, +\infty]$, if it is valid in the p-norm for all $t \in \mathcal{T}_\lambda$ and for all $\lambda \ge 1$ with some positive constant $c(d;p;\mu)$ that does not depend on t and λ.

The main problem connected with the inequality $(*)$ is to find its range of validity $\Theta \equiv \Theta(d;\mu)$, that is, a set of points $p \in (0, +\infty]$, such that $p \in \Theta$ if and only if the inequality is valid for p.

We give some examples.

1. Let $d = 1$, $\gamma > 0$, $\mu(\xi) = (i\xi)^\gamma = |\xi|^\gamma \exp\left(\frac{i\pi\gamma}{2} \cdot \operatorname{sgn}\xi\right)$, that is $A_\lambda(\mu) = \lambda^{-\gamma} \cdot \mathcal{D}^\gamma$, where $\mathcal{D}^\gamma \equiv \frac{d^\gamma}{dx^\gamma}$ is the Weil derivative. Then $(*)$ is a Bernstein type inequality which has been intensively studied by many authors. In the case $\gamma \in \mathbb{N}$, $1 \le p \le +\infty$ it was obtained with the best possible constant 1 by S.B. Stechkin [11]. The case $\gamma \in \mathbb{N}$, $0 < p < 1$ was considered by P. Oswald, E.A. Storozhenko [13] and others. The sharp constant 1 was obtained by V.V. Arestov [1]. The Bernstein inequality for non-natural γ and $1 < p < +\infty$ is an immediate consequence of the Marcinkiewicz multiplier theorem (see, for instance, [15], pp. 178-179). Its validity for $p = 1, +\infty$ is mentioned in [3], p. 427. The general case $\gamma > 0,\ \gamma \notin \mathbb{N},\ 0 < p < 1$ can be found in [12] and [2].

2. The Bernstein inequality is a special case of the inequality of type $(*)$ with

$$\mu(\xi) = \varepsilon_1 \cdot \xi_-^\gamma + \varepsilon_2 \cdot \xi_+^\gamma\,, \quad \varepsilon_1, \varepsilon_2 \in \mathbb{C}; \quad \varepsilon_1^2 + \varepsilon_2^2 \neq 0, \tag{2.2}$$

where

$$\xi_-^\gamma = \begin{cases} 0, & \xi \geq 0 \\ |\xi|^\gamma, & \xi < 0 \end{cases} \quad , \quad \xi_+^\gamma = \begin{cases} |\xi|^\gamma, & \xi > 0 \\ 0, & \xi \leq 0 \end{cases}$$

and $\gamma > 0$. If, for example, $\varepsilon_1 = \varepsilon_2 = e^{i\pi\beta} = (-1)^\beta,\ \ \gamma = 2\beta > 0$, the operator $A_\lambda(\mu)$ can be naturally denoted by $\lambda^{-2\beta} \cdot (\mathcal{D}^2)^\beta$. It coincides (up to the multiplier $(-1)^\beta$) with the Riesz derivative of order 2β ([3], p. 427). Since $(\mathcal{D}^2)^\beta \neq \mathcal{D}^{2\beta}$, the corresponding inequality is not of Bernstein type. However, as it will be shown in section 5, the range of validity of inequality $(*)$ generated by (2.2) does not depend on ε_1 and ε_2.

3. In the multivariate case the situation is different. It will be proved in section 5 that the inequality of type $(*)$

$$\left\| \sum_{j=1}^d \frac{\partial^\gamma t}{\partial x_j^\gamma} \right\|_p \leq c(d; p; \gamma) \cdot \lambda^\gamma \|t\|_p\,, \quad t \in \mathcal{T}_\lambda\,, \quad \lambda \geq 1 \tag{2.3}$$

with $\mu(\xi) = \sum\limits_{j=1}^d |\xi_j|^\gamma \exp\left(\frac{i\pi\gamma}{2} \operatorname{sgn}\xi_j\right), \gamma > 0$, that is one of the possible extensions of the Bernstein type inequality to the case of several variables, is valid for $\gamma \notin \mathbb{N}$ if and only if $\frac{1}{\gamma+1} < p \leq +\infty$, that is, the answer does not depend on the dimension.

On the other hand, the inequality

$$\|\Delta^\beta t\|_p \leq c(d; p; \beta) \cdot \lambda^{2\beta} \|t\|_p\,, \quad t \in \mathcal{T}_\lambda\,, \quad \lambda \geq 1 \tag{2.4}$$

with $\mu(\xi) = (-1)^\beta \cdot |\xi|^{2\beta},\ \beta > 0$, (that is also an extension of type (2.2) with $\varepsilon_1 = \varepsilon_2 = (-1)^\beta$) is valid for $\beta \notin \mathbb{N}$ if and only if $\frac{d}{d+2\beta} < p \leq +\infty$.

Analyzing the inequalities described above, we can observe that they are generated by homogeneous functions. Further studying of such type inequalities is one of our main aims. We will amplify a series of appropriate examples in section 5. On the other hand, all the material of sections 3 and 4 is applicable to inequalities of type $(*)$ without any supplementary restrictions. One of possible applications to inequalities generated by non-homogeneous functions will be given at the end of the paper.

3. General assertions

By C^d we denote the class of complex valued C^∞-functions with a compact support contained in D_1. We use the symbol $\mathcal{R}^d_{a,b}$, where $0 < a < b + \infty$, to denote the class of real valued non-negative radial C^∞-functions that are equal to 1 on D_a and 0 outside of D_b. By $\mathcal{R}^d$ we denote the sum of all $\mathcal{R}^d_{a,b}$.

As usual, the Fourier transform of a function $f(\xi)$ in $L_1(\mathbb{R}^d)$ is given by

$$\hat{f}(x) = (2\pi)^{-d/2} \cdot \int_{\mathbb{R}^d} f(\xi) e^{-i\xi x} d\xi \ .$$

In this section we study the inequality of type $(*)$, where $\{A_\lambda(\mu)\}_{\lambda \geq 1}$ is generated by the function $\mu(\xi)$ which is defined everywhere on $\mathbb{R}^d$. By $\Theta \equiv \Theta(d;\mu) \subset (0, +\infty]$ we denote the range of validity of inequality $(*)$.

First we treat a trivial case.

Lemma 3.1. *If $\mu(\xi)$ is unbounded on $\overline{D}_1$, Θ is empty.*

Proof. There exist sequences $\{k_s\}_{s=1}^{+\infty} \subset \mathbb{Z}^d$, $\{n_s\}_{s=1}^{+\infty} \subset \mathbb{N}$ satisfying

$$|k_s| \leq n_s, \quad s \in \mathbb{N}; \quad \lim_{s \to +\infty} \frac{k_s}{n_s} = x_0 \in \overline{D}_1; \quad \lim_{s \to +\infty} \left| \mu\left(\frac{k_s}{n_s}\right) \right| = +\infty \ .$$

Then

$$\lim_{s \to +\infty} \frac{\|A_{n_s}(\mu)(e^{ik_s \cdot})\|_p}{\|e^{ik_s \cdot}\|_p} = \lim_{s \to +\infty} \left| \mu\left(\frac{k_s}{n_s}\right) \right| = +\infty$$

and $(*)$ is not valid. ■

The following theorem gives the necessary condition for the validity of $(*)$. We put

$$\widetilde{p} = \begin{cases} p, & 0 < p \leq 2 \\ \dfrac{p}{p-1}, & 2 < p \leq +\infty \end{cases} .$$

Theorem 3.1. *Let $\mu(\xi)$ be integrable in the Riemannian sense on $\overline{D}_1$. If inequality $(*)$ is valid for some $p \in (0, +\infty]$, then $\widehat{\mu\psi}(x) \in L_{\widetilde{p}}(\mathbb{R}^d)$ for any $\psi \in C^d$.*

Proof. Let first $0 < p \leq 2$. We consider the sequence of functions $\{F_n(x)\}_{n=1}^{+\infty}$ given by

$$F_n(x) = \begin{cases} n^{-dp} \cdot \left| \sum_{k \in \mathbb{Z}^d} \mu\left(\frac{k}{n}\right) \psi\left(\frac{k}{n}\right) e^{i\frac{k}{n}x} \right|^p, & x \in [-\pi n, \pi n]^d \\ 0, & \text{otherwise} \end{cases} . \tag{3.1}$$

Clearly, the functions $F_n(x)$, $n \in \mathbb{N}$, are non-negative and measurable. Let $x_0 \in \mathbb{R}^d$. Then there exists $n_0 \in \mathbb{N}$, such that $x_0 \in [-\pi n, \pi n]^d$ for $n \geq n_0$.

The function $\mu(\xi)\psi(\xi)e^{i\xi x_0}$ of variable ξ is integrable in the Riemann sense on $[-1,1]^d$. By definition of the Riemannian integral we get

$$\lim_{n\to+\infty} n^{-d}\cdot\sum_{k\in\mathbb{Z}^d}\mu\left(\frac{k}{n}\right)\psi\left(\frac{k}{n}\right)e^{i\frac{k}{n}x_0} = \int\limits_{[-1,1]^d}\mu(\xi)\psi(\xi)e^{i\xi x_0}d\xi$$
$$= (2\pi)^{d/2}\cdot\widehat{\mu\psi}(-x_0)\ .$$

Therefore,

$$\lim_{n\to+\infty} F_n(x_0) = (2\pi)^{dp/2}\cdot|\widehat{\mu\psi}(-x_0)|^p\ . \tag{3.2}$$

Now we use the fact that

$$\left\|\sum_{k\in\mathbb{Z}^d}\psi\left(\frac{k}{n}\right)e^{ikx}\right\|_p \le cn^{d(1-1/p)}\ ,\quad n\in\mathbb{N}, \tag{3.3}$$

where c does not depend on n. For $0<p\le 1$ inequality (3.3) is proved in [7] with the help of the Poisson summation formula. The case $1<p\le+\infty$ is an immediate consequence of the Nikolskii inequality (see, for instance, [10], p. 147) and (3.3) for $p=1$. Using inequality $(*)$ for $W_n(x)=\sum\limits_{k\in\mathbb{Z}^d}\psi\left(\frac{k}{n}\right)e^{ikx}$ and (3.3) we have

$$\int\limits_{\mathbb{R}^d}F_n(x)dx = n^{-dp}\cdot\int\limits_{[-\pi n,\pi n]^d}\left|\sum_{k\in\mathbb{Z}^d}\mu\left(\frac{k}{n}\right)\psi\left(\frac{k}{n}\right)\cdot e^{ik\frac{x}{n}}\right|^p dx =$$
$$= n^{d(1-p)}\cdot\left\|\sum_{k\in\mathbb{Z}^d}\mu\left(\frac{k}{n}\right)\psi\left(\frac{k}{n}\right)e^{iky}\right\|_p^p \le$$
$$\le c\cdot n^{d(1-p)}\left\|\sum_{k\in\mathbb{Z}^d}\psi\left(\frac{k}{n}\right)e^{iky}\right\|_p^p \le c'\ ,$$

where c' does not depend on n.

Thus, we have proved that the sequence $\{F_n(x)\}_{n=1}^{+\infty}$ fulfils all conditions of Fatou's lemma. Therefore, the integral of its limit can be estimated by the same constant, that is,

$$(2\pi)^{\frac{dp}{2}}\cdot\int\limits_{\mathbb{R}^d}|\widehat{\mu\psi}(-x)|^p dx \le c'$$

and $\widehat{\mu\psi}\in L_p(\mathbb{R}^d)$.

Let now $2<p\le+\infty$. As it follows from above, it is enough to check that

$$\|A_n(\mu)W_n\|_{\tilde{p}} \le c\cdot\|W_n\|_{\tilde{p}}\ . \tag{3.4}$$

To prove (3.4) we will apply the principle of duality. We choose $\delta > 0$, such that $supp\ \psi \subset D_{1-\delta}$ and we consider the polynomial $\Phi_n(x) = \sum\limits_{k\in\mathbb{Z}^d} \varphi\left(\frac{k}{n}\right) e^{ikx}$, where $\varphi \in \mathcal{R}^d_{1-\delta,1}$. For $g \in L_p$ we have

$$A_n(\varphi)(g;x) = \sum_{k\in\mathbb{Z}^d} \varphi\left(\frac{k}{n}\right) g^\wedge(k)e^{ikx} = (2\pi)^{-d}\cdot \int\limits_{\mathbb{T}^d} g(x+h)\Phi_n(h)dh\ ,$$

where

$$g^\wedge(k) = (2\pi)^{-d}\cdot \int\limits_{\mathbb{T}^d} g(x)e^{-ikx}dx, \quad k\in\mathbb{Z}^d,$$

and by virtue of (3.3)

$$\begin{aligned} \|A_n(\varphi)g\|_p &\le (2\pi)^{-d}\cdot \int\limits_{\mathbb{T}^d} \|g(x+h)\|_p\cdot|\Phi_n(h)|dh = \\ &= (2\pi)^{-d}\cdot\|g\|_p\cdot\|\Phi_n\|_1 \le c'\cdot\|g\|_p\ , \end{aligned} \tag{3.5}$$

where c' does not depend on g and n.

Noticing that the inequality (*) being valid for μ is also valid for $\overline{\mu}$ (complex conjugation of μ) we get from (3.5)

$$\begin{aligned} \|A_n(\mu)W_n\|_{\widetilde{p}} &= \sup_{\|g\|_p\le 1} |(A_n(\mu)W_n, g)| = \\ &= (2\pi)^d\cdot \sup_{\|g\|_p\le 1}\left|\sum_{k\in\mathbb{Z}^d}\mu\left(\frac{k}{n}\right)\psi\left(\frac{k}{n}\right)\overline{g^\wedge(k)}\right| \\ &= (2\pi)^d\cdot \sup_{\|g\|_p\le 1}\left|\sum_{k\in\mathbb{Z}^d}\psi\left(\frac{k}{n}\right)\overline{\overline{\mu}\left(\frac{k}{n}\right)\varphi\left(\frac{k}{n}\right)g^\wedge(k)}\right| = \\ &= \sup_{\|g\|_p\le 1}|(W_n, A_n(\overline{\mu})(A_n(\varphi)g))| \le \\ &\le \|W_n\|_{\widetilde{p}}\cdot \sup_{\|g\|_p\le 1}\|(A_n(\overline{\mu})(A_n(\varphi)g)\|_p \le c\cdot\|W_n\|_{\widetilde{p}}\ . \end{aligned}$$

The proof is complete. ■

Remark 3.1. The idea to represent $A_n(\mu)W_n(x)$ as an integral sum of the Fourier transform of the function $\mu\psi$ is not new. It goes back to [3] and [17], where it was applied to some problems in the case $p = +\infty$.

Now we establish the sufficient condition for the validity of inequality (*).

Theorem 3.2 . *Let* $\mu(\xi)$ *be continuous,* $0 < p \le +\infty$, $p^* = \min(1,p)$. *If* $\widehat{\mu\psi}(x) \in L_{p^*}(\mathbb{R}^d)$ *for some* $\psi \in \mathcal{R}^d_{1,1+\delta}$, $\delta > 0$, *then inequality* (*) *is valid for* p.

The proof of Theorem 3.2 repeats the proof of the theorem on the description of Fourier multipliers in terms of the Bessel potential spaces [10], p. 150-151 with some obvious modifications.

Remark 3.2. The requirement of continuity μ is essential in Theorem 3.2 for the case $0 < p < 1$. Indeed, we consider

$$\mu(\xi) = \begin{cases} 1, & \xi = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\widehat{\mu\psi}(x) \equiv 0$, but for $W_n(x) = \sum\limits_{k \in \mathbf{Z}^d} \psi\left(\frac{k}{n}\right) e^{ikx}$ we get from (3.3)

$$\lim_{n\to+\infty} \frac{\|A_n(\mu)W_n\|_p}{\|W_n\|_p} = \lim_{n\to+\infty} \frac{(2\pi)^{d/p}}{\|W_n\|_p} \geq c \cdot \lim_{n\to+\infty} n^{-d(1-1/p)} = +\infty ,$$

that is, $(*)$ fails.

We will see that in many cases the condition "$\widehat{\mu\psi} \in L_q(\mathbb{R}^d)$" does not depend on $\psi \in \mathcal{R}^d$. In particular, this holds for infinitely differentiable on $\mathbb{R}^d \setminus \{0\}$ functions. This observation gives the following criterion of the validity of inequality $(*)$ in the case $\widetilde{p} = p^*$, that is, for $0 < p \leq 1$ and $p = +\infty$.

Theorem 3.3. *Let $0 < p \leq 1$ or $p = +\infty$. Let also the function $\mu(\xi)$ be continuous on $\mathbb{R}^d$ and infinitely differentiable on $\mathbb{R}^d \setminus \{0\}$. Then inequality $(*)$ is valid for p if and only if $\widehat{\mu\psi} \in L_{p^*}(\mathbb{R}^d)$ for some (any) $\psi \in \mathcal{R}^d$.*

Proof. It is enough to prove that the condition "$\widehat{\mu\psi} \in L_{p^*}(\mathbb{R}^d)$" does not depend on $\psi \in \mathcal{R}^d$. Then Theorem 2.3 will immediately follow from Theorems 3.1 and 3.2.

Let $\widehat{\mu\psi} \in L_{p^*}(\mathbb{R}^d)$ for $\psi \in \mathcal{R}^d_{a,b}$, $0 < a < b < +\infty$ and $\varphi \in \mathcal{R}^d_{a',b'}$, $0 < a' < b' < +\infty$.

Then $\mu\varphi = \mu\psi + \mu(\varphi - \psi)$. Clearly, $\varphi(\xi) - \psi(\xi) = 0$ for $\xi \in \overline{D}_{\min(a,a')}$ and $\xi \in \mathbb{R}^d \setminus D_{\max(b,b')}$.

Therefore, $\mu(\varphi - \psi)$ and its Fourier transform belongs to the Schwartz space $\mathcal{S}$ of rapidly decreasing test functions. Therefore, the Fourier transform of $\mu(\varphi - \psi)$ is in $L_{p^*}(\mathbb{R}^d)$. The proof is complete. ∎

Theorem 3.4. *Let $\mu(\xi)$ be continuous on $\mathbb{R}^d$ and infinitely differentiable on $\mathbb{R}^d \setminus \{0\}$. Then*

1. *If $(*)$ is valid for $p = +\infty$, it is also valid for all $1 \leq p \leq +\infty$;*
2. *If $(*)$ is valid for some $0 < p_0 \leq 1$, it is also valid for all $p_0 \leq p \leq +\infty$.*

Proof. Part 1. follows immediately from Theorems 3.3 and 3.2, If $(*)$ is valid for some $0 < p_0 \leq 1$, then by Theorem 3.3 $\widehat{\mu\psi} \in L_{p_0}(\mathbb{R}^d)$ for some $\psi \in \mathcal{R}^d$. Since $\widehat{\mu\psi}$ decreases to 0 at infinity, $\widehat{\mu\psi}(x) \in L_p(\mathbb{R}^d)$ for $p_0 \leq p \leq 1$ and by Theorem 3.3 $(*)$ is valid for $p_0 \leq p \leq 1$ and $p = +\infty$. In view of part 1. it is also valid for $1 < p < +\infty$. ∎

Remark 3.3. As a rule, assertions like Theorem 3.4 are proved by applying interpolation theorems, whereby the information on the boundedness of a given operator in two "limiting" spaces is needed. We notice that in difference to standard methods only one space (L_{p_0} or L_∞) was enough for us. Moreover, our approach does not use any specific tools of harmonic analysis. Practically, it is based on the definition of the Riemann integral and elementary properties of the Fourier transform.

4. The Fourier transform of some functions

In this section we will deal with the Fourier transforms of some homogeneous distributions. We will study the influence of multipliers belonging to the Schwartz space $\mathcal{S}$ of rapidly decreasing test functions and satisfying some additional conditions on their asymptotic behavior. We use some facts of the theory of homogeneous distributions that can be found in [6]. It should be noticed that the technique we use here was partly developed in [5]. For the sake of convenience and for better reading we adopt Lemma 4.1 and Theorem 4.1 from [9] with full proofs.

Definition 4.1. A function $f(\xi)$ defined on $\mathbb{R}^d\{\backslash 0\}$ is called homogeneous of order $a \in \mathbb{R}$, if

$$f(t\xi) = t^a f(\xi) \tag{4.1}$$

for $t > 0$ and $\xi \in \mathbb{R}^d \setminus \{0\}$.

In the spherical coordinates

$$f(\xi) = r^a \Phi(u), \quad r = |\xi| > 0, \quad u \in \mathcal{S}^{d-1}, \tag{4.2}$$

where $\mathcal{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ (as usual, we admit that $\mathcal{S}^0$ consists of two points: 1 and -1). If in addition, $f \in C^\infty(\mathbb{R}^d \setminus \{0\})$, then $\Phi(u)$ is bounded on $\mathcal{S}^{d-1}$ and f has at most polynomial growth at infinity; therefore, it is a regular element of the space $\mathcal{S}'$ of distributions on $\mathcal{S}$, that is

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(\xi)\varphi(\xi)d\xi, \quad \varphi \in \mathcal{S} .$$

We recall that for $g \in \mathcal{S}'$ and $\nu \in \mathbb{N}_0^d$ the derivative $\mathcal{D}^\nu g$ is defined by

$$\langle \mathcal{D}^\nu g, \varphi \rangle = (-1)^{|\nu|_1} \cdot \langle g, \mathcal{D}^\nu \varphi \rangle, \ \varphi \in \mathcal{S} , \tag{4.3}$$

where

$$\mathcal{D}^\nu \varphi = \frac{\partial^{|\nu|_1} \varphi}{\partial x_1^{\nu_1} \ldots \partial x_d^{\nu_d}} .$$

The Fourier transform of $g \in \mathcal{S}'$ is given by

$$\langle \widehat{g}, \varphi \rangle = \langle g, \widehat{\varphi} \rangle, \quad \varphi \in \mathcal{S} .$$

We notice that if $g \in \mathcal{S}' \cap C^\infty(\mathbb{R}^d \setminus \{0\})$, the restriction of $\mathcal{D}^\nu g$ defined by (4.3) to $\mathcal{S}_0 = \{\varphi \in \mathcal{S} \ : \ \operatorname{supp} \varphi \subset \mathbb{R}^d \setminus \{0\}\}$ coincides as an element of the dual space $\mathcal{S}_0'$ with the pointwise derivative of g.

The preliminary estimate of the asymptotic behavior of the Fourier transform of $f\psi$, where $\psi \in \mathcal{S}$ and f is homogeneous of a non-negative order, is given by the following

Lemma 4.1. *If* $f \in C^\infty(\mathbb{R}^d \setminus \{0\})$ *is homogeneous of order* $a \in \mathbb{R}$, $a \geq 0$, *then for any* $\psi \in \mathcal{S}$

$$|\widehat{f\psi}(x)| \leq c \cdot (1+|x|)^{-[a]-d+1}\ , \quad x \in \mathbb{R}^d\ , \tag{4.4}$$

where c *does not depend on* x.

Proof. We put $m = [a]+d-1$. Let $\nu \in \mathbb{N}_0^d$ and $|\nu|_1 \leq m$. Then $\mathcal{D}^\nu f$ is homogeneous of order $a - |\nu|_1$ as an element of $\mathcal{S}'$ [6], p. 75. With the help of remarks given above, $\mathcal{D}^\nu f$ is a regular element of $\mathcal{S}_0'$ and

$$\mathcal{D}^\nu f(\xi) = r^{a-|\nu|_1} \cdot \Phi_\nu(u), \quad r = |\xi| > 0, \quad u \in S^{d-1}\ .$$

Since $\Phi_\nu(u)$ is bounded on S^{d-1},

$$\begin{aligned} \|\mathcal{D}^\nu f\|_{L_1(D_1)} &= \int\limits_{S^{d-1}} \int\limits_0^1 r^{a-|\nu|_1+d-1} \cdot \Phi_\nu(u)\, dr\, d\mathcal{S}(u) \\ &\leq c(\nu) \int\limits_0^1 r^{a-|\nu|_1+d-1} dr < +\infty\ , \end{aligned} \tag{4.5}$$

where $d\mathcal{S}(u)$ is a surface element of $\mathcal{S}^{d-1}$. Applying the Leibniz formula for the derivative of the product we deduce from (4.5) that $\mathcal{D}^\nu(f\psi) \in L_1(\mathbb{R}^d)$ for $|\nu|_1 \leq m$.

Since $f\psi \in L_1(\mathbb{R}^d)$, the inequality (4.4) is valid for $|x| \leq 1$. Let now $|x| > 1$. Since

$$x^\nu \widehat{f\psi}(x) = (-i)^{|\nu|_1} \widehat{\mathcal{D}^\nu(f\psi)}(x), \quad x \in \mathbb{R}^d\ ,$$

we obtain for $|\nu|_1 \leq m$, $x \in \mathbb{R}^d$

$$|x^\nu| \cdot |\widehat{f\psi}(x)| \leq \|\mathcal{D}^\nu(f\psi)\|_{L_1(\mathbb{R}^d)}$$

and

$$|\widehat{f\psi}(x)| \leq \left(\sum_{|\nu|_1=m} |x^\nu|\right)^{-1} \cdot \sum_{|\nu|_1=m} \|\mathcal{D}^\nu(f_\psi)\|_{L_1(\mathbb{R}^d)} \leq c \cdot |x|^{-m}\ ,$$

that completes the proof. ∎

By $\mathcal{X}^d$ we denote the space of real valued radial functions ψ, such that $\psi(0) = 1$.

Theorem 4.1. *Suppose* $f(\xi) \in C^\infty(\mathbb{R}^d \setminus \{0\})$ *is homogeneous of order* $a \in \mathbb{R}$, $a \geq 0$, *it is not a polynomial and* $\psi(\xi) \in \mathcal{X}^d$. *Then* $\widehat{f\psi}(x) \in L_p(\mathbb{R}^d)$ *if and only if* $\frac{d}{d+a} < p \leq +\infty$.

Proof. We will prove that

$$|\widehat{f\psi}(x)| \leq c_1(1+|x|)^{-(d+a)}\ , \quad x \in \mathbb{R}^d \tag{4.6}$$

and there exist $\varrho > 0$, $\theta > 0$ and $u_0 \in \mathcal{S}^{d-1}$, such that

$$|\widehat{f\psi}(x)| \geq c_2 \cdot |x|^{-(d+a)}, \quad x \in \Omega, \tag{4.7}$$

where

$$\Omega \equiv \Omega(\varrho, \theta, u_0) = \{x = ru : r \geq \varrho, \quad u \in \mathcal{S}^{d-1}, \cos\theta \leq (u, u_0) \leq 1\}.$$

Clearly, Theorem 4.1 follows from (4.6) and (4.7). Indeed, since $a \geq 0$, f is bounded on $\overline{D}_1$ and $\widehat{f\psi} \in L_\infty(\mathbb{R}^d)$. If $\frac{d}{d+a} < p < +\infty$, then $\sigma = d - 1 - p(d+a) < -1$ and by (4.6) we get after the transfering to the spherical coordinates

$$\|\widehat{f\psi}\|^p_{L_p(\mathbb{R}^d)} \leq c \cdot \left\{1 + \int_{\mathcal{S}^{d-1}} \int_1^{+\infty} r^{-p(d+a)} \cdot r^{d-1} dr dS(u)\right\}$$
$$\leq c' \cdot \left\{1 + \int_1^{+\infty} r^\sigma dr\right\} < +\infty.$$

Let now $0 < p \leq \frac{d}{d+a}$. Then $\sigma \geq -1$ and we obtain from (4.7)

$$\|\widehat{f\psi}\|^p_{L_p(\mathbb{R}^d)} \geq \|\widehat{f\psi}\|^p_{L_p(\Omega)} \geq c \cdot \int_{\cos\theta \leq (u,u_0) \leq 1} \int_\varrho^{+\infty} r^{-p(d+a)} \cdot r^{d-1} dr dS(u)$$
$$= c' \cdot \int_\varrho^{+\infty} r^\sigma dr = +\infty.$$

First we prove (4.6) and (4.7) for functions ψ in

$$\mathcal{X}_0^d = \left\{\psi \in \mathcal{X}^d : \widehat{\psi} \geq 0, \ \operatorname{supp} \widehat{\psi} \subset D_{3/4}\right\}. \tag{4.8}$$

Since f is bounded on $\overline{D}_1 \setminus \{0\}$, $f\psi$ belongs to $L_1(\mathbb{R}^d)$. This implies (4.6) for $|x| \leq 1$. Let now $|x| > 1$. On the basis of the Fourier transform properties of homogeneous distributions [6], Theorems 7.1.16, 7.1.18, pp. 167-168, $\widehat{f} \in \mathcal{S}'$ is homogeneous of order $-(d+a)$ and it belongs to $C^\infty(\mathbb{R}^\lceil \setminus \{\prime\})$, in particular, it is regular on $\mathcal{S}_0$ and

$$\widehat{f}(x) = r^{-(d+a)} \cdot \Psi(u), \quad r = |x| > 0, \quad u \in \mathcal{S}^{d-1}. \tag{4.9}$$

Noticing that $\widehat{\psi}(x - \cdot)$ belongs to $\mathcal{S}_0$ for $|x| > 1$ and applying the properties of convolution [6], Theorem 7.1.15, p. 166 as well as (4.8) and (4.9) we obtain

$$|\widehat{f\psi}(x)| = |<\widehat{f},\widehat{\psi}(x-\cdot)>| = \left|\int_{\mathbb{R}^d} \widehat{f}(y)\widehat{\psi}(x-y)dy\right| =$$

$$= \left|\int_{|x-y|\le 3/4} \widehat{f}(y)\widehat{\psi}(x-y)dy\right| \le \max_{|x-y|\le 3/4} |\widehat{f}(y)| \cdot \int_{\mathbb{R}^d} \widehat{\psi}(y)dy \le$$

$$\le \mathrm{c} \cdot \max_{|x-y|\le 3/4} |y|^{-(d+a)} \le \mathrm{c}' \cdot |x|^{-(d+a)}$$

that proves (4.6).

To prove the lower estimate we notice first that since f is not a polynomial, $\widehat{f}$ can not be concentrated at 0 and, therefore, there exists $u_0 \in S^{d-1}$, such that $\Psi(u_0) \neq 0$. Without loss of generality we may assume that $\mathrm{Re}\Psi(u_0) > 0$. We choose $\theta > 0$ from the condition

$$\mathrm{Re}\Psi(u) \ge \frac{1}{2}\mathrm{Re}\Psi(u_0), \quad u \in S^{d-1}, \quad \cos 2\theta \le (u, u_0) \le 1.$$

Let $\varrho > 1$ be so large that the conditions $x \in \Omega(\varrho, \theta, u_0)$, $|y - x| \le \frac{3}{4}$ imply $y \in \Omega(1, 2\theta, u_0)$. Then we obtain for $x \in \Omega(\varrho, \theta, u_0)$

$$|\widehat{f\psi}(x)| = \left|\int_{|x-y|\le 3/4} \widehat{f}(y)\widehat{\psi}(x-y)dy\right| \ge$$

$$\ge \int_{|x-y|\le 3/4} |y|^{-(d+a)} \cdot \mathrm{Re}\Psi\left(\frac{y}{|y|}\right)\widehat{\psi}(x-y)dy \ge$$

$$\ge \frac{1}{2}\mathrm{Re}\psi(u_0) \cdot \int_{|x-y|\le 3/4} |y|^{-(d+a)}\widehat{\psi}(x-y)dy \ge$$

$$\ge 2^{-(d+a)-1} \cdot \mathrm{Re}\Psi(x) \cdot |x|^{-(d+a)} \int_{\mathbb{R}^d} \widehat{\psi}(y)dy =$$

$$= \mathrm{c} \cdot |x|^{-(d+a)} ,$$

where $\mathrm{c} = 2^{-(d+a)-1} \cdot \mathrm{Re}\Psi(u_0)(2\pi)^{d/2} \cdot \psi(0) > 0$. The inequality (4.7) is proved.

Let now ψ be an arbitrary function in $\mathcal{X}^d$. We set

$$f\psi = f\varphi + f(\psi - \varphi), \quad \varphi \in \mathcal{X}_0^d .$$

Clearly,

$$\psi(\xi) - \varphi(\xi) = \alpha|\xi|^2\psi_1(\xi) ,$$

where $\psi_1 \in \mathcal{X}^d$ and

$$\alpha = \lim_{\xi \to 0} \frac{\psi(\xi) - \varphi(\xi)}{|\xi|^2} .$$

Therefore,

$$f(\xi)\psi(\xi) = f(\xi)\varphi(\xi) + \alpha f(\xi) \cdot |\xi|^2 \cdot \psi_1(\xi) . \tag{4.10}$$

Noticing that the function $g(\xi) \equiv f(\xi)|\xi|^2$ belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$ and is homogeneous of order $a+2$, we obtain from Lemma 4.1 that

$$|\widehat{g\psi_1}(x)| \leq c(1+|x|)^{-(d+[a]+1)} , \quad x \in \mathbb{R}^d . \tag{4.11}$$

From (4.6) for function in $\mathcal{X}_0^d$, (4.10) and (4.11) we get for $x \in \mathbb{R}^d$

$$|\widehat{f\psi}(x)| \leq |\widehat{f\varphi}(x)| + |\alpha| \cdot |\widehat{g\psi_1}(x)| \leq$$
$$\leq c' \left(1 + |x|^{-(d+a)} + |x|^{-(d+[a]+1)}\right) \leq c'' \left(1 + |x|^{-(d+a)}\right) .$$

The estimate (4.6) is proved for all $\psi \in \mathcal{X}^d$.

We put

$$\widetilde{\varrho} = \max \left\{ \varrho, \left(\frac{2|\alpha|c}{c_2}\right)^{\frac{1}{1-\{a\}}} \right\} ,$$

where c and c_2 are the constants from (4.11) and (4.7) respectively. Then for $|x| \geq \widetilde{\varrho}$

$$c_2 - c|\alpha| \cdot |x|^{-(1-\{a\})} \geq \frac{c_2}{2} . \tag{4.12}$$

From (4.7) for functions in $\mathcal{X}_0^d$, (4.11) and (4.12) we have for $x \in \Omega(\widetilde{\varrho}, \theta, u_0)$

$$|\widehat{f\psi}(x)| \geq |\widehat{f\varphi}(x)| - |\alpha| \cdot |\widehat{g\psi_1}(x)| \geq$$
$$\geq c_2|x|^{-(d+a)} - c|\alpha| \cdot |x|^{-(d+[a]+1)} \geq$$
$$\geq |x|^{-(d+a)} \cdot \left(c_2 - c|\alpha| \cdot |x|^{-(1-\{a\})}\right) \geq \frac{c_2}{2}|x|^{-(d+a)} .$$

The proof of Theorem 4.1 is complete. ∎

By $\mathcal{Y}^d$ we denote the set of functions $\psi(\xi) = \prod_{j=1}^{d} \psi_j(\xi_j)$, where $\psi_j \in \mathcal{X}^1$, $j = 1, \ldots, d$.

Theorem 4.2. *Let* $f(\xi) = \sum_{j=1}^{d} f_j(\xi_j)$, $f_j \in C^\infty(\mathbb{R} \setminus \{0\})$, $j = 1 \ldots, d$, f_j *be homogeneous of order* $\gamma \in \mathbb{R}$, $\gamma \geq 0$ *and at least one of them be not a polynomial. Let also* $\psi(\xi) \in \mathcal{Y}^d$. *Then* $\widehat{f\psi}(x) \in L_p(\mathbb{R}^d)$ *if and only if* $\frac{1}{1+\gamma} < p \leq +\infty$.

Proof. Using the Fourier transform properties for the tensor product (see, for instance, [16], p. 134), we get

$$\widehat{f\psi}(x) = \sum_{j=1}^{d} \widehat{f_j\psi_j}(x_j) \cdot \prod_{\substack{\nu=1 \\ \nu \neq j}}^{d} \widehat{\psi_\nu}(x_\nu) . \tag{4.13}$$

Let $\frac{1}{1+\gamma} < p \le +\infty$. We put $N = \left[\frac{1}{p}\right] + 1$, so that $Np > 1$. Since $\widehat{\psi_\nu} \in \mathcal{S}(\mathbb{R})$, we derive from (4.13) and (4.6) with $d = 1$ (obviously, it is valid for polynomials as well)

$$\|\widehat{f\psi}\|^p_{L_p(\mathbb{R}^d)} \le c \cdot \sum_{j=1}^{d} \int_{\mathbb{R}} (1+|x_j|)^{-p(1+\gamma)} dx_j \cdot \prod_{\substack{\nu=1 \\ \nu \ne j}}^{d} \int_{\mathbb{R}} (1+|x_\nu|)^{-Np} dx_\nu < +\infty .$$

Let now $0 < p \le \frac{1}{1+\gamma}$. Without loss of generality, f_1 is not a polynomial. Because of (4.7) with $d = 1$, we get

$$|\widehat{f_1\psi_1}(t)| \ge c_2 \cdot |t|^{-(1+\gamma)} \tag{4.14}$$

for $t \ge \varrho$ or for $t \le -\varrho$, where $\varrho \ge 1$. We will assume that (4.14) is valid for $t \ge \varrho$. We consider a point $X^0 = (x_2^0, \dots, x_d^0) \in \mathbb{R}^{d-1}$ such that $\widehat{\psi_\nu}(x_\nu^0) \ne 0$, $\nu = 2, \dots, d$. Then for some $\sigma > 0$

$$\prod_{\nu=2}^{d} |\widehat{\psi_\nu}(x_\nu)| \ge \alpha > 0, \quad x_\nu \in U(x_\nu^0), \quad \nu = 2, \dots, d, \tag{4.15}$$

where $U(x_\nu^0) = \{t : \ |t - x_\nu^0| < \sigma\}$. From (4.14) and (4.15) we obtain for $x \in [\varrho, +\infty) \times U(X^0)$, where $U(X^0) = U(x_2^0) \times \dots \times U(x_d^0) \subset \mathbb{R}^{d-1}$, that

$$|\widehat{f_1\psi_1}(x_1)| \cdot \prod_{\nu=2}^{d} |\widehat{\psi_\nu}(x_\nu)| \ge c_2\alpha |x_1|^{-(1+\gamma)} . \tag{4.16}$$

In (4.16) for $d = 1$ we put $\prod_{\nu=2}^{d} \equiv 1$, $\alpha = 1$, that is, (4.16) coincides with (4.14) in this case.

Let

$$M = \max_{(x_2,\dots,x_d) \in U(X^0)} \sum_{j=2}^{d} (1+|x_j|)^{-(1+\gamma)} \cdot \prod_{\substack{\nu=2 \\ \nu \ne j}}^{d} (1+|x_\nu|)^{-(2+\gamma)} .$$

By (4.6) we have for $x \in [\varrho, +\infty) \times U(X^0)$

$$\begin{aligned} J \equiv \sum_{j=2}^{d} |\widehat{f_j\psi_j}(x_j)| \cdot \prod_{\substack{\nu=1 \\ \nu \ne j}}^{d} |\widehat{\psi_\nu}(x_\nu)| \ \le \\ \le c_1 \sum_{j=2}^{d} (1+|x_j|)^{-(1+\gamma)} \cdot |\widehat{\psi_1}(x_1)| \cdot \prod_{\substack{\nu=2 \\ \nu \ne j}}^{d} |\widehat{\psi_\nu}(x_\nu)| \ \le \\ \le c(1+|x_1|^{-(2+\gamma)}) \cdot \sum_{j=2}^{d} (1+|x_j|)^{-(1+\gamma)} \cdot \prod_{\substack{\nu=2 \\ \nu \ne j}}^{d} (1+|x_\nu|^{-(2+\gamma)}) \ \le \\ \le cM(1+|x_1|^{-(2+\gamma)}) \le c' \cdot |x_1|^{-(2+\gamma)} . \end{aligned} \tag{4.17}$$

We put

$$\widetilde{\varrho} = \max\left\{\varrho,\ \frac{2c'}{c_2 \cdot \alpha}\right\} .$$

Then for $|x_1| \geq \widetilde{\varrho}$

$$c_2\alpha - c'|x_1|^{-1} \geq \frac{c_2\alpha}{2} . \tag{4.18}$$

From (4.13), (4.16)-(4.18) we get for $x \in [\widetilde{\varrho}, +\infty) \times U(X^0)$

$$|\widehat{f\psi}(x)| \geq |\widehat{f_1\psi_1}(x)| \cdot \prod_{\nu=2}^{d} |\widehat{\psi_\nu}(x_\nu)| - J \geq$$
$$\geq |x_1|^{-(1+\gamma)} \cdot (c_2\alpha - c'|x_1|^{-1}) \geq \frac{c_2\alpha}{2} \cdot |x_1|^{-(1+\gamma)}$$

and, therefore,

$$\|\widehat{f\psi}(x)\|^p_{L_p(\mathbb{R}^d)} \geq \int\limits_{U(X^0)} \int\limits_{\widetilde{\varrho}}^{+\infty} |\widehat{f\psi}(x)|^p dx_1 dx_2 \ldots dx_d$$
$$\geq \left(\frac{c_2\alpha}{2}\right)^p \cdot \int\limits_{U(X^0)} dx_2 \ldots dx_d \int\limits_{\widetilde{\varrho}}^{+\infty} x_1^{-p(1+\gamma)} dx_1 = +\infty.$$

The proof is complete. ■

5. Inequalities for trigonometric polynomials

Theorem 5.1. *Let $\mu(\xi) \in C^\infty(\mathbb{R}^d \backslash \{0\})$ be homogeneous of order $a \in \mathbb{R}$, $a > 0$ and $\mu(0) = 0$. If $\mu(\xi)$ is not identical with a polynomial on $\mathbb{R}^d \setminus \{0\}$, then inequality $(*)$ is valid if and only if $\frac{d}{d+a} < p \leq +\infty$.*

Taking into account that $\mathcal{X}^d \subset \mathcal{R}^d$ we immediately obtain Theorem 5.1 from Theorems 3.3, 3.4 and 4.1. ■

Theorem 5.2. *Let $\mu(\xi) = \sum_{j=1}^{d} \mu_j(\xi_j)$, where $\mu_j \in C^\infty(\mathbb{R} \setminus \{0\})$, $\mu_j(0) = 0$, $j = 1, \ldots, d$, μ_j be homogeneous of order $\gamma \in \mathbb{R}$, $\gamma > 0$ and at least one of them do not be identical with a polynomial on $\mathbb{R} \setminus \{0\}$. Then inequality $(*)$ is valid if and only if $\frac{1}{\gamma+1} < p \leq +\infty$.*

Taking into accout that $\mathcal{Y}^d \subset R^d$ we immediately obtain Theorem 5.2 from Theorems 3.3, 3.4 and 4.2. ■

Before we formulate some consequences of Theorems 5.1 and 5.2, we recall the exact definitions of operators we need. For our purposes it will be enough to define them on the space of trigonometric polynomials $\mathcal{T}$. Henceforth,

$$t(x) = \sum_{k \in \mathbf{Z}^d} c_k e^{ikx} \in \mathcal{T} .$$

1. *The partial Weil derivative of order* $\gamma > 0$ on x_j is an operator given by

$$t_j^{(\gamma)}(x) \equiv \frac{\partial^\gamma t(x)}{\partial x_j^\gamma} = \sum_{k \in \mathbf{Z}^d} (ik_j)^\gamma c_k e^{ikx}, \left((\pm i)^\gamma = e^{-\frac{i\pi\gamma}{2}}\right) .$$

2. *The power of the Laplacian* Δ^γ, $\gamma \in \mathbb{R}$, $\gamma > 0$ is defined by

$$\Delta^\gamma t(x) = \sum_{k \in \mathbf{Z}^d} (-1)^\gamma |k|^{2\gamma} c_k e^{ikx}, ((-1)^\gamma = e^{i\pi\gamma}) .$$

We notice that in the one-dimensional case this operator is called the Riesz derivative (see [3], p. 427).

3. *The conjugate operator* is defined for polynomials of one variable by

$$\widetilde{t}(x) = -i \cdot \sum_{k \in \mathbf{Z}^d} \operatorname{sgn} k \cdot c_k e^{ikx} .$$

Theorem 5.3. *Suppose* $d = 1$, $\gamma \in \mathbb{R}$, $\gamma > 0$. *Then the inequality*

$$\|\widetilde{t}^{(\gamma)}\|_p \le c(p;\gamma)\lambda^\gamma \cdot \|t\|_p, \ t \in \mathcal{T}_\lambda, \ \lambda \ge 1 \tag{5.1}$$

is valid if and only if $\frac{1}{\gamma+1} < p \le +\infty$.

Proof. The inequality (5.1) is of type $(*)$. The operator $A_\lambda(\mu) = \lambda^{-\gamma} \cdot (\widetilde{\ })^{(\gamma)}$ is generated by the function $\mu(\xi) = i \cdot \operatorname{sgn}\xi \cdot (e^{\frac{i\pi\gamma}{2}} \cdot \xi_+^\gamma + e^{-\frac{i\pi\gamma}{2}} \cdot \xi_-^\gamma)$ that is homogeneous of order γ and it is not a polynomial for each real $\gamma > 0$. Now inequality (5.1) follows immediately from Theorem 5.1. ■

Theorem 5.3 is a direct extension of a classical result on the conjugate function, namely, if $\gamma = 0$, the inequality (5.1) is valid if and only if $1 < p < +\infty$ (see, for instance, ([8], Chapter 4). For $\gamma > 0$ this inequality seems to be new even for natural γ.

Theorem 5.4. *Suppose* $\gamma > 0$, $\gamma \notin \mathbb{N}$. *Then the inequality*

$$\left\| \sum_{j=1}^d \frac{\partial^\gamma t}{\partial x_j^\gamma} \right\|_p \le c(d;p;\gamma) \cdot \lambda^\gamma \|t\|_p, \quad t \in \mathcal{T}_\lambda, \ \lambda \ge 1 \tag{5.2}$$

is valid if and only if $\frac{1}{\gamma+1} < p \le +\infty$.

Proof. The inequality (5.2) is of type (*), where $A_\lambda(\mu) = \lambda^{-\gamma} \cdot \sum_{j=1}^{d} \frac{\partial^\gamma}{\partial x_j^\gamma}$, and $\mu(\xi) = \sum_{j=1}^{d} (e^{\frac{i\pi\gamma}{2}} \cdot (\xi_j)_+^\gamma + e^{-\frac{i\pi\gamma}{2}} \cdot (\xi_j)_-^\gamma)$ satisfy the conditions of Theorem 5.2. ■

In the case $d = 1$ inequality (5.2) is of Bernstein type (see, for references, section 1). Theorem 5.4 is one of its possible extensions to the case of several variables.

The following theorem gives another extension, in which the answer will already depend on the dimension.

Theorem 5.5. *Suppose* $\gamma > 0$, $\gamma \notin \mathbb{N}$. *Then the inequality*

$$\|\Delta^\gamma t\|_p \le c(d;p;\gamma) \cdot \lambda^{2\gamma} \|t\|_p, \quad t \in \mathcal{T}_\lambda, \ \lambda \ge 1 \tag{5.3}$$

is valid if and only if $\frac{d}{d+2\gamma} < p \le +\infty$.

Proof. (5.2) is of type (*), where $A_\lambda(\mu) = \lambda^{-2\gamma} \cdot \Delta^\gamma$, $\mu(\xi) = e^{i\pi\gamma} \cdot |\xi|^{2\gamma}$ is homogeneous of order 2γ and it is not a polynomial for $\gamma \notin \mathbb{N}$. Thus, Theorem 5.5 is the special case of Theorem 5.1. ■

The approach to treating inequalities for trigonometric polynomials we have worked out in this paper find further applications apart from inequalities generated by homogeneous functions. We give only one example.

Let $\mathcal{G} \subset \mathbb{R}^d$ be a bounded set. For 2π-periodic functions f in L_1 we consider the partial sums of its Fourier series

$$\mathcal{S}_\lambda^{\mathcal{G}}(f;x) = \sum_{k \in \lambda\mathcal{G}} f^\wedge(k) e^{ikx}, \quad \lambda > 0 \ , \tag{5.5}$$

where $\lambda\mathcal{G} = \{k \in \mathbb{Z}^d \ : \ \frac{k}{\lambda} \in \mathcal{G}\}$. As is well-known (see, for instance, [8], Chapter 4), in the classical case $d = 1$, $\mathcal{G} = [-1,1]$ the sequence of norms of operators $\mathcal{S}_\lambda^{\mathcal{G}}$ in L_1 and L_∞ is unbounded. The same result is valid if $\mathcal{G} = [-1,1]^d$ or $\mathcal{G} = \overline{D}_1$. We will show that this fact remains valid for a wide class of sets $\mathcal{G}$.

Theorem 5.6. *Suppose* $\mathcal{G} \subset \mathbb{R}^d$ *has a positive measure and the measure of its boundary is equal to* 0. *Then for* $X = L_1$ *or* $X = L_\infty$

$$\overline{\lim_{\lambda \to +\infty}} \|\mathcal{S}_\lambda^{\mathcal{G}}\|_{X \to X} = +\infty \ .$$

Proof. Without loss of generality, $\overline{\mathcal{G}} \subset D_1$. Clearly,

$$\mathcal{S}_\lambda^{\mathcal{G}}(f;x) = \sum_{k \in \mathbb{Z}^d} \mathcal{X}_{\mathcal{G}}\left(\frac{k}{\lambda}\right) f^\wedge(k) e^{ikx} \ ,$$

where $\mathcal{X}_{\mathcal{G}}(\xi)$ is the characteristic function of $\mathcal{G}$. Noticing that $\mathcal{X}_{\mathcal{G}}$ is discontinuous only on $\partial\mathcal{G}$ with measure 0 we obtain by the Lebesgue criterion that

$\mathcal{X}_{\mathcal{G}}$ is integrable in the Riemann sense on $\overline{D}_1$. Obviously, $\mathcal{X}_{\mathcal{G}}$ does not coincide almost everywhere with a continuous function; therefore, the Fourier transform of the function $\mathcal{X}_{\mathcal{G}}(\xi) \equiv \mathcal{X}_{\mathcal{G}}(\xi)\psi(\xi)$, where $\psi \in \mathcal{R}^d_{1-\varepsilon,1}$, $0 < \varepsilon < 1$, $\overline{\mathcal{G}} \subset D_{1-\varepsilon}$, can not belong to $L_1(\mathbb{R}^d)$. Hence, by Theorem 3.1 the inequality of type $(*)$ generated by $\mu \equiv \mathcal{X}_{\mathcal{G}}$ fails for $p = 1$ and $p = +\infty$, that is, for each $c > 0$ there exists $\lambda \equiv \lambda(c) \geq 1$ and $t \in \mathcal{T}_\lambda$, such that,

$$\|\mathcal{S}^{\mathcal{G}}_\lambda t\|_X > \mathrm{c} \cdot \|t\|_X . \tag{5.6}$$

Noticing that for $f \in X$ and $\lambda \geq 1$

$$\begin{aligned} \|\mathcal{S}^{\mathcal{G}}_\lambda f\|_X &= \left\| (2\pi)^{-d} \int_{\pi^d} f(x+\xi) \left\{ \sum_{k \in \lambda\mathcal{G}} e^{ikx} \right\} d\xi \right\|_X \leq \\ &\leq (2\pi)^d \cdot \|f\|_X \cdot \left\| \sum_{k \in \lambda\mathcal{G}} e^{ikx} \right\|_1 \leq \\ &\leq \operatorname{card}\{\lambda\mathcal{G} \cap \mathbb{Z}^d\} \cdot \|f\|_X \leq (2\lambda+1)^d \cdot \|f\|_X , \end{aligned}$$

we obtain that $\overline{\lim_{\mathrm{c} \to +\infty}} \lambda(\mathrm{c}) = +\infty$. Then we get from (5.6) that

$$\overline{\lim_{\lambda \to +\infty}} \|\mathcal{S}^{\mathcal{G}}_\lambda\|_{X \to X} \geq \overline{\lim_{\lambda \to +\infty}} \left(\sup_{t \in \mathcal{T}_\lambda} \frac{\|\mathcal{S}^{\mathcal{G}}_\lambda\|_X}{\|t\|_X} \right) = +\infty .$$

The proof is complete. ■

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