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FOURTH POWER MOMENT OF DEDEKIND ZETA-FUNCTIONS OF REAL QUADRATIC NUMBER FIELDS WITH CLASS NUMBER ONE

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1. The problem

The Dedekind zeta function of a real quadratic number field F is defined by

$$\zeta_F(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s}, \quad \mathrm{Re}\, s > 1, \tag{1.1}$$

with $\mathfrak{a}$ running over all integral ideals of F. It continues to $\mathbb{C}$, and $(s-1)\zeta_F(s)$ turns out to be entire. Our principal aim is to establish a spectral decomposition of the fourth power moment of ζ_F:

$$\mathcal{Z}_2(g, F) = \int_{-\infty}^{\infty} |\zeta_F(\tfrac{1}{2} + it)|^4 g(t)dt, \tag{1.2}$$

where g is assumed to be entire, and of rapid decay in any fixed horizontal strip. With this, we extend to ζ_F the discussion that are developed in [10] on the Riemann zeta-function, and in [2] on the Dedekind zeta-function of the Gaussian number field. The relevant spectral theories in [10] and [2] are, respectively, on the full modular group and on the Picard group; here it is on the Hilbert modular group over F, as is to be made precise in Section 3.

Basic convention. We assume that F is of class number one, and that the fundamental unit $\epsilon_0 > 1$ of F has norm equal to -1. Thus each ideal in F has a totally positive generator. The first assumption is essential for our argument, but the second is mainly for the sake of simplicity. Notations are introduced at the places where they are needed first time, and thereafter continue to be effective.

To begin with, we put

$$\mathcal{J}(z_1, z_2, z_3, z_4; g) = \int_{-\infty}^{\infty} \zeta_F(z_1 + it)\zeta_F(z_2 + it)\zeta_F(z_3 - it)\zeta_F(z_4 - it)g(t)dt, \tag{1.3}$$

with all $\operatorname{Re} z_j > 1$. This continues meromorphically to $\mathbb{C}^4$. In particular, it is regular at $p_{\frac{1}{2}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and we have

$$\mathcal{Z}_2(g, \mathrm{F}) = \mathcal{I}(p_{\frac{1}{2}}; g) + a_0 g(\tfrac{1}{2}i) + b_0 g(-\tfrac{1}{2}i) + a_1 g'(\tfrac{1}{2}i) + b_1 g'(-\tfrac{1}{2}i) \tag{1.4}$$

with constants a_0, b_0, a_1, b_1 which depend on F, and could be made explicit. On the other hand, in the region of absolute convergence, we have the expression

$$\mathcal{I}(z_1, z_2, z_3, z_4; g) = \sum_{\mathfrak{a}, \mathfrak{b}} (N\mathfrak{a})^{-z_1} (N\mathfrak{b})^{-z_3} \sigma_{z_1 - z_2}(\mathfrak{a}) \sigma_{z_3 - z_4}(\mathfrak{b}) \hat{g}\left(\log \frac{N\mathfrak{b}}{N\mathfrak{a}}\right), \tag{1.5}$$

where

$$\hat{g}(y) = \int_{-\infty}^{\infty} g(t) e^{ity} dt, \quad \sigma_\xi(\mathfrak{a}) = \sum_{\mathfrak{c} | \mathfrak{a}} (N\mathfrak{c})^\xi, \tag{1.6}$$

with $\mathfrak{c}$ being an integral ideal. A sum much similar to (1.5) is treated in [2], but over the Gaussian number field. There an application is made of a natural extension of a dissection argument that is employed in Section 4.6 of [10]. It exploits the lattice structure of the ring of integers in the field. We are going to use the same device. But then we have to transform the sum over ideals in (1.5) into a sum over the elements of $\mathcal{O}$, the ring of integers in F. In the real quadratic case, that is a problem, as there are infinitely many generators for each ideal in $\mathcal{O}$.

2. Initial reduction

To overcome this difficulty, we shall appeal to an instance of partition of one:

Lemma 2.1. *Let* p *be such that its Fourier transform* $\hat{\mathrm{p}}$ *(see (1.6)) is even, real-valued, smooth, supported on a neighbourhood of 0 contained in* $(-\pi, \pi)$, *and moreover* $\hat{\mathrm{p}}(0) = 1$. *Then* p *is even, real-valued, smooth and of rapid decay on* $\mathbb{R}$, *and we have, for any* $x \in \mathbb{R}$,

$$\sum_{n \in \mathbf{Z}} \mathrm{p}(x + n) = 1. \tag{2.1}$$

Also we have, for any $x, y \in \mathbb{R}$,

$$\sum_{n \in \mathbf{Z}} \mathrm{p}(x + n)\mathrm{p}(y + n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\mathrm{p}}(\xi)^2 e^{i(x-y)\xi} d\xi. \tag{2.2}$$

Proof. These identities are results of applications of the Poisson sum formula; in (2.2) the Parseval formula is also used.

We put, for a non-zero $x \in \mathrm{F}$,

$$\chi(x) = \frac{1}{2}\mathrm{p}\left(\frac{\log|x/x'|}{2\log\epsilon_0}\right), \tag{2.3}$$

where x' is the conjugate of x over $\mathbb{Q}$. The identities (2.1) and (2.2) give, for any non-zero $x \in \mathrm{F}$,

$$\sum_{\epsilon}\chi(\epsilon x) = 1, \tag{2.4}$$

$$\sum_{\epsilon}\chi(\epsilon x)^2 = c_\chi, \tag{2.5}$$

respectively. Here ϵ runs over all units in $\mathcal{O}$; that is, $\epsilon = \pm\epsilon_0^\nu$, $\nu \in \mathbb{Z}$. Also we have put

$$c_\chi = \frac{1}{4\pi}\int_{-\pi}^{\pi}\hat{\mathrm{p}}^2(\xi)d\xi. \tag{2.6}$$

Then, for any function f defined over positive reals, we have

$$f(N\mathfrak{a}) = \sum_a f(|N(a)|)\chi(a), \tag{2.7}$$

where a runs over all generators of $\mathfrak{a}$, and $N(a) = aa'$. Thus, formally,

$$\sum_{\mathfrak{a}}\sigma_\xi(\mathfrak{a})f(N\mathfrak{a}) = \sum_{a\in\mathcal{O}_*}\sigma_\xi(a)f(|N(a)|)\chi(a), \tag{2.8}$$

with $\mathcal{O}_* = \mathcal{O}\backslash\{0\}$ and $\sigma_\xi(a) = \sigma_\xi((a))$.

Applying (2.8) to (1.5) we have

$$\begin{aligned}\mathcal{I}(z_1, z_2, z_3, z_4; g) &= \sum_{a,\,b\in\mathcal{O}_*}\frac{\sigma_{z_1-z_2}(a)\sigma_{z_3-z_4}(b)}{|N(a)|^{z_1}|N(b)|^{z_3}}\chi(a)\chi(b)\hat{g}(\log|N(b/a)|)\\ &= \Big\{\sum_{a=b}+\sum_{a\neq b}\Big\}\cdots\\ &= \{\mathcal{I}_0 + \mathcal{I}_+\}(z_1, z_2, z_3, z_4; g),\end{aligned} \tag{2.9}$$

say. By virtue of (2.5)

$$\mathcal{I}_0(z_1, z_2, z_3, z_4; g) = c_\chi\hat{g}(0)\frac{\zeta_{\mathrm{F}}(z_1+z_3)\zeta_{\mathrm{F}}(z_1+z_4)\zeta_{\mathrm{F}}(z_2+z_3)\zeta_{\mathrm{F}}(z_2+z_4)}{\zeta_{\mathrm{F}}(z_1+z_2+z_3+z_4)}. \tag{2.10}$$

On the other hand

$$\begin{aligned}\mathcal{I}_+(z_1, z_2, z_3, z_4; g) &= \sum_{m\in\mathcal{O}_*}|N(m)|^{-z_1-z_3}\\ &\times\sum_{\substack{n\in\mathcal{O}_*\\ n+m\neq 0}}\frac{\sigma_{z_1-z_2}(n)\sigma_{z_3-z_4}(n+m)}{|N(n/m)|^{z_1}|N(1+n/m)|^{z_3}}\chi(n)\chi(n+m)\hat{g}(\log|N(1+m/n)|).\end{aligned} \tag{2.11}$$

Classifying m according to the ideal $\mathfrak{m}=(m)$, we have

$$\mathcal{J}_+(z_1,z_2,z_3,z_4;g) = \sum_{\substack{\mathfrak{m}\\ \mathfrak{m}=(m)}} (N\mathfrak{m})^{-z_1-z_3}$$
$$\times \sum_{\substack{n\in\mathcal{O}_* \\ n+m\neq 0}} \frac{\sigma_{z_1-z_2}(n)\sigma_{z_3-z_4}(n+m)}{|N(n/m)|^{z_1}|N(1+n/m)|^{z_3}} \hat{g}(\log|N(1+m/n)|)$$
$$\times \sum_{\epsilon} \chi(\epsilon n)\chi(\epsilon(n+m)). \tag{2.12}$$

The formula (2.2) gives

$$\sum_{\epsilon} \chi(\epsilon n)\chi(\epsilon(n+m)) = c_\chi(1+m/n), \tag{2.13}$$

where

$$c_\chi(x) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \hat{\mathfrak{p}}(\xi)^2 \exp\left(i\xi\frac{\log|x/x'|}{2\log\epsilon_0}\right) d\xi, \tag{2.14}$$

so that $c_\chi^{\cdot} = c_\chi(1)$.

In this way we are led to binary additive divisor sums over F:

$$\mathcal{B}_m(\alpha,\beta;h) = \sum_{\substack{n\in\mathcal{O}_* \\ n+m\neq 0}} \sigma_\alpha(n)\sigma_\beta(n+m)h(n/m), \quad m \succ 0. \tag{2.15}$$

The condition $m\succ 0$, i.e., totally positive, causes no loss of generality under the present assumptions on F. From (2.12)–(2.13)

$$\mathcal{J}_+(z_1,z_2,z_3,z_4;g) = \sum_{\substack{\mathfrak{m}\\ \mathfrak{m}=(m)}} (N\mathfrak{m})^{-z_1-z_3}\mathcal{B}_m(z_1-z_2,z_3-z_4;g_*(\cdot;z_1,z_3)), \tag{2.16}$$

with

$$g_*(x;\gamma,\delta) = \frac{\hat{g}(\log|N(1+1/x)|)c_\chi(1+1/x)}{|N(x)|^\gamma |N(1+x)|^\delta}. \tag{2.17}$$

Following the argument in [9] basically, we shall, in the next section, transform $\mathcal{B}_m(\alpha,\beta;h)$ into a sum of Kloosterman sums over F, and consequentially, in Section 5, decompose it spectrally with the geometric sum formula for the Hilbert modular group over F, provided α, β lie in an appropriate domain. A condition on h that makes our procedure legitimate is to be given in (2.35). In Section 6 we shall examine g_* if it meets this condition, and apply the result on $\mathcal{B}_m(\alpha,\beta;h)$ to obtain a spectral decomposition of $\mathcal{Z}_2(g,\mathrm{F})$. Because of this, the bulk of our paper is devoted to the study of $\mathcal{B}_m(\alpha,\beta;h)$. We begin it with invoking the Ramanujan expansion, i.e., (2.19), of σ_β in terms of additive characters over F.

Thus, let us put, for $x \in \mathrm{F}$,

$$e[x] = \exp\left(\frac{2\pi i}{\sqrt{\mathrm{D_F}}}(x - x')\right), \tag{2.18}$$

where $\mathrm{D_F}$ is the fundamental discriminant of F. We have

$$\sum_{\substack{\mathfrak{c} \\ \mathfrak{c}=(c)}} \frac{1}{(N\mathfrak{c})^s} \sum_{\substack{a \bmod c \\ (a,c)=1}} e[an/c] = \frac{1}{\zeta_{\mathrm{F}}(s)} \begin{cases} \zeta_{\mathrm{F}}(s-1) & \text{if } n = 0, \\ \sigma_{1-s}(n) & \text{if } n \neq 0, \end{cases} \tag{2.19}$$

provided $\mathrm{Re}\, s > 2$. This can of course be formulated as

$$\sum_{c \in \mathcal{O}_*} \frac{\chi(c)}{|N(c)|^s} \sum_{\substack{a \bmod c \\ (a,c)=1}} e[an/c] = \frac{1}{\zeta_{\mathrm{F}}(s)} \begin{cases} \zeta_{\mathrm{F}}(s-1) & \text{if } n = 0, \\ \sigma_{1-s}(n) & \text{if } n \neq 0. \end{cases} \tag{2.20}$$

Hence, for $\mathrm{Re}\,\beta < -1$,

$$\mathcal{B}_m(\alpha, \beta; h) = \zeta_{\mathrm{F}}(1-\beta) \sum_{c \in \mathcal{O}_*} \frac{\chi(c)}{|N(c)|^{1-\beta}} \sum_{\substack{a \bmod c \\ (a,c)=1}} e[am/c] \mathcal{D}_m(\alpha, a/c\,; h). \tag{2.21}$$

Here

$$\mathcal{D}_m(\alpha, a/c; h) = \sum_{n \in \mathcal{O}_*/\mathrm{U}_c} \sigma_\alpha(n) e[an/c] h(n/m\,; c), \tag{2.22}$$

with

$$h(x; c) = \sum_{\epsilon \in \mathrm{U}_c} h(\epsilon x), \tag{2.23}$$

$$\mathrm{U}_c = \{\epsilon : \text{totally positive unit congruent to } 1 \bmod c\}. \tag{2.24}$$

We fix a generator of the group:

$$\mathrm{U}_c = [\epsilon_c], \quad \epsilon_c > 1,\ \epsilon_c' > 0. \tag{2.25}$$

We assume that h is such that (2.23) is absolutely convergent, and, for $x \in \mathrm{F}$,

$$h(x; c) \ll |N(x)|^{-1-\max(0, \mathrm{Re}\,\alpha)-\mu} \tag{2.26}$$

uniformly in c, with an arbitrary fixed $\mu > 0$. On this and $\mathrm{Re}\,\beta < -1$, the expansion (2.21) holds.

The use of the Ramanujan formula (2.19)–(2.20) is to separate the parameters n and m in $\sigma_\beta(n+m)$ of (2.15). We need to do the same separation for the factor $h(n/m; c)$ too. To this end we extend h to $(\mathbb{R}\backslash\{0\})^2$ via the embedding $x \mapsto (x, x')$, so that in place of (2.23)

$$h(x, x'; c) = \sum_{\nu \in \mathbf{Z}} h(\epsilon_c^\nu x, \epsilon_c^{-\nu} x'). \tag{2.27}$$

Following Hecke, we apply the Poisson sum formula to the right side, getting

$$h(x,x';c)=\sum_{\nu\in\mathbf{Z}}\int_{-\infty}^{\infty}h(\epsilon_c^t x,\epsilon_c^{-t}x')e^{-2\pi i\nu t}dt$$
$$=\frac{1}{\log\epsilon_c}\sum_{\nu\in\mathbf{Z}}\int_0^{\infty}h(\xi x,\xi^{-1}x')\xi^{-2\pi i\nu/\log\epsilon_c}\frac{d\xi}{\xi}. \tag{2.28}$$

The change of variable $\xi\mapsto\xi\sqrt{|x'/x|}$ gives

$$h(e_1|x|,e_2|x'|;c)$$
$$=\frac{1}{\log\epsilon_c}\sum_{\nu\in\mathbf{Z}}\left|\frac{x}{x'}\right|^{\nu\varpi_c i}\int_0^{\infty}h\left(e_1\xi\sqrt{|N(x)|},e_2\xi^{-1}\sqrt{|N(x)|}\right)\xi^{-2\nu\varpi_c i}\frac{d\xi}{\xi}, \tag{2.29}$$

where $e_j=\pm1$, and $\varpi_c=\pi/\log\epsilon_c$. This integral is a function of $\sqrt{|N(x)|}$. Considering its Mellin transform in each quadrant separately, we have, with a certain vertical line $(a)=\{s:\operatorname{Re}s=a\}$,

$$h(e_1|x|,e_2|x'|;c)=\frac{1}{2\pi i\log\epsilon_c}\sum_{\nu\in\mathbf{Z}}\left|\frac{x}{x'}\right|^{\nu\varpi_c i}$$
$$\times\int_{(a)}\widetilde{h}(s-\nu\varpi_c i,s+\nu\varpi_c i;\mathrm{e})\,|N(x)|^{-s}ds, \tag{2.30}$$

where $\mathrm{e}=(e_1,e_2)$, and

$$\widetilde{h}(s_1,s_2;\mathrm{e})=\int_0^{\infty}\int_0^{\infty}h(e_1u_1,e_2u_2)u_1^{s_1-1}u_2^{s_2-1}du_1du_2. \tag{2.31}$$

Thus, a rearrangement gives

$$\mathcal{D}_m(\alpha,a/c;h)=\frac{1}{2\pi i\log\epsilon_c}\sum_{\ell}\sum_{\nu\in\mathbf{Z}}\left|\frac{m}{m'}\right|^{-\nu\varpi_c i}$$
$$\times\int_{(a)}|N(m)|^s\widetilde{h}_\ell(s-\nu\varpi_c i,s+\nu\varpi_c i)\,D_\ell(s,\alpha;\nu;a/c)ds, \tag{2.32}$$

where $1+\max(0,\operatorname{Re}\alpha)<a$, and $\ell=(l_1,l_2)$, $l_j=0,1$. Here

$$\widetilde{h}_\ell(s_1,s_2)=\frac{1}{4}\sum_{\mathrm{e}}e_1^{l_1}e_2^{l_2}\widetilde{h}(s_1,s_2;\mathrm{e}), \tag{2.33}$$

and

$$D_\ell(s,\alpha;\nu;a/c)=\sum_{n\in\mathcal{O}_*/\mathrm{U}_c}\frac{\sigma_\alpha(n)}{|N(n)|^s}\left|\frac{n}{n'}\right|^{\nu\varpi_c i}\operatorname{sgn}^\ell[n]e[an/c],\quad (a,c)=1, \tag{2.34}$$

with $\operatorname{sgn}^\ell[x]=\operatorname{sgn}^\ell[x,x']=(x/|x|)^{l_1}(x'/|x'|)^{l_2}$.

We assume that h is such that

$$\begin{gathered}\widetilde{h}(s_1, s_2; \mathrm{e}) \text{ is regular and } \ll (1+|s_1|+|s_2|)^{-C_0} \\ \text{in the domain } |\mathrm{Re}\, s_1|, |\mathrm{Re}\, s_2| < C_0,\end{gathered} \tag{2.35}$$

with a sufficiently large $C_0 > 0$. Then (2.26) holds for $|\mathrm{Re}\,\alpha| < \frac{1}{2}C_0$, for instance. Namely, (2.32), with an appropriate (a), is valid under (2.35). This assumption on h is quite drastic; a more refined formulation is of course possible, which seems, however, not to be essential for our present purpose.

3. Sum of Kloosterman sums

We are going to show a functional equation for D_ℓ, from which emerges a representation of $\mathcal{B}_m$ in terms of a sum of Kloosterman sums over F (see (3.53) below). To achieve this, there are at least two ways to follow. One is to extend Lemma 3.7 [10], which is originally due to Hecke and Estermann, by using a theta-transformation formula over F. The other is more functional, and it employs the Eisenstein series for the Hilbert modular group over F. We shall take the latter, in order to indicate the existence of an intrinsic geometric structure behind the mean value $\mathcal{Z}_2(g, \mathrm{F})$, and thus possibilities of extension, as well. It should, however, be stressed that the direct treatment developed in this section is based on our local specifications, and the deduction of those results from the general theory of automorphic forms on semisimple Lie groups is naturally possible (see e.g., [6]).

Thus, we shall work with the Lie group

$$G = \mathrm{PSL}_2(\mathbb{R})^2. \tag{3.1}$$

The Hilbert modular group Γ is the discrete subgroup in G resulting from the embedding $g \mapsto (g, g')$ of $\mathrm{PSL}_2(\mathcal{O})$ into G, where the conjugation is applied to matrices element-wise. Write

$$\begin{aligned}\mathrm{n}[x] &= \left(\begin{bmatrix}1 & x_1\\ & 1\end{bmatrix}, \begin{bmatrix}1 & x_2\\ & 1\end{bmatrix}\right),\\ \mathrm{a}[y] &= \left(\begin{bmatrix}\sqrt{y_1} & \\ & 1/\sqrt{y_1}\end{bmatrix}, \begin{bmatrix}\sqrt{y_2} & \\ & 1/\sqrt{y_2}\end{bmatrix}\right),\\ \mathrm{k}[\theta] &= \left(\begin{bmatrix}\cos\theta_1 & \sin\theta_1\\ -\sin\theta_1 & \cos\theta_1\end{bmatrix}, \begin{bmatrix}\cos\theta_2 & \sin\theta_2\\ -\sin\theta_2 & \cos\theta_2\end{bmatrix}\right),\end{aligned} \tag{3.2}$$

and put

$$N = \left\{\mathrm{n}[x] : x \in \mathbb{R}^2\right\}, A = \left\{\mathrm{a}[y] : y \in (0,\infty)^2\right\}, K = \left\{\mathrm{k}[\theta] : \theta \in (\mathbb{R}/\pi\mathbb{Z})^2\right\}. \tag{3.3}$$

Then we have the Iwasawa decomposition

$$G = NAK, \quad G \ni \mathrm{g} = \mathrm{n}[x]\mathrm{a}[y]\mathrm{k}[\theta]. \tag{3.4}$$

In the sequel we shall use this coordinate system on G without mention. If either $\mathrm{n}[x]$ or $\mathrm{a}[y]$ or both contain expressions involving elements of F, then they should be understood as results of the embedding. Haar measures on these groups are defined by

$$d\mathrm{n} = dx_1 dx_2,\ d\mathrm{a} = (y_1y_2)^{-1}dy_1dy_2, d\mathrm{k} = \pi^{-2}d\theta_1 d\theta_2, d\mathrm{g} = (y_1y_2)^{-1}d\mathrm{n}d\mathrm{a}d\mathrm{k}, \quad (3.5)$$

with Lebesgue measures dx_j, dy_j, $d\theta_j$. Elements of the Lie algebra $\mathfrak{g}$ of G are identified with corresponding right-differential operators on G. The algebra $\mathfrak{g}$ has the basis:

$$\mathbf{w}_j = \tfrac{1}{2}\partial_{\theta_j}, \quad \mathbf{e}_j^+ = e^{2i\theta_j}\left(iy_j\partial_{x_j} + y_j\partial_{y_j} - \tfrac{1}{2}i\partial_{\theta_j}\right), \quad \mathbf{e}_j^- = \overline{\mathbf{e}_j^+}, \quad j = 1,2. \quad (3.6)$$

We have the relations

$$[\mathbf{w}_j, \mathbf{e}_j^\pm] = \pm i\mathbf{e}_j^\pm, \quad [\mathbf{e}_j^+, \mathbf{e}_j^-] = -2i\mathbf{w}_j, \quad (3.7)$$

and also $[\mathbf{x}_1, \mathbf{x}_2] = 0$ for $\mathbf{x}_j \in \{\mathbf{w}_j, \mathbf{e}_j^+, \mathbf{e}_j^-\}$. These imply in particular that the center of the universal enveloping algebra of $\mathfrak{g}$ is the polynomial ring on two Casimir elements:

$$\Omega_j = -\mathbf{e}_j^+\mathbf{e}_j^- + \mathbf{w}_j^2 - i\mathbf{w}_j = -y_j^2(\partial_{x_j}^2 + \partial_{y_j}^2) + y_j\partial_{x_j}\partial_{\theta_j}. \quad (3.8)$$

Let f be a function on G that is *left* Γ-automorphic and of weight $2\mathrm{q} = 2(q_1, q_2)$, $q_j \in \mathbb{Z}$; that is, for any $\mathrm{g} \in G$,

$$f(\gamma\mathrm{g}) = f(\mathrm{g}), \quad \gamma \in \Gamma; \quad f(\mathrm{g}\mathrm{k}[\theta]) = e^{2i\mathrm{q}\theta}f(\mathrm{g}), \quad \mathrm{q}\theta = q_1\theta_1 + q_2\theta_2. \quad (3.9)$$

The latter is obviously equivalent to

$$\mathbf{w}_j f = iq_j f, \quad j = 1,2. \quad (3.10)$$

Then the first relation in (3.7) implies that

$$\begin{gathered} \mathbf{e}_j^\pm f \text{ are } \Gamma\text{-automorphic and of weight } 2(\mathrm{q} \pm 1_j), \\ \text{with } 1_1 = (1,0),\ 1_2 = (0,1) \end{gathered} \quad (3.11)$$

Such an f satisfies naturally $f(\mathrm{n}[n]\mathrm{g}) = f(\mathrm{g})$ for any $n \in \mathcal{O}$, and thus, under an appropriate smoothness condition, f should admit a Fourier expansion in terms of the additive characters

$$\psi_n(\mathrm{g}) = \exp\left(\frac{2\pi i}{\sqrt{\mathrm{D_F}}}(nx_1 - n'x_2)\right), \quad n \in \mathcal{O}. \quad (3.12)$$

Having said this, we introduce the Eisenstein series: It is defined, initially for $\mathrm{Re}\, s > \frac{1}{2}$, by

$$E_\mathrm{q}(\mathrm{g}; s, \nu) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \phi_\mathrm{q}(\gamma\mathrm{g}; (s+\nu\varpi i, s-\nu\varpi i)), \quad \mathrm{q} = (q_1, q_2) \in \mathbb{Z}^2,\ \nu \in \mathbb{Z}, \quad (3.13)$$

with $\varpi = \pi/(2\log\epsilon_0)$. Here

$$\phi_{\mathrm{q}}(\mathrm{g};\mathrm{s}) = y_1^{s_1+\frac{1}{2}} y_2^{s_2+\frac{1}{2}} e^{2iq\theta}, \quad \mathrm{s}=(s_1,s_2), \tag{3.14}$$

and $\Gamma_\infty = \Gamma\cap N\cdot\Gamma\cap A$ is the stabilizer in Γ of the point infinity. To describe the Fourier expansion of $E_{\mathrm{q}}(\mathrm{g};s,\nu)$, we need also to introduce the twists of ζ_{F} and σ_ξ with Grössencharakters:

$$\zeta_{\mathrm{F}}(s,\nu) = \sum_{\mathfrak{n}\neq 0} |n/n'|^{\nu\varpi i}(N\mathfrak{n})^{-s}, \quad \sigma_\xi(n,\nu) = \sum_{\mathfrak{d}|n} |d/d'|^{\nu\varpi i} N(\mathfrak{d})^{\xi}, \quad \nu\in\mathbb{Z}, \tag{3.15}$$

where $\mathfrak{n}=(n)$, $\mathfrak{d}=(d)$ with $n\succ 0$, $d\succ 0$ run over integral ideals. The L-function $\zeta_{\mathrm{F}}(s,\nu)$ continues to an entire function, provided $\nu\neq 0$.

Lemma 3.1. *As a function of s, $E_{\mathrm{q}}(\mathrm{g};s,\nu)$ continues meromorphically to $\mathbb{C}$, satisfying the functional equation*

$$E_{\mathrm{q}}(\mathrm{g};-s,-\nu) = \left(\frac{\pi}{\sqrt{\mathrm{D_F}}}\right)^{-4s} \times\frac{\Gamma(\frac{1}{2}+s+\nu\varpi i+|q_1|)\Gamma(\frac{1}{2}+s-\nu\varpi i+|q_2|)}{\Gamma(\frac{1}{2}-s-\nu\varpi i+|q_1|)\Gamma(\frac{1}{2}-s+\nu\varpi i+|q_2|)}\frac{\zeta_{\mathrm{F}}(1+2s,-2\nu)}{\zeta_{\mathrm{F}}(1-2s,2\nu)}E_{\mathrm{q}}(\mathrm{g};s,\nu). \tag{3.16}$$

When it is of finite value, $E_{\mathrm{q}}(\mathrm{g};s,\nu)$ is Γ-automorphic and of weight $2\mathrm{q}$. In the half plane $\mathrm{Re}\, s>0$, singularities occur only when $\nu=0$ and $\mathrm{q}=(0,0)$, and $E_{(0,0)}(\mathrm{g};s,0)$ has a simple pole at $s=\frac{1}{2}$ with the residue $(\pi^2\log\epsilon_0)/(\mathrm{D_F}\zeta_{\mathrm{F}}(2))$ as its sole singularity. Further,

$$\begin{aligned} e^{-2iq\theta}E_{\mathrm{q}}(\mathrm{g};s,\nu) &= (y_1y_2)^{\frac{1}{2}+s}(y_1/y_2)^{\nu\varpi i} \\ &+(-1)^{q_1+q_2}\frac{\pi}{\sqrt{\mathrm{D_F}}}(y_1y_2)^{\frac{1}{2}-s}(y_1/y_2)^{-\nu\varpi i}\frac{\Gamma(s+\nu\varpi i)\Gamma(s-\nu\varpi i)}{\Gamma(\frac{1}{2}+s+\nu\varpi i)\Gamma(\frac{1}{2}+s-\nu\varpi i)} \\ &\times\prod_{j_1=0}^{|q_1|-1}\left(\frac{s+\nu\varpi i-j_1-\frac{1}{2}}{\frac{1}{2}+s+\nu\varpi i+j_1}\right)\prod_{j_2=0}^{|q_2|-1}\left(\frac{s-\nu\varpi i-j_2-\frac{1}{2}}{\frac{1}{2}+s-\nu\varpi i+j_2}\right)\frac{\zeta_{\mathrm{F}}(2s,-2\nu)}{\zeta_{\mathrm{F}}(1+2s,-2\nu)} \\ &+\pi\frac{(-1)^{q_1+q_2}(\pi/\sqrt{\mathrm{D_F}})^{2s}}{\zeta_{\mathrm{F}}(1+2s,-2\nu)}\sum_{n\in\mathcal{O}_*}\frac{|n/n'|^{-\nu\varpi i}}{|N(n)|^{s+\frac{1}{2}}}\sigma_{2s}(n,2\nu)\psi_n(\mathrm{n}[x]) \\ &\times\frac{\mathrm{W}_{q_1\mathrm{sgn}(n),s+\nu\varpi i}(4\pi|n|y_1/\sqrt{\mathrm{D_F}})}{\Gamma(\frac{1}{2}+s+\nu\varpi i+q_1\mathrm{sgn}(n))}\frac{\mathrm{W}_{-q_2\mathrm{sgn}(n'),s-\nu\varpi i}(4\pi|n'|y_2/\sqrt{\mathrm{D_F}})}{\Gamma(\frac{1}{2}+s-\nu\varpi i-q_2\mathrm{sgn}(n'))}, \end{aligned} \tag{3.17}$$

where $\mathrm{W}_{a,b}$ is the Whittaker function. The sum over $n\in\mathcal{O}_*$ converges absolutely and uniformly for all parameters involved, and moreover it is of exponential decay as y_1y_2 tends to infinity.

Proof. Obviously it is enough to prove the expansion (3.17). By the Bruhat decomposition, we have

$$E_{\mathrm{q}}(\mathrm{g};s,\nu) = \phi_{\mathrm{q}}(\mathrm{g};\mathrm{s}) + \sum_{\substack{\mathfrak{c}=(c)\\ c\succ 0}}\sum_{\substack{a \bmod c\\ (a,c)=1}}\sum_{n\in\mathcal{O}}\phi_{\mathrm{q}}(\mathrm{a}[1/c^2]\mathrm{w}\mathrm{n}[a/c+n]\mathrm{g};\mathrm{s}), \tag{3.18}$$

where $\mathrm{s} = (s + \nu\varpi, s - \nu\varpi)$, $\mathrm{w} = \mathrm{k}[\frac{1}{2}\pi, \frac{1}{2}\pi]$. The sum over n is, by the Poisson sum formula, equal to

$$\begin{aligned} &\frac{1}{\sqrt{\mathrm{D_F}}} \sum_{n\in\mathcal{O}} e[an/c] \int_N \psi_n^{-1}(\mathrm{n})\phi_{\mathrm{q}}(\mathrm{a}[1/c^2]\mathrm{wng};\mathrm{s})dn \\ &= \frac{e^{2iq\theta}}{\sqrt{\mathrm{D_F}}N(\mathfrak{c})^2} \sum_{n\in\mathcal{O}} e[an/c]\psi_n(\mathrm{n}[x])\mathcal{A}_{n/c^2}\phi_{\mathrm{q}}(\mathrm{a}[c^2y];\mathrm{s}), \end{aligned} \tag{3.19}$$

where $\mathcal{A}_n$ is the Jacquet operator:

$$\mathcal{A}_n f(\mathrm{g}) = \int_N \psi_n^{-1}(\mathrm{n}) f(\mathrm{wng})dn. \tag{3.20}$$

Computing the coordinates of $\mathrm{wna}[y]$, we have

$$\begin{aligned} \mathcal{A}_n\phi_{\mathrm{q}}(\mathrm{a}[y];\mathrm{s}) = (y_1y_2)^{\frac{1}{2}-s}(y_1/y_2)^{-\nu\varpi i} &\int_{-\infty}^{\infty} \frac{\exp(2\pi i n y_1\xi_1/\sqrt{\mathrm{D_F}})}{(1+\xi_1^2)^{\frac{1}{2}+s+\nu\varpi i}} \left(\frac{i+\xi_1}{|i+\xi_1|}\right)^{2q_1} d\xi_1 \\ \times &\int_{-\infty}^{\infty} \frac{\exp(2\pi i n' y_2\xi_2/\sqrt{\mathrm{D_F}})}{(1+\xi_2^2)^{\frac{1}{2}+s-\nu\varpi i}} \left(\frac{i+\xi_2}{|i+\xi_2|}\right)^{-2q_2} d\xi_2. \end{aligned} \tag{3.21}$$

These integrals are tabulated: For $\mathrm{Re}\, s > 0$, $\mathbb{R} \ni u$,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{e^{\frac{1}{2}iu\xi}}{(\xi^2+1)^{s+\frac{1}{2}}} \left(\frac{\xi+i}{|\xi+i|}\right)^{2q} d\xi \\ &= \begin{cases} (-1)^q\pi\left(\dfrac{|u|}{4}\right)^{s-\frac{1}{2}} \dfrac{W_{q\,\mathrm{sgn}(u),s}(|u|)}{\Gamma(s+\frac{1}{2}+q\,\mathrm{sgn}(u))} & \text{if } u \neq 0, \\ (-1)^q\pi 2^{1-2s} \dfrac{\Gamma(2s)}{\Gamma(s+\frac{1}{2}+q)\Gamma(s+\frac{1}{2}-q)} & \text{if } u = 0. \end{cases} \end{aligned} \tag{3.22}$$

Ignoring the convergence issue temporarily, we insert (3.21)–(3.22) into (3.18) via (3.19), rearrange the summation, and use the twist of (2.19) with Grössencharakters:

$$\begin{aligned} &\sum_{\substack{\mathfrak{c} \\ \mathfrak{c}=(c)}} |c/c'|^{2\nu\varpi i}(N(\mathfrak{n}))^{-s} \sum_{\substack{a \bmod c \\ (a,c)=1}} e[an/c] \\ &= \frac{1}{\zeta_{\mathrm{F}}(s,2\nu)} \begin{cases} \zeta_{\mathrm{F}}(s-1,2\nu) & \text{if } n=0, \\ \sigma_{1-s}(n,2\nu) & \text{if } n\neq 0. \end{cases} \end{aligned} \tag{3.23}$$

This leads us to the expansion (3.17). As to the convergence, we shift vertically the two contours in (3.21) appropriately, and see that uniformly for $\mathrm{Re}\, s > 0$

$$\mathcal{A}_{n/c^2}\phi_{\mathrm{q}}(\mathrm{a}[c^2y];\mathrm{s}) \ll (N(\mathfrak{c})\sqrt{y_1y_2})^{1-2\mathrm{Re}\, s} \exp(-a(|ny_1| + |n'y_2|)) \tag{3.24}$$

with $a > 0$ and the implicit constant depending only on F. Then, provided $\mathrm{Re}\, s > 0$, the sum over $n \in \mathcal{O}_*$ in (3.19) is

$$\ll (N(\mathfrak{c})\sqrt{y_1y_2})^{1-2\mathrm{Re}\, s} \sum_{\mathfrak{n}\neq(0)} K_0\left(a\sqrt{N(\mathfrak{n})y_1y_2}\right), \tag{3.25}$$

where $\mathfrak{n}$ runs over integral ideals of F, and K_ν is the K-Bessel function of order ν. This shows that (3.18) converges absolutely for $\mathrm{Re}\, s > \frac{1}{2}$, and moreover yields the last assertion in the lemma.

Lemma 3.2. *The function D_ℓ of s continues meromorphically to $\mathbb{C}$. If $\alpha \neq 0$, then it has simple poles at $s = 1,\ 1+\alpha$ with residues*

$$4\delta_{\nu,0}\delta_{\ell,0}|N(c)|^{\alpha-1}\frac{\log \epsilon_c}{\sqrt{\mathrm{D_F}}}\zeta_{\mathrm{F}}(1-\alpha), \quad 4\delta_{\nu,0}\delta_{\ell,0}|N(c)|^{-\alpha-1}\frac{\log \epsilon_c}{\sqrt{\mathrm{D_F}}}\zeta_{\mathrm{F}}(1+\alpha), \tag{3.26}$$

respectively, and elsewhere it is regular; i.e., it is entire unless $\nu = 0$ and $\ell = (0,0)$ simultaneously. Further, it holds, for all s, that

$$D_\ell(s,\alpha;\nu;a/c) = \frac{1}{\pi^2}\left(2\pi/\sqrt{\mathrm{D_F}|N(c)|}\right)^{2(2s-\alpha-1)}|c/c'|^{2\nu\varpi_c i} \\ \times \Gamma_\ell\left(s-\nu\varpi_c i, s+\nu\varpi_c i;\alpha\right) D_\ell(1-s,-\alpha;-\nu;a^*/c), \tag{3.27}$$

where $a^*a \equiv 1 \bmod c$, and

$$\Gamma_\ell(s_1,s_2;\alpha) = \prod_{j=1}^{2}\left(\cos\tfrac{1}{2}\pi\alpha - (-1)^{l_j}\cos\pi(s_j - \tfrac{1}{2}\alpha)\right)\Gamma(1-s_j)\Gamma(1+\alpha-s_j). \tag{3.28}$$

Proof. Put

$$f_\alpha(\mathrm{g};a/c) = (\pi/\sqrt{\mathrm{D_F}})^{-\alpha}\Gamma^2(\tfrac{1}{2}(1+\alpha))\zeta_{\mathrm{F}}(1+\alpha) \\ \times (\mathrm{e}_1^+ - \mathrm{e}_1^-)^{l_1}(\mathrm{e}_2^+ - \mathrm{e}_2^-)^{l_2}E_{(0,0)}(\mathrm{n}[a/c]\mathrm{g};\tfrac{1}{2}\alpha,0). \tag{3.29}$$

Note that

$$\mathrm{n}[a/c]\mathrm{a}[y]^{-1} \in \Gamma\mathrm{n}[-a^*/c]\mathrm{a}[y/c^2]\mathrm{k}[\tfrac{1}{2}\pi,\tfrac{1}{2}\pi]. \tag{3.30}$$

Thus, by (3.11), we have

$$f_\alpha(\mathrm{a}[y]^{-1};a/c) = (-1)^{l_1+l_2}f_\alpha(\mathrm{a}[y/c^2];-a^*/c). \tag{3.31}$$

On the other hand, (3.17) gives the expansion

$$f_\alpha(\mathrm{a}[y];a/c) = \left\{f_\alpha^{(0)} + f_\alpha^{(1)}\right\}(\mathrm{a}[y];a/c), \tag{3.32}$$

where

$$f_\alpha^{(0)}(\mathrm{a}[y];a/c) = \delta_{\ell,0}(\pi/\sqrt{\mathrm{D_F}})^{-\alpha}\Gamma^2(\tfrac{1}{2}(1+\alpha))\zeta_{\mathrm{F}}(1+\alpha)(y_1y_2)^{\frac{1}{2}(1+\alpha)} \\ +\delta_{\ell,0}(\pi/\sqrt{\mathrm{D_F}})^{\alpha}\Gamma^2(\tfrac{1}{2}(1-\alpha))\zeta_{\mathrm{F}}(1-\alpha)(y_1y_2)^{\frac{1}{2}(1-\alpha)} \tag{3.33}$$

and

$$f_\alpha^{(1)}(\mathrm{a}[y];a/c) = (-1)^{l_1}(4\pi/\sqrt{\mathrm{D_F}})^{l_1+l_2+1}{y_1}^{l_1+\frac{1}{2}}{y_2}^{l_2+\frac{1}{2}} \\ \times \sum_{n\in\mathcal{O}_*}\frac{\sigma_\alpha(n)}{|N(n)|^{\frac{1}{2}\alpha}}n^{l_1}n'^{l_2}\mathrm{e}[an/c]K_{\frac{1}{2}\alpha}(2\pi|n|y_1/\sqrt{\mathrm{D_F}}) \\ \times K_{\frac{1}{2}\alpha}(2\pi|n'|y_2/\sqrt{\mathrm{D_F}}). \tag{3.34}$$

We then put

$$I(s,\alpha;\nu;a/c) = \iint_{1\le y_1/y_2\le \epsilon_c^2} f_\alpha^{(1)}(\mathrm{a}[y/|\mathrm{c}|];a/c)(y_1y_2)^{s-\frac12(\alpha+3)}(y_1/y_2)^{-\nu\varpi_c i}dy_1dy_2. \quad (3.35)$$

The expansion (3.34) gives

$$I(s,\alpha;\nu;a/c) = \frac{(-1)^{l_1}}{|c|^{l_1+\frac12}|c'|^{l_2+\frac12}}\left(\frac{4\pi}{\sqrt{D_F}}\right)^{l_1+l_2+1}\sum_{n\in\mathcal{O}_*/U_c}\frac{\sigma_\alpha(n)}{|N(n)|^{\frac12\alpha}}n^{l_1}n'^{l_2}e[an/c]$$
$$\times\iint_{1\le y_1/y_2\le\epsilon_c^2}\sum_{p\in\mathbb{Z}}K_{\frac12\alpha}\left(\frac{2\pi\epsilon_c^p y_1}{\sqrt{D_F}}\left|\frac{n}{c}\right|\right)K_{\frac12\alpha}\left(\frac{2\pi\epsilon_c'^p y_2}{\sqrt{D_F}}\left|\frac{n'}{c'}\right|\right)$$
$$\times(\epsilon_c^p y_1)^{s+l_1-1-\frac12\alpha-\nu\varpi_c i}(\epsilon_c'^p y_2)^{s+l_2-1-\frac12\alpha+\nu\varpi_c i}dy_1dy_2, \quad (3.36)$$

where the convergence is absolute throughout. Hence, unfolding the integral, we get

$$I(s,\alpha;\nu;a/c) = (-1)^{l_1}2^{2(l_1+l_2-1)}\left(\frac{\pi}{\sqrt{D_F|N(c)|}}\right)^{-2s+\alpha+1}\left|\frac{c}{c'}\right|^{-\nu\varpi_c i}$$
$$\times\Gamma\left(\tfrac12(s+l_1-\nu\varpi_c i)\right)\Gamma\left(\tfrac12(s+l_1-\alpha-\nu\varpi_c i)\right)\Gamma\left(\tfrac12(s+l_2+\nu\varpi_c i)\right)$$
$$\times\Gamma\left(\tfrac12(s+l_2-\alpha+\nu\varpi_c i)\right)D_\ell(s,\alpha;\nu;a/c). \quad (3.37)$$

We then divide I into two parts:

$$I(s,\alpha;\nu;a/c) = \iint_{\substack{1\le y_1/y_2\le\epsilon_c^2\\ y_1y_2\ge1}} + \iint_{\substack{1\le y_1/y_2\le\epsilon_c^2\\ y_1y_2\le1}}\cdots = \{I^+ + I^-\}(s,\alpha;\nu;a/c). \quad (3.38)$$

Obviously I^+ is entire in s. By (3.31)–(3.33), we have

$$I^-(s,\alpha;\nu;a/c)$$
$$= (-1)^{l_1+l_2}I^+(1-s,-\alpha;-\nu;-a^*/c)$$
$$+\delta_{\ell,0}\delta_{\nu,0}\frac{\log\epsilon_c}{|N(c)|^{\frac12(1+\alpha)}}\left(\frac{\pi}{\sqrt{D_F}}\right)^{-\alpha}\Gamma^2(\tfrac12(1+\alpha))\zeta_F(1+\alpha)\left(\frac{1}{s-\alpha-1}-\frac1s\right)$$
$$+\delta_{\ell,0}\delta_{\nu,0}\frac{\log\epsilon_c}{|N(c)|^{\frac12(1-\alpha)}}\left(\frac{\pi}{\sqrt{D_F}}\right)^{\alpha}\Gamma^2(\tfrac12(1-\alpha))$$
$$\times\zeta_F(1-\alpha)\left(\frac{1}{s-1}-\frac{1}{s-\alpha}\right), \quad (3.39)$$

provided $\mathrm{Re}\, s$ is sufficiently large. From (3.37)–(3.39), D_ℓ is meromorphic over $\mathbb{C}$. The assertion (3.26) is now immediate. Also (3.39) implies that

$$I(s,\alpha;\nu;a/c) = (-1)^{l_1+l_2} I(1-s,-\alpha;-\nu;-a^*/c), \tag{3.40}$$

which is equivalent to (3.27)–(3.28). This ends the proof.

Now, we return to (2.32). We assume that (2.35) holds. We have

$$\mathcal{D}_m(\alpha, a/c; h) = \sum_\ell Y_\ell(m;\alpha,a/c;h), \tag{3.41}$$

where

$$Y_\ell(m;\alpha,a/c;h) = \frac{1}{2\pi i \log \epsilon_c} \sum_{\nu\in\mathbb{Z}} \left|\frac{m}{m'}\right|^{-\nu\varpi_c i} \times \int_{(a)} |N(m)|^s \widetilde{h}_\ell\,(s-\nu\varpi_c i, s+\nu\varpi_c i)\, D_\ell(s,\alpha;\nu;a/c)ds, \tag{3.42}$$

with $0 < a - 1 - \max(0, \mathrm{Re}\,\alpha) < C_0$. Inserting this into (2.21) we have also

$$\mathcal{B}_m(\alpha,\beta;h) = \sum_\ell Z_\ell(m;\alpha,\beta;h), \tag{3.43}$$

where

$$Z_\ell(m;\alpha,\beta;h) = \zeta_{\mathrm{F}}(1-\beta) \sum_{c\in\mathcal{O}_*} \frac{\chi(c)}{|N(c)|^{1-\beta}} \sum_{\substack{a \bmod c\\ (a,c)=1}} e[am/c] Y_\ell(m;\alpha,a/c;h). \tag{3.44}$$

Shifting the contour in (3.42) to the left, we have, by Lemma 3.2

$$\begin{aligned} Y_\ell(m;\alpha,a/c;h) &= 4\frac{\delta_{0,\ell}}{\sqrt{\mathrm{D_F}}}|N(c)|^{\alpha-1}|N(m)|\widetilde{h}_\ell(1,1)\zeta_{\mathrm{F}}(1-\alpha) \\ &+ 4\frac{\delta_{0,\ell}}{\sqrt{\mathrm{D_F}}}|N(c)|^{-\alpha-1}|N(m)|^{1+\alpha}\widetilde{h}_\ell(1+\alpha,1+\alpha)\zeta_{\mathrm{F}}(1+\alpha) \\ &+ \frac{|N(c)|^{\alpha+1}}{2\pi^3 i\log\epsilon_c}\left(\frac{2\pi}{\sqrt{\mathrm{D_F}}}\right)^{-2\alpha-2} \sum_{n\in\mathcal{O}_*/\mathrm{U}_c} \frac{\sigma_{-\alpha}(n)}{|N(n)|}\mathrm{sgn}^\ell[n]e[a^*n/c]\sum_{\nu\in\mathbb{Z}}\left|\frac{mnc'^2}{m'n'c^2}\right|^{-\nu\varpi_c i} \\ &\times \int_{(b)} \Gamma_\ell(s-\nu\varpi_c i, s+\nu\varpi_c i;\alpha)\widetilde{h}_\ell\,(s-\nu\varpi_c i, s+\nu\varpi_c i) \\ &\times \left(\frac{4\pi^2\sqrt{|N(mn)|}}{\mathrm{D_F}|N(c)|}\right)^{2s} ds, \end{aligned} \tag{3.45}$$

with

$$b < \min(0, \mathrm{Re}\,\alpha). \tag{3.46}$$

Here we have used a bound for $D_\ell(s,\alpha;\nu:a/c)$ which follows from (3.27); thus, it is implicitly assumed that $|\mathrm{Re}\,\alpha|$ and $|b|$ are sufficiently smaller than the C_0 in (2.35). Denoting the last integral as $L(\nu\varpi_c)$, we have, by the Poisson sum formula,

$$\sum_\nu = \frac{1}{\varpi_c}\sum_{\epsilon\in\mathrm{U}_c}\int_{-\infty}^{\infty} L(\tau)\left|\frac{\epsilon mnc'^2}{\epsilon' m'n'c^2}\right|^{-i\tau} d\tau. \tag{3.47}$$

That is, in (3.45)

$$\sum_{n\in\mathcal{O}_*/\mathrm{U}_c} = \frac{1}{\varpi_c}\sum_{n\in\mathcal{O}_*}\frac{\sigma_{-\alpha}(n)}{|N(n)|}\mathrm{sgn}^\ell[n]e[a^*n/c]\int_{-\infty}^{\infty} L(\tau)\left|\frac{mnc'^2}{m'n'c^2}\right|^{-i\tau} d\tau. \tag{3.48}$$

This integral can be put as

$$[h]_\ell(x(m,n;c);\alpha),\quad x(m,n;c) = \left(\frac{4\pi\sqrt{|mn|}}{|c|\sqrt{\mathrm{D_F}}},\frac{4\pi\sqrt{|m'n'|}}{|c'|\sqrt{\mathrm{D_F}}}\right) \tag{3.49}$$

with

$$[h]_\ell(x;\alpha) = \frac{1}{2i}\int_{(b)}\int_{(b)}\Gamma_\ell\left(s_1,s_2;\alpha\right)\tilde{h}_\ell(s_1,s_2)\left|\frac{x}{2}\right|^{2s_1}\left|\frac{x'}{2}\right|^{2s_2} ds_1 ds_2. \tag{3.50}$$

Then

$$\begin{aligned}Y_\ell(m;\alpha,a/c;h) &= 4\frac{\delta_{0,\ell}}{\sqrt{\mathrm{D_F}}}|N(c)|^{\alpha-1}|N(m)|\tilde{h}_\ell(1,1)\zeta_{\mathrm F}(1-\alpha)\\ &+ 4\frac{\delta_{0,\ell}}{\sqrt{\mathrm{D_F}}}|N(c)|^{-\alpha-1}|N(m)|^{1+\alpha}\tilde{h}_\ell(1+\alpha,1+\alpha)\zeta_{\mathrm F}(1+\alpha)\\ &+ \frac{1}{\pi^4 i}|N(c)|^{\alpha+1}\left(\frac{2\pi}{\sqrt{\mathrm{D_F}}}\right)^{-2\alpha-2}\sum_{n\in\mathcal{O}_*}\frac{\sigma_{-\alpha}(n)}{|N(n)|}\mathrm{sgn}^\ell[n]e[a^*n/c][h]_\ell(x(m,n;c);\alpha).\end{aligned} \tag{3.51}$$

This, together with (2.20) and (3.44), gives

$$\begin{aligned}Z_\ell(m;\alpha,\beta;h) &= 4\frac{\delta_{0,\ell}}{\sqrt{\mathrm{D_F}}}\frac{\zeta_{\mathrm F}(1-\alpha)\zeta_{\mathrm F}(1-\beta)}{\zeta_{\mathrm F}(2-\alpha-\beta)}\tilde{h}_\ell(1,1)|N(m)|\sigma_{\alpha+\beta-1}(m)\\ &+ 4\frac{\delta_{0,\ell}}{\sqrt{\mathrm{D_F}}}\frac{\zeta_{\mathrm F}(1+\alpha)\zeta_{\mathrm F}(1-\beta)}{\zeta_{\mathrm F}(2+\alpha-\beta)}\tilde{h}_\ell(1+\alpha,1+\alpha)|N(m)|^{1+\alpha}\sigma_{\beta-\alpha-1}(m)\\ &+ \frac{1}{\pi^4 i}\zeta_{\mathrm F}(1-\beta)\left(\frac{2\pi}{\sqrt{\mathrm{D_F}}}\right)^{-2\alpha-2}\sum_{n\in\mathcal{O}_*}\frac{\sigma_{-\alpha}(n)}{|N(n)|}\mathrm{sgn}^\ell[n]\\ &\times\sum_{c\in\mathcal{O}_*}\chi(c)|N(c)|^{\alpha+\beta}S_{\mathrm F}(m,n;c)[h]_\ell(x(m,n;c);\alpha),\end{aligned} \tag{3.52}$$

where

$$S_{\mathrm{F}}(m,n;c) = \sum_{\substack{a \bmod c \\ (a,c)=1}} e[(ma+na^*)/c] \tag{3.53}$$

is a Kloosterman sum over F.

In the sequel we shall always assume, for the sake of simplicity, that

$$|\alpha|,\ |\beta| < C_1, \tag{3.54}$$

where $C_1 > 0$ is sufficiently large but C_1/C_0 with C_0 as in (2.35) is sufficiently small; note the remark made after (3.46). On this and (2.35), the double sum on the right side of (3.52) converges absolutely for

$$|\mathrm{Re}\,\alpha| + \mathrm{Re}\,\beta < -2. \tag{3.55}$$

In fact, an appropriate shift of contours in (3.50) gives

$$[h]_\ell(x;\alpha) \ll |xx'|^{2b}(|x|+|x'|)^{-\mu} \tag{3.56}$$

with a $\mu > 0$ and b as in (3.46), from which the assertion follows immediately.

In Section 5, we shall decompose spectrally the interior sum over c in (3.52). For that purpose we make here a little rearrangement of the sum. By the definition (2.3) we have

$$\begin{aligned}&\sum_{c\in\mathcal{O}_*} \chi(c)|N(c)|^{\alpha+\beta} S_{\mathrm{F}}(m,n;c)[h]_\ell(x(m,n;c);\alpha)\\ &= \frac{1}{4\pi}\left(\frac{4\pi^2}{\mathrm{D_F}}\sqrt{|N(mn)|}\right)^{\alpha+\beta+1}\\ &\times \int_{-\infty}^{\infty} \hat{\mathrm{p}}(\xi)\left|\frac{mn}{m'n'}\right|^{i\xi/(4\log\epsilon_0)} \mathrm{S}_{m,n}(\alpha,\beta,\xi;[h]_\ell)d\xi,\end{aligned} \tag{3.57}$$

where

$$\mathrm{S}_{m,n}(\alpha,\beta,\xi;[h]_\ell) = \sum_{c\in\mathcal{O}_*}\frac{1}{|N(c)|}S_{\mathrm{F}}(m,n;c)[[h]]_\ell(x(m,n;c);\alpha,\beta,\xi), \tag{3.58}$$

with

$$[[h]]_\ell(x;\alpha,\beta,\xi) = \left|\tfrac{1}{4}xx'\right|^{-\alpha-\beta-1}|x/x'|^{-i\xi/(2\log\epsilon_0)}[h]_\ell(x;\alpha). \tag{3.59}$$

Note that under (2.35) and (3.54) we have the bounds

$$[[h]]_\ell(x;\alpha,\beta,\xi) \ll (|x|+|x'|)^{-\mu}\cdot\begin{cases}|xx'|^{-C_0/2} & \text{if } |xx'|\ge 1,\\ |xx'|^{-|\mathrm{Re}\,\alpha|-\mathrm{Re}\,\beta+1} & \text{if } |xx'|<1,\end{cases} \tag{3.60}$$

with $\mu > 0$ as in (3.56). The extra integration in (3.57) will eventually be eliminated (see (5.4) below).

Remark. One may consider, more generally than $\mathrm{Z}_2(g,\mathrm{F})$, the mean value

$$\int_{-\infty}^{\infty} \left|\zeta_{\mathrm{F}}(\tfrac{1}{2}+it,\nu_1)\zeta_{\mathrm{F}}(\tfrac{1}{2}+it,\nu_2)\right|^2 g(t)dt. \tag{3.61}$$

with arbitrary ν_1, $\nu_2 \in \mathbb{Z}$. The relevant reduction to a sum of Kloosterman sums can also be performed with Lemma 3.1.

4. Geometric sum formula

The next step is to decompose spectrally the sum $\mathrm{S}_{m,n}(\alpha,\beta,\xi;[h]_\ell)$. This will be accomplished with the *geometric* sum formula for the Hilbert modular group Γ. In the present section we shall describe this principal tool for our purpose.

Let $L^2(\Gamma\backslash G)$ be the Hilbert space composed of all left Γ-automorphic functions on G which are square integrable against the measure dg. Let ${}^0L^2(\Gamma\backslash G)$ be its cuspidal subspace, i.e., the one spanned by those elements with zero constant terms in their Fourier expansions (see (4.17)). One can show the decompositions

$$L^2(\Gamma\backslash G) = \mathbb{C} \oplus {}^0L^2 \oplus \mathcal{E}, \tag{4.1}$$

$${}^0L^2(\Gamma\backslash G) = \overline{\bigoplus \mathrm{V}}, \tag{4.2}$$

$$\mathcal{E} = \overline{\bigoplus_{\nu\in\mathbb{Z}} \mathcal{E}_\nu}, \tag{4.3}$$

where V runs over an orthogonal system of right irreducible subspaces, and $\mathcal{E}_\nu$ is generated by the values of the Eisenstein series $E_{\mathrm{q}}(\mathrm{g};s,\nu)$ as in (4.6) below. The action of the subgroup K leads to a further decomposition

$$\mathrm{V} = \overline{\bigoplus_{\mathrm{q}\in\mathbb{Z}^2} \mathrm{V}_{\mathrm{q}}}, \tag{4.4}$$

where $\mathrm{k}[\theta]$ acts in V_{q} as the multiplication by $e^{2i\mathrm{q}\theta}$, and $\dim \mathrm{V}_{\mathrm{q}} \le 1$. Analogously we have

$$\mathcal{E}_\nu = \overline{\bigoplus_{\mathrm{q}\in\mathbb{Z}^2} \mathcal{E}_{\nu,\mathrm{q}}}, \tag{4.5}$$

where

$$\mathcal{E}_{\nu,\mathrm{q}} = \left\{ \int_{-\infty}^{\infty} u(t) E_{\mathrm{q}}(\mathrm{g};\tfrac{1}{2}+it,\nu)dt \;:\; \int_{-\infty}^{\infty} |u(t)|^2 dt < \infty \right\}. \tag{4.6}$$

We note that the operator Ω_j acts as a multiplication by a constant in each V, i.e.,

$$\Omega_j|\mathrm{V} = (\tfrac{1}{4}+\kappa_j^2)\cdot 1, \quad \kappa_{\mathrm{V}} = (\kappa_1,\kappa_2), \tag{4.7}$$

with certain $\kappa_j \in \mathbb{C}$, and that

$$\mathrm{w}_j|\mathrm{V}_{\mathrm{q}} = iq_j \cdot 1, \quad \mathrm{w}_j|\mathcal{E}_{\nu,\mathrm{q}} = iq_j \cdot 1, \tag{4.8}$$

which is the same as (3.10).

The fundamental assertions (4.1)–(4.6) follows from the general theory of automorphic forms on semisimple Lie groups, see, e.g., [4] or [6]. However, in the present situation, it is of no difficulties to derive them in the same way as for the full modular group acting on the upper half plane, for instance as in Chapter One of [10].

In the geometric sum formula the spaces V of (4.2) occur, along with a certain classification among them. Because of this, we need to show a structure in the decomposition (4.4). Thus, observe that the basis elements $\mathbf{e}_j^{\pm}$ of $\mathfrak{g}$ act in V, and the assertion (3.11) implies the diagram within V:

$$\begin{array}{ccccc} & & (\mathbf{e}_2^+) & & \\ & & \mathrm{V}_{\mathrm{q}+\mathrm{1}_2} & & \\ & & \uparrow & & \\ (\mathbf{e}_1^-)\ \mathrm{V}_{\mathrm{q}-\mathrm{1}_1} & \leftarrow & \mathrm{V}_{\mathrm{q}} & \rightarrow & \mathrm{V}_{\mathrm{q}+\mathrm{1}_1}\ (\mathbf{e}_1^+)\,. \\ & & \downarrow & & \\ & & \mathrm{V}_{\mathrm{q}-\mathrm{1}_2} & & \\ & & (\mathbf{e}_2^-) & & \end{array} \tag{4.9}$$

The classification of V concerns how the diagram extends to a rectangular grid in which the V_{q} are placed. To this end, we could rely on the fact that $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, but we shall use a direct argument:

We pick up a generator φ of a non-trivial V_{q}. The function φ is a cusp-form over $\Gamma\backslash G$ in the sense that it is an element of ${}^0L^2(\Gamma\backslash G)$, and a simultaneous eigenfunction of Ω_j and $\mathbf{w}_j$ as indicated in (4.7) and (4.8). Then, observe that $\mathbf{e}_j^{\mp}\mathbf{e}_j^{\pm}\varphi = -((\pm q_j + \frac12)^2 + \kappa_j^2)\varphi$ via (3.7) and (3.8), and that integration by parts gives $\|\mathbf{e}_j^{\pm}\varphi\|^2 = -\langle \mathbf{e}_j^{\mp}\mathbf{e}_j^{\pm}\varphi, \varphi\rangle$, with an obvious usage of notation. Hence we have

$$\|\mathbf{e}_j^{\pm}\varphi\|^2 = ((\pm q_j + \tfrac12)^2 + \kappa_j^2)\|\varphi\|^2. \tag{4.10}$$

In particular $(\pm q_j + \frac12)^2 + \kappa_j^2$ is non-negative, which allows us to choose κ_j so that

$$\text{either}\ \ \kappa_j \ge 0 \ \ \text{or}\ \ 0 \le i\kappa_j \le \left||q_j| - \tfrac12\right|. \tag{4.11}$$

The relation (4.10) shows that the mappings in (4.9) are bijective in general. Exceptions can occur only if $i\kappa_j = l_j - \frac12$, with an integer $l_j \ge 1$. More precisely,

$$\mathrm{V}_{\mathrm{q}} \ne \{0\} \ \text{ and } \ \mathbf{e}_j^{\pm}\mathrm{V}_{\mathrm{q}} = \{0\} \quad \Longleftrightarrow \quad q_j = \mp l_j \ \text{ or } \ l_j = 1,\, q_j = 0\,. \tag{4.12}$$

A combination of (4.11)–(4.12) shows that the $l_j - \frac12$ are the only values that $i\kappa_j$ can take if $i\kappa_j \ge \frac12$. Then, the irreducibility of V implies that the set $\{\mathrm{q} :\ \mathrm{V}_{\mathrm{q}} \ne \{0\}\}$ is the direct product of two intervals in $\mathbb{Z}$. The possibilities are as follows, with corresponding technical terms: (I) $q_j \in \mathbb{Z}$ if $\kappa_j \ge 0$ (unitary principal series), (II) $q_j \in \mathbb{Z}$ if $0 \le i\kappa_j < \frac12$ (complementary series), (III) $q_j \ge l_j$ with $i\kappa_j = l_j - \frac12$ (holomorphic discrete series), (IV) $q_j \le -l_j$ with $i\kappa_j = l_j - \frac12$ (anti-holomorphic discrete series), and (V) $q_j = 0$ with $i\kappa_j = \frac12$ (trivial representation). But the last case cannot occur, because we are dealing with spaces of cusp forms. In this way we are led to a classification of the spaces V:

Lemma 4.1. *Let* V *be an irreducible subspace of* ${}^0L^2(\Gamma\backslash G)$, *for which* (4.4) *and* (4.7) *hold. We have the following possibilities:*

1. *either* $\kappa_j \ge 0$ *or* $0 \le i\kappa_j < \frac{1}{2}$, *for both* $j = 1, 2$,
2. *either* $\kappa_2 \ge 0$ *or* $0 \le i\kappa_2 < \frac{1}{2}$, *and* $i\kappa_1 = l - \frac{1}{2}$, $\eta_1 q_1 \ge l$ *with an integer* $l \ge 1$,
3. *either* $\kappa_1 \ge 0$ *or* $0 \le i\kappa_1 < \frac{1}{2}$, *and* $i\kappa_2 = l - \frac{1}{2}$, $\eta_2 q_2 \ge l$ *with an integer* $l \ge 1$,
4. $i\kappa_1 = l_1 - \frac{1}{2}$, $i\kappa_2 = l_2 - \frac{1}{2}$, $\eta_3 q_1 \ge l_1$, $\eta_4 q_2 \ge l_2$ *with integers* $l_1, l_2 \ge 1$,

$$(4.13)$$

where $\eta_\nu = \pm 1$, *and* q_j *without constraint runs over all integers. We may choose a cusp form* φ_{V} *in* V *of weight* $2\mathrm{q}_{\mathrm{V}}$ *with*

$$\mathrm{q}_{\mathrm{V}} = (0,0),\ (\eta_1 l, 0),\ (0, \eta_2 l),\ (\eta_3 l_1, \eta_4 l_2), \tag{4.14}$$

respectively, for which

$$\mathrm{V} = \overline{\mathfrak{U} \cdot \varphi_{\mathrm{V}}}, \tag{4.15}$$

with the universal enveloping algebra $\mathfrak{U}$ *of* $\mathfrak{g}$.

Thus, starting from $\mathrm{V}_{\mathrm{q_V}}$, a multiple application of $\mathbf{e}_j^{\pm}$ fills the grid. In case *1*, all V_{q} are non-trivial. In case *4*, a sole quadrant of the grid is filled with V_{q} all non-trivial; and other three quadrants contain only trivial spaces. In the remaining cases *2* and *3*, we have a mixed situation. Non-trivial V_{q} are, respectively, in a vertical and a horizontal halves of the grid that are fixed by V.

Remark. The $\frac{1}{4} + \kappa_j^2$ with $0 < i\kappa_j < \frac{1}{2}$ are called exceptional eigenvalues. It is known that they satisfy non-trivial lower bounds. The best published result

$$i\kappa_j \le \frac{1}{5} \tag{4.16}$$

is due to Rudnick, Luo and Sarnak [7]; in the preprint [5], Kim and Shahidi give $i\kappa_j \le \frac{5}{34}$.

As to the Fourier expansion of φ_{V}, we have

$$\begin{aligned}\varphi_{\mathrm{V}}(\mathrm{g}) &= (-1)^{q_1+q_2} e^{2iq\theta} \sum_{n \in \mathcal{O}} \frac{\varrho_{\mathrm{V}}(n)}{\sqrt{|N(n)|}} \psi_n(\mathrm{g}) \\ &\times \frac{\mathrm{W}_{q_1 \mathrm{sgn}(n), i\kappa_1}\left(4\pi|n|y_1/\sqrt{\mathrm{D_F}}\right)}{\Gamma(\frac{1}{2} + i\kappa_1 + q_1\mathrm{sgn}(n))} \frac{\mathrm{W}_{-q_2 \mathrm{sgn}(n'), i\kappa_2}\left(4\pi|n'|y_2/\sqrt{\mathrm{D_F}}\right)}{\Gamma(\frac{1}{2} + i\kappa_2 - q_2\mathrm{sgn}(n'))},\end{aligned} \tag{4.17}$$

with certain complex numbers $\varrho_{\mathrm{V}}(n)$ and the Whittaker function $\mathrm{W}_{a,b}$. This is in fact the specialization $\varphi = \varphi_{\mathrm{V}}$ of the result of solving, on the side condition $\varphi \in \mathrm{V}_{\mathrm{q}}$, the differential equation $\Omega_j \varphi = (\frac{1}{4} + \kappa_j^2)\varphi$ in the coordinate system (3.4). The gamma factors do not produce zeros for the combinations of q and (κ_1, κ_2) that really occur in ${}^0L^2(\Gamma\backslash G)$. Hereafter we shall assume that

$$\{\varphi_{\mathrm{V}} : \mathrm{V}\} \text{ is an orthonormal system in } L^2(\Gamma\backslash G). \tag{4.18}$$

Then we call $\varrho_V(n)$ the *Fourier coefficients* of V. The vector $\{\varrho_V(n) : n \in \mathcal{O}_*\}$ is well-defined, save for an arbitrary constant multiplier of unit absolute value.

Next, we turn to Hecke operators over the space $L^2(\Gamma\backslash G)$. Because of the particular importance of this concept for our purpose, we shall dwell on certain details, again exploiting our local situation. Thus, let $\mathfrak{n} = (n)$, $n \succ 0$, be an integral ideal of F; note our initial assumption that $N(\epsilon_0) = -1$. We define the action of the Hecke operator $\mathrm{T}_\mathfrak{n}$ over a left Γ-automorphic function f on G by

$$\mathrm{T}_\mathfrak{n} f(\mathrm{g}) = \frac{1}{\sqrt{N\mathfrak{n}}} \sum_{\substack{\mathfrak{d}|\mathfrak{n} \\ \mathfrak{d}=(d), d\succ 0}} \sum_{b \bmod \mathfrak{d}} f(\mathrm{n}[b/d]\mathrm{a}[n/d^2]\cdot \mathrm{g}), \tag{4.19}$$

where $\mathfrak{d}$ is an integral ideal. The orthogonal decomposition (4.1) is obviously preserved by any $\mathrm{T}_\mathfrak{n}$, and the same can be arranged for (4.2) and (4.4). Further, it can be shown in a standard way that $\mathrm{T}_\mathfrak{m}\mathrm{T}_\mathfrak{n} = \mathrm{T}_\mathfrak{n}\mathrm{T}_\mathfrak{m}$ for any $\mathfrak{m}$, $\mathfrak{n}$, and each $\mathrm{T}_\mathfrak{n}$ is symmetric over $L^2(\Gamma\backslash G)$. Thus we may assume that V is such that

$$\mathrm{T}_\mathfrak{n}|_\mathrm{V} = \mathrm{t_V}(\mathfrak{n})\cdot 1, \quad \mathrm{t_V}(\mathfrak{n}) \in \mathbb{R}. \tag{4.20}$$

Before taking this into (4.17), we note that $\varrho_V(\epsilon_0^{2\nu} n) = \varrho_V(n)$, $\nu \in \mathbb{Z}$, which follows from $\varphi_V(\mathrm{a}[\epsilon_0^{2\nu}]\mathrm{g}) = \varphi_V(\mathrm{g})$. Then, computing the Fourier coefficients of $\mathrm{T_n}\varphi_V$, we have, for any $m \in \mathcal{O}_*$,

$$\varrho_V(m)\mathrm{t_V}(\mathfrak{n}) = \sum_{(d)|(m,n)} \varrho_V(mn/d^2). \tag{4.21}$$

Hence, for any unit ϵ and $\mathcal{O}_* \ni n \succ 0$,

$$\varrho_V(\epsilon n) = \varrho_V(\epsilon)\mathrm{t_V}((n)). \tag{4.22}$$

In other words, for any $n \in \mathcal{O}_*$,

$$\varrho_V(n) = \mathrm{t_V}((n)) \cdot \begin{cases} \varrho_V(1) & \text{if } n \succ 0, \\ \varrho_V(\epsilon_0) & \text{if } \epsilon_0 n \succ 0, \\ \varrho_V(\epsilon_0') & \text{if } \epsilon_0' n \succ 0 \\ \varrho_V(-1) & \text{if } -n \succ 0. \end{cases} \tag{4.23}$$

Thus there exists at least one unit ϵ such that $\varrho_V(\epsilon) \neq 0$, since otherwise we would have $\varphi_V \equiv 0$, and the relation (4.21) implies the multiplicative property of Hecke eigenvalues:

$$\mathrm{t_V}((m))\mathrm{t_V}((n)) = \sum_{(d)|(m,n)} \mathrm{t_V}((mn/d^2)), \quad m, n \in \mathcal{O}_*. \tag{4.24}$$

As in the modular case, Hecke operators $\mathrm{T}_\mathfrak{n}$ are to be supplemented with involutions with which one may distinguish the parities or the four cases in (4.23). To this end we put, for any unit ϵ,

$$\mathfrak{J}_\epsilon = \left(\begin{bmatrix} \mathrm{sgn}(\epsilon) & \\ & 1 \end{bmatrix}, \begin{bmatrix} \mathrm{sgn}(\epsilon') & \\ & 1 \end{bmatrix} \right) \in \mathrm{PGL}_2(\mathbb{R})^2, \tag{4.25}$$

We have

$$\jmath_\epsilon G \jmath_\epsilon = G, \quad \mathrm{a}[|\epsilon|]\jmath_\epsilon \Gamma(\mathrm{a}[|\epsilon|]\jmath_\epsilon)^{-1} = \Gamma. \tag{4.26}$$

We then put, for any left Γ-automorphic function f on G,

$$\begin{aligned} \mathrm{i}_\epsilon f(\mathrm{g}) &= f(\mathrm{a}[|\epsilon|]\jmath_\epsilon \mathrm{g} \jmath_\epsilon) \\ &= f\big(\mathrm{n}[(\epsilon x_1, \epsilon' x_2)]\mathrm{a}[|\epsilon| y_1, |\epsilon'| y_2]\mathrm{k}[\mathrm{sgn}(\epsilon)\theta_1, \mathrm{sgn}(\epsilon')\theta_2]\big). \end{aligned} \tag{4.27}$$

We see readily that the left Γ-automorphy is preserved by i_ϵ because of (4.26), and that

$$\mathrm{i}_\epsilon^2 = 1, \quad \mathrm{i}_{\epsilon_1}\mathrm{i}_{\epsilon_2} = \mathrm{i}_{\epsilon_1\epsilon_2}; \quad \Omega_j \mathrm{i}_\epsilon = \mathrm{i}_\epsilon \Omega_j, \quad \mathrm{T}_n \mathrm{i}_\epsilon = \mathrm{i}_\epsilon \mathrm{T}_n. \tag{4.28}$$

But weights are not preserved in general. If f is of weight $2\mathrm{q}$, then $\mathrm{i}_\epsilon f$ is of weight $2(\mathrm{sgn}(\epsilon)q_1, \mathrm{sgn}(\epsilon')q_2)$. To get an involution we need either to restrict the weights or to choose ϵ appropriately.

Let us suppose first that $\kappa = (\kappa_1, \kappa_2)$ comes under Case *1* in Lemma 4.1. Then the finite number of V with $\kappa_{\mathrm{V}} = \kappa$ all have $\mathrm{q}_{\mathrm{V}} = (0,0)$, and the φ_{V} span a space in which i_ϵ is an involution or the identity. The commutativity in (4.28) implies that the V with $\kappa_{\mathrm{V}} = \kappa$ can be chosen such that $\mathrm{i}_\epsilon \varphi_{\mathrm{V}} = \xi_{\mathrm{V}}(\epsilon)\varphi_{\mathrm{V}}$ for all ϵ in the unit group. Then each ξ_{V} is a character of the unit group $\mathrm{mod}[\epsilon_0^2]$, with values in $\{\pm 1\}$. Hence, in the case $\mathrm{q}_{\mathrm{V}} = (0,0)$, the relation (4.23) is refined with

$$\varrho_{\mathrm{V}}(\epsilon) = \varrho_{\mathrm{V}}(1)\lambda_{\mathrm{V}}(\epsilon) \tag{4.29}$$

for any unit ϵ. As to the mixed cases, let us assume, for instance, that $\mathrm{q}_{\mathrm{V}} = (q_1, 0)$, $q_1 \neq 0$. The expansion (4.17) is actually over those n such that $q_1 n > 0$. Thus we need to use i_ϵ with $\epsilon > 0$, i.e., $\epsilon = 1$ or ϵ_0 mod $[\epsilon_0^2]$. It is an involution, and we can again choose a φ_{V} satisfying $\mathrm{i}_\epsilon \varphi_{\mathrm{V}} = \lambda_{\mathrm{V}}(\epsilon)\varphi_{\mathrm{V}}$ with $\lambda_{\mathrm{V}}(\epsilon) = \pm 1$. Hence, if $\mathrm{q}_{\mathrm{V}} = (q_1, 0)$, $q_1 \neq 0$, then

$$\varrho_{\mathrm{V}}(\epsilon_{\mathrm{V}}\epsilon) = \varrho_{\mathrm{V}}(\epsilon_{\mathrm{V}})\lambda_{\mathrm{V}}(\epsilon); \quad \lambda_{\mathrm{V}}(\epsilon) = 0 \text{ if } \epsilon \not\equiv 1, \epsilon_0 \bmod [\epsilon_0^2]. \tag{4.30}$$

with $\epsilon_{\mathrm{V}} = \mathrm{sgn}(q_1)$. Similarly, if $\mathrm{q}_{\mathrm{V}} = (0, q_2)$, $q_2 \neq 0$, then

$$\varrho_{\mathrm{V}}(\epsilon_{\mathrm{V}}\epsilon) = \varrho_{\mathrm{V}}(\epsilon_{\mathrm{V}})\lambda_{\mathrm{V}}(\epsilon); \quad \lambda_{\mathrm{V}}(\epsilon) = 0 \text{ if } \epsilon \not\equiv 1, \epsilon_0' \bmod [\epsilon_0^2], \tag{4.31}$$

with $\epsilon_{\mathrm{V}} = -\mathrm{sgn}(q_2)$. Further, if $\mathrm{q}_{\mathrm{V}} = (q_1, q_2)$, $q_1 q_2 \neq 0$, then the expansion (4.17) reduces to the one over the integers n such that $q_1 n > 0$ and $q_2 n' < 0$. Thus in (4.23) only one case is in fact possible:

$$\varrho_{\mathrm{V}}(\epsilon_{\mathrm{V}}\epsilon) = \varrho_{\mathrm{V}}(\epsilon_{\mathrm{V}})\lambda_{\mathrm{V}}(\epsilon); \quad \lambda_{\mathrm{V}} \text{ is the characteristic function of the set } [\epsilon_0^2], \tag{4.32}$$

where $\epsilon_{\mathrm{V}} = \epsilon_0^{\frac{1}{2}(1+\mathrm{sgn}(q_2))}\epsilon_0'^{\frac{1}{2}(1-\mathrm{sgn}(q_1))}$.

These definitions and (4.23) imply that we may put, for any $n \in \mathcal{O}_*$ and for any unit ϵ such that $\epsilon n \succ 0$,

$$\varrho_{\mathrm{V}}(n) = \varrho_{\mathrm{V}}\eta_{\mathrm{V}}(n)\mathrm{t}_{\mathrm{V}}((n)), \quad \eta_{\mathrm{V}}(n) = \lambda_{\mathrm{V}}(\epsilon/\epsilon_{\mathrm{V}}), \tag{4.33}$$

where

$$\varrho_{\mathrm{V}} = \varrho_{\mathrm{V}}(\epsilon_{\mathrm{V}}) \tag{4.34}$$

with ϵ_{V} as above, and with $\epsilon_{\mathrm{V}} = 1$ in the case (4.29).

In this way we have defined the function λ_{V} on the set of units via i_ϵ. The action of i_ϵ is, however, not limited to cuspidal subspaces. In fact, (3.17) implies that $\mathrm{i}_\epsilon E_{(0,0)}(\mathrm{g}; s, \nu) = |\epsilon/\epsilon'|^{\nu\varpi i} E_{(0,0)}(\mathrm{g}; s, \nu)$. This and (4.6) give an extension of λ and η:

$$\lambda_{\mathcal{E}_\nu}(\epsilon) = \eta_{\mathcal{E}_\nu}(\epsilon) = \left|\frac{\epsilon}{\epsilon'}\right|^{\nu\varpi i} = \pm 1. \tag{4.35}$$

Now, we are ready to state the spectral results that are essential for our purpose:

Lemma 4.2. *Let $\kappa_{\mathrm{V}} = (\kappa_1, \kappa_2)$ be defined by (4.7) and (4.13); $q_{\mathrm{V}} = (q_1, q_2)$ by (4.14); $\mathrm{t}_{\mathrm{V}}((n))$ by (4.20); $\eta_{\mathrm{V}}(n)$ by (4.33), and ϱ_{V} by (4.34). Let*

$$a_{\mathrm{V}} = |\varrho_{\mathrm{V}}|^2 \frac{\Gamma(\frac{1}{2} + |q_1| + i\overline{\kappa_1})\Gamma(\frac{1}{2} + |q_2| + i\overline{\kappa_2})}{\Gamma(\frac{1}{2} + |q_1| + i\kappa_1)\Gamma(\frac{1}{2} + |q_2| + i\kappa_2)}. \tag{4.36}$$

Further, let w be defined for all κ_{V}, and satisfy $w(\kappa_{\mathrm{V}}) \ll ((1+|\kappa_1|)(1+|\kappa_2|))^{-2-\mu}$ with an arbitrary small constant $\mu > 0$. Then we have, for any $n \in \mathcal{O}_*$,

$$\sum_{\mathrm{V}} a_{\mathrm{V}} |\eta_{\mathrm{V}}(n)| \mathrm{t}_{\mathrm{V}}((n))^2 w(\kappa_{\mathrm{V}}) \ll |N(n)|^{\frac{1}{2}+\mu}, \tag{4.37}$$

where V runs over all cuspidal irreducible subspaces, and the implicit constant depends only on μ. This implies, in particular, that

$$\mathrm{t}_{\mathrm{V}}(\mathfrak{n}) \ll (N(\mathfrak{n}))^{\frac{1}{4}+\mu}, \tag{4.38}$$

with the same dependency on μ.

Lemma 4.3. *Let f be sufficiently smooth over $(0,\infty)^2$, and decay sufficiently fast as one or both of the two variables tend either to 0^+ or to $+\infty$. Let*

$$\mathrm{B}_{\mathrm{e}} f(r_1, r_2) = -2^3 \int_0^\infty \int_0^\infty \frac{J^{(e_1)}_{2ir_1}(u_1) - J^{(e_1)}_{-2ir_1}(u_1)}{\sinh \pi r_1} \frac{J^{(e_2)}_{2ir_2}(u_2) - J^{(e_2)}_{-2ir_2}(u_2)}{\sinh \pi r_2} f(u_1, u_2) \frac{du_1 du_2}{u_1 u_2}, \tag{4.39}$$

where $\mathrm{e} = (e_1, e_2)$ with $e_j = \pm$, and $J^+_\nu = J_\nu$, $J^-_\nu = I_\nu$ in the usual notation for Bessel functions. Then we have, for any m_1, $m_2 \in \mathcal{O}_*$,

$$\begin{aligned}
&\sum_{c \in \mathcal{O}_*} \frac{S_{\mathrm{F}}(m_1, m_2; c)}{|N(c)|} f\left(\frac{4\pi}{|c|\sqrt{\mathrm{D_F}}}\sqrt{|m_1 m_2|}, \frac{4\pi}{|c'|\sqrt{\mathrm{D_F}}}\sqrt{|m'_1 m'_2|}\right) \\
&= \sum_{\mathrm{V}} a_{\mathrm{V}} \eta_{\mathrm{V}}(m_1) \mathrm{t}_{\mathrm{V}}((m_1)) \eta_{\mathrm{V}}(m_2) \mathrm{t}_{\mathrm{V}}((m_2)) \mathrm{B}_{[m_1 m_2]} f(\kappa_{\mathrm{V}}) \\
&+ \frac{\pi}{2^3 \sqrt{\mathrm{D_F}} \log \epsilon_0} \sum_{\nu=-\infty}^{\infty} \left|\frac{m_1 m_2}{m'_1 m'_2}\right|^{-\nu\varpi i} \\
&\times \int_{-\infty}^{\infty} \frac{\sigma_{2it}(m_1, \nu)\sigma_{2it}(m_2, \nu)}{|N(m_1 m_2)|^{it} |\zeta_{\mathrm{F}}(1+2it, 2\nu)|^2} \mathrm{B}_{[m_1 m_2]} f(t + \nu\varpi, t - \nu\varpi) dt,
\end{aligned} \tag{4.40}$$

where $[m_1m_2] = (\mathrm{sgn}(m_1m_2), \mathrm{sgn}(m'_1m'_2))$, *and other symbols are the same as in the previous lemma.*

Proof. Note that the classification of V enters into these assertions through κ_V and η_V. The identity (4.40) is the version of the geometric sum formula for the Hilbert modular group Γ that we shall apply in the next section. This is an adaptation of Theorem 2.7.1 of [1] to our present situation. Also, the statistical result (4.37) follows from Proposition 3.3.1 there. It should be worth remarking that the argument in Chapter 2 of [10] can readily be extended so as to yield these two principal results. The way to deduce the bound (4.38) from (4.37) is analogous to the modular case (see e.g., Section 3.1 of [10]). Better results are known, but for our purpose the conventional bound (4.38) is more than sufficient. It is important that (4.38) is uniform in V.

For the sake of a later purpose, we need to make it clear that the expansion (4.40) converges rapidly under the assumption on f given in the lemma: To this end, we remark first that by the definition (4.39) the function $\mathrm{B}_e f(r_1, r_2)$ is regular for $|\mathrm{Im}\, r_j| < C_f$, $j = 1, 2$. This constant C_f can be assumed to be sufficiently large. Put

$$\widetilde{f}(s_1, s_2) = \int_0^\infty \int_0^\infty f(u_1, u_2) \left(\frac{u_1}{2}\right)^{2s_1-1} \left(\frac{u_2}{2}\right)^{2s_2-1} du_1 du_2. \tag{4.41}$$

This is holomorphic for $|\mathrm{Re}\, s_j| < C_f$, $j = 1, 2$, and can be assumed to decay sufficiently fast there. By the Mellin inversion,

$$f(u_1, u_2) = \frac{1}{(2\pi i)^2} \int_{(a_1)} \int_{(a_2)} \widetilde{f}(s_1, s_2) \left(\frac{u_1}{2}\right)^{-2s_1} \left(\frac{u_2}{2}\right)^{-2s_2} ds_1 ds_2, \tag{4.42}$$

with appropriately chosen a_1, a_2. Inserting this into (4.39), we get formally

$$\mathrm{B}_e f(r_1, r_2) = \frac{1}{2\pi^2} \int_{(a_1)} \int_{(a_2)} \widetilde{f}(s_1, s_2) J^{(e_1)}(s_1, r_1) J^{(e_2)}(s_2, r_2) ds_1 ds_2, \tag{4.43}$$

where

$$J^{(e)}(s, r) = \int_0^\infty \frac{J^{(e)}_{2ir}(u) - J^{(e)}_{-2ir}(u)}{\sinh \pi r} \left(\frac{u}{2}\right)^{-2s-1} du. \tag{4.44}$$

Assume temporarily that $|\mathrm{Im}\, r_j| < \frac{1}{4}$, $j = 1, 2$. Then set $-\frac{1}{4} < a_j < -|\mathrm{Im}\, r_j|$. With this the quadruple integral involved in (4.43) converges absolutely, and the expression (4.43) holds in this domain of (r_1, r_2). On the other hand, we have

$$J^{\pm}(s, r) = \frac{1}{\pi i}\{(1 \pm 1)\cos \pi s + (1 \mp 1)\cosh \pi r\}\Gamma(ir - s)\Gamma(-ir - s), \tag{4.45}$$

provided $-\frac{1}{4} < \mathrm{Re}\, s < -|\mathrm{Im}\, r|$ (for the plus case, which is more delicate, see pp. 183–184 of [10]). Then, replace $J^{\pm}(s_j, r_j)$ in (4.43) by these, and shift the contours to the left appropriately. We see that the representation (4.43) holds in the much wider domain $|\mathrm{Im}\, r_j| < C_f$. Having done this, we shift the contours to the right. We find that for instance

$$\mathrm{B}_{\mathrm{e}} f(r_1, r_2) \ll (1+|r_1|+|r_2|)^{-C_f/2}, \quad |\mathrm{Im}\, r_j| < \tfrac{1}{4} C_f. \tag{4.46}$$

It remains to consider the case where $e = +$ and $ir = l - \frac{1}{2}$, $1 \le l \in \mathbb{Z}$, in (4.44). We have then

$$J^+\left(s, i(\tfrac{1}{2} - l)\right) = 2i(-1)^{l-1} \frac{\Gamma(l - \frac{1}{2} - s)}{\Gamma(l + \frac{1}{2} + s)}, \tag{4.47}$$

which is of course a special case of (4.45). With this, we see that (4.46) holds for all relevant combinations of e and (r_1, r_2). Hence Lemma 4.2 implies that the right side of (4.40) converges rapidly.

Before moving to our application of the geometric sum formula, we shall briefly discuss the Hecke series H_{V} associated with a cuspidal irreducible subspace V. Thus, assume (4.20), and put

$$H_{\mathrm{V}}(s) = \sum_{\mathfrak{n}} \mathrm{t}_{\mathrm{V}}(\mathfrak{n})(N\mathfrak{n})^{-s}, \tag{4.48}$$

which converges absolutely at least for $\mathrm{Re}\, s > \frac{5}{4}$, and is bounded there uniformly in V, because of (4.38). In fact it is convergent for $\mathrm{Re}\, s > 1$ as can be seen via a use of the Rankin zeta-function attached to V, but this fact is irrelevant to our purpose. The formula (4.24) implies an Euler product expression for H_{V}, and also the relation

$$H_{\mathrm{V}}(s_1) H_{\mathrm{V}}(s_2) = \zeta_{\mathrm{F}}(s_1 + s_2) \sum_{\mathfrak{n}} \sigma_{s_1 - s_2}(\mathfrak{n}) \mathrm{t}_{\mathrm{V}}(\mathfrak{n})(N\mathfrak{n})^{-s_1} \tag{4.49}$$

in the region of absolute convergence. Further, we have

Lemma 4.4. *The function H_{V} is entire, and satisfies the functional equation:*

$$\begin{aligned} H_{\mathrm{V}}(s) = \pi^{-2} \left(\frac{2\pi}{\sqrt{\mathrm{D_F}}}\right)^{2(2s-1)} & H_{\mathrm{V}}(1-s) \\ \times \prod_{j=1}^{2} \Big[(\lambda_{\mathrm{V}}(\epsilon_j) \cosh \pi\kappa_j - \cos \pi s) & \Gamma(1 - s + i\kappa_j)\Gamma(1 - s - i\kappa_j) \Big], \end{aligned} \tag{4.50}$$

where $\epsilon = \epsilon_0'$ with $(\epsilon_1, \epsilon_2) = (\epsilon, \epsilon')$, and (κ_1, κ_2) is defined by (4.7) and (4.13). In particular, we have the bound

$$H_{\mathrm{V}}(s) \ll (1 + |s| + |\kappa_1| + |\kappa_2|)^c, \tag{4.51}$$

where c depends only on $\mathrm{Re}\, s$, and the implicit constant only on F and $\mathrm{Re}\, s$.

Proof. The second assertion is the consequence of the first and (4.38) via the Phragmén–Lindelöf convexity principle. The functional equation is a special case of Theorem 11.1 in [3]. But we shall give a direct treatment. We apply the method in Section 3.2 of [10] to the case $q_V = (0,0)$. In fact the other cases are simpler. Let $\lambda_1 = \frac{1}{2}(1-\lambda_V(\epsilon_0'))$, $\lambda_2 = \frac{1}{2}(1-\lambda_V(\epsilon_0))$, and put $\varphi_V^* = (\mathrm{e}_1^+)^{\lambda_1}(\mathrm{e}_2^+)^{\lambda_2}\varphi_V$. Then consider the integral

$$A_V(s) = \iint\limits_{1\le y_1/y_2\le\epsilon_0^4} \varphi_V^*(\mathrm{a}[y])(y_1y_2)^{s-\frac{3}{2}}dy_1dy_2, \tag{4.52}$$

with $\mathrm{Re}\, s$ being sufficiently large. The relation (4.29) implies that

$$\begin{aligned}\varphi_V^*(\mathrm{a}[y]) &= 4(-1)^{\lambda_1}\frac{(2\pi/\sqrt{\mathrm{D_F}})^{\lambda_1+\lambda_2}}{\sqrt{\mathrm{D_F}}\Gamma(\frac{1}{2}+i\kappa_1)\Gamma(\frac{1}{2}+i\kappa_2)}\\ &\times y_1^{\lambda_1+\frac{1}{2}}y_2^{\lambda_2+\frac{1}{2}}\\ &\times \sum_{n\in\mathcal{O}_*}\varrho_V^*(n)K_{i\kappa_1}\left(2\pi|n|y_1/\sqrt{\mathrm{D_F}}\right)K_{i\kappa_2}\left(2\pi|n'|y_2/\sqrt{\mathrm{D_F}}\right),\end{aligned} \tag{4.53}$$

where

$$\varrho_V^*(n) = n^{\lambda_1}n'^{\lambda_2}\varrho_V(n) = \varrho_V|n|^{\lambda_1}|n'|^{\lambda_2}t_V((n)). \tag{4.54}$$

In fact, $\lambda_V(\epsilon)$ is a nontrivial character of the unit group, provided $\lambda_1+\lambda_2\neq 0$, and the terms caused by the derivative ∂_{y_j} in (3.6) cancel out each other in (4.53). Thus, we have

$$\begin{aligned}A_V(s) &= 4(-1)^{\lambda_1}\varrho_V\frac{(2\pi/\sqrt{\mathrm{D_F}})^{\lambda_1+\lambda_2}}{\sqrt{\mathrm{D_F}}\Gamma(\frac{1}{2}+i\kappa_1)\Gamma(\frac{1}{2}+i\kappa_2)}\\ &\times \sum_{n\in\mathcal{O}_*\bmod[\epsilon_0^2]} t_V((n))|n|^{\lambda_1}|n'|^{\lambda_2}\iint\limits_{(0,\infty)^2}K_{i\kappa_1}\left(2\pi|n|y_1/\sqrt{\mathrm{D_F}}\right)\\ &\times K_{i\kappa_2}\left(2\pi|n'|y_2/\sqrt{\mathrm{D_F}}\right)y_1^{\lambda_1}y_2^{\lambda_2}(y_1y_2)^{s-1}dy_1dy_2,\end{aligned} \tag{4.55}$$

in which the convergence is absolute throughout, at least for $\mathrm{Re}\, s > \frac{5}{4}$. We find that

$$\begin{aligned}A_V(s) &= \varrho_V\frac{(-1)^{\lambda_1}2^{\lambda_1+\lambda_2-2}}{\pi\Gamma(\frac{1}{2}+i\kappa_1)\Gamma(\frac{1}{2}+i\kappa_2)}\left(\frac{\pi}{\sqrt{\mathrm{D_F}}}\right)^{1-2s}\\ &\times\Gamma(\tfrac{1}{2}(s+\lambda_1+i\kappa_1))\Gamma(\tfrac{1}{2}(s+\lambda_1-i\kappa_1))\Gamma(\tfrac{1}{2}(s+\lambda_2+i\kappa_2))\\ &\times\Gamma(\tfrac{1}{2}(s+\lambda_2-i\kappa_2))H_V(s).\end{aligned} \tag{4.56}$$

On the other hand, arguing as (3.30)–(3.31), we see that $\varphi_V^*(a[y]) = (-1)^{\lambda_1+\lambda_2}\varphi_V^*(a[y]^{-1})$. Dividing the range of integration in (4.52) into two parts, according as $y_1y_2\le 1$ and $y_1y_2\ge 1$, we find that $A_V(s)$ is an entire function, and satisfies the functional equation $A_V(s) = (-1)^{\lambda_1+\lambda_2}A_V(1-s)$. This ends the proof.

5. Binary additive divisor

Now, we are ready to return to (3.58); thus we shall work on the assumptions (2.35) and (3.54). Note that we have $m \succ 0$ (see (2.15)), and $\eta_{\rm V}(m) = \eta_{\rm V}(1)$ by the definition (4.33).

The formulas (3.50), (3.59) and the bound (3.60) imply that the function $[[h]]_\ell(x;\alpha,\beta,\xi)$ is so smooth that the geometric sum formula (4.40) can safely be applied to $\mathrm{S}_{m,n}(\alpha,\beta,\xi;[h]_\ell)$, provided

$$|\mathrm{Re}\,\alpha| + \mathrm{Re}\,\beta < -\frac{1}{2}C_1, \tag{5.1}$$

with C_1 as in (3.54); for instance, take $\mathrm{Re}\,\beta$ negative and large, and keep $|\mathrm{Re}\,\alpha|$ relatively small. We shall assume this for the time being; it will be eventually eliminated. Then the discussion following Lemma 4.3 yields that we have a fast converging spectral decomposition:

$$\begin{aligned}\mathrm{S}_{m,n}(\alpha,\beta,\xi;[h]_\ell) &= \sum_{\rm V} a_{\rm V}\eta_{\rm V}(1)t_{\rm V}((m))\eta_{\rm V}(n)t_{\rm V}((n))\mathrm{B}_{[n]}[[h]]_\ell(\kappa_{\rm V};\alpha,\beta,\xi)\\ &+\frac{\pi}{2^3\sqrt{\mathrm{D_F}}\log\epsilon_0}\sum_{\nu=-\infty}^{\infty}\left|\frac{mn}{m'n'}\right|^{-\nu\varpi i}\int_{-\infty}^{\infty}\frac{\sigma_{2it}(m,2\nu)\sigma_{2it}(n,2\nu)}{|N(mn)|^{it}|\zeta_{\rm F}(1+2it,2\nu)|^2}\\ &\qquad\times \mathrm{B}_{[n]}[[h]]_\ell(t+\nu\varpi,t-\nu\varpi;\alpha,\beta,\xi)dt\\ &= \{\mathrm{S}^c_{m,n}+\mathrm{S}^e_{m,n}\}(\alpha,\beta,\xi;[h]_\ell),\end{aligned} \tag{5.2}$$

say. It is easy to check the uniformity of the convergence with respect to all involved parameters. The contribution, via (3.57), of $\mathrm{S}^c_{m,n}$ to (3.52) is equal to

$$\begin{aligned}&-i\pi^{-4}\zeta_{\rm F}(1-\beta)\left(\frac{2\pi}{\sqrt{\mathrm{D_F}}}\right)^{2\beta}N(m)^{\frac12(\alpha+\beta+1)}\\ &\times\sum_{n\in\mathcal{O}_*\bmod[\epsilon_0^2]}\frac{\sigma_{-\alpha}(n)}{|N(n)|^{\frac12(1-\alpha-\beta)}}\mathrm{sgn}^\ell[n]\sum_{\rm V}a_{\rm V}\eta_{\rm V}(1)t_{\rm V}((m))\eta_{\rm V}(n)t_{\rm V}((n))\\ &\times\sum_{\nu=-\infty}^{\infty}\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{i\nu\xi}\hat{\mathrm{p}}(\xi)\left|\frac{mn}{m'n'}\right|^{i\xi/(4\log\epsilon_0)}\mathrm{B}_{[n]}[[h]]_\ell(\kappa_{\rm V};\alpha,\beta,\xi)d\xi.\end{aligned} \tag{5.3}$$

The exchange of the order of summation is legitimate: The function $\mathrm{B}_{[n]}[[h]]_\ell(\kappa_{\rm V};\alpha,\beta,\xi)$ is smooth in ξ, and the bound (4.46) with the present choice of f and with $C_f = C_1$ holds even after differentiating with respect to ξ several times. Thus the last integral decays sufficiently fast in ν and $\kappa_{\rm V}$. By virtue of Lemma 4.2, the triple sum in (5.3) converges absolutely, which confirms our claim.

But, Poisson's sum formula gives, in (5.3),

$$\sum_\nu \cdots = \frac{1}{2}\mathrm{B}_{[n]}[[h]]_\ell(\kappa_{\rm V};\alpha,\beta,0), \tag{5.4}$$

because of our assumption on $\hat{\mathfrak{p}}$. Hence, by (4.49), we see that (5.3) is equal to

$$\frac{1}{2\pi^4 i}\left(\frac{2\pi}{\sqrt{\mathrm{D_F}}}\right)^{2\beta} N(m)^{\frac12(\alpha+\beta+1)}\sum_{\mathrm{V}} a_{\mathrm{V}}\eta_{\mathrm{V}}(1)\mathrm{t_V}((m))$$
$$\times H_{\mathrm{V}}\left(\tfrac12(1+\alpha-\beta)\right) H_{\mathrm{V}}\left(\tfrac12(1-\alpha-\beta)\right) \mathrm{B}^{\mathrm{V},\ell}[[h]]_\ell(\kappa_{\mathrm{V}};\alpha,\beta,0), \tag{5.5}$$

where

$$\mathrm{B}^{\mathrm{V},\ell} = \sum_{\epsilon \bmod [\epsilon_0^2]} \mathrm{sgn}^\ell[\epsilon]\eta_{\mathrm{V}}(\epsilon)\mathrm{B}_{[\epsilon]}, \tag{5.6}$$

with ϵ running over units. Here we have used the fact that $\eta_{\mathrm{V}}(\epsilon n) = \eta_{\mathrm{V}}(\epsilon)$, if $n \succ 0$. Similarly the contribution of $\mathrm{S}^{e}_{m,n}$ to (3.52) is equal to

$$\frac{(2\pi/\sqrt{\mathrm{D_F}})^{2\beta+1}}{2^5\pi^4 i\log\epsilon_0} N(m)^{\frac12(\alpha+\beta+1)} \sum_{\nu=-\infty}^{\infty}\left|\frac{m}{m'}\right|^{-\nu\varpi i}\int_{-\infty}^{\infty}\frac{\sigma_{2it}(m,2\nu)}{N(m)^{it}|\zeta_{\mathrm{F}}(1+2it,2\nu)|^2}$$
$$\times \zeta_{\mathrm{F}}\left(\tfrac12(1+\alpha-\beta)-it,\nu\right)\zeta_{\mathrm{F}}\left(\tfrac12(1+\alpha-\beta)+it,-\nu\right)\zeta_{\mathrm{F}}\left(\tfrac12(1-\alpha-\beta)-it,\nu\right)$$
$$\times \zeta_{\mathrm{F}}\left(\tfrac12(1-\alpha-\beta)+it,-\nu\right)\mathrm{B}^{\mathcal{E}_\nu,\ell}[[h]]_\ell(t+\nu\varpi,t-\nu\varpi;\alpha,\beta;0)dt, \tag{5.7}$$

where we have used (4.35). We insert these expressions into (3.43) via (3.52). We get a spectral decomposition of $\mathcal{B}_m(\alpha,\beta;h)$, provided (2.35), (3.54) and (5.1).

The domain (5.1) is, however, not suitable for the application in our mind, i.e., that to $\mathcal{Z}_2(g,\mathrm{F})$. We have to continue the decomposition to a neighbourhood of the point $(\alpha,\beta)=(0,0)$. Because of this, we shall study the transform

$$\Phi_*(r_1,r_2;\alpha,\beta;h) = \frac{1}{2\pi^4 i}\sum_\ell\sum_{\epsilon \bmod [\epsilon_0^2]}\mathrm{sgn}^\ell[\epsilon]\eta_*(\epsilon)\mathrm{B}_{[\epsilon]}[[h]]_\ell(r_1,r_2;\alpha,\beta,0). \tag{5.8}$$

Here $* = \mathrm{V}$ or $\mathcal{E}_\nu$, and (r_1,r_2) is initially equal to κ_{V} or $(t+\nu\varpi, t-\nu\varpi)$, respectively, but after (5.10) it will be regarded as a variable point in $\mathbb{C}^2$. From (2.33), (3.28), (3.50), and (3.59),

$$\sum_\ell \mathrm{sgn}^\ell[\epsilon][[h]]_\ell(x;\alpha,\beta,0) = \frac{1}{8i}\left|\frac{xx'}{4}\right|^{-\alpha-\beta-1}$$
$$\times\sum_{\mathrm{e}}\int_{(b)}\int_{(b)}\prod_{j=1}^{2}\Big[\left((1+\mathrm{e}_j\mathrm{sgn}(\epsilon_j))\cos\tfrac12\pi\alpha-(1-\mathrm{e}_j\mathrm{sgn}(\epsilon_j))\cos\pi(s_j-\tfrac12\alpha)\right)$$
$$\times\Gamma(1-s_j)\Gamma(1+\alpha-s_j)\Big]\widetilde{h}(s_1,s_2;\mathrm{e})\left|\frac{x}{2}\right|^{2s_1}\left|\frac{x'}{2}\right|^{2s_2}ds_1ds_2, \tag{5.9}$$

where b is as in (3.46), and $(\epsilon_1,\epsilon_2)=(\epsilon,\epsilon')$ as before. Apply $\mathrm{B}_{[\epsilon]}$ to this, and argue as in (4.42)–(4.45). We find that

$$\Phi_*(r_1,r_2;\alpha,\beta;h) = -\frac{1}{8\pi^6}\sum_{\mathrm{e}}\sum_{\epsilon \bmod [\epsilon_0^2]}\eta_*(\epsilon)$$

$$\times \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \widetilde{h}(s_1,s_2;\mathrm{e}) \prod_{j=1}^{2} \Big[\Delta_{e_j}^{\mathrm{sgn}(\epsilon_j)}(s_j,r_j;\alpha,\beta)\Gamma(s_j-\tfrac12(\alpha+\beta+1)-ir_j)$$
$$\times \Gamma(s_j-\tfrac12(\alpha+\beta+1)+ir_j)\Gamma(1-s_j)\Gamma(1+\alpha-s_j)\Big] ds_1 ds_2, \tag{5.10}$$

where

$$\Delta_{e_j}^{\pm}(s_j,r_j;\alpha,\beta) = \left\{(1\pm \mathrm{e}_j)\cos\tfrac12\pi\alpha - (1\mp \mathrm{e}_j)\cos\pi(s_j-\tfrac12\alpha)\right\}$$
$$\times \left\{(1\mp 1)\cosh\pi r_j + (1\pm 1)\sin\pi(s_j - \tfrac12(\alpha+\beta))\right\}. \tag{5.11}$$

The expression (5.10) is in fact a result of an application of analytic continuation: The s_j-contour separates the poles of $\Delta_{e_j}^{\mathrm{sgn}(\epsilon_j)}(s_j,r_j;\alpha,\beta)\Gamma(s_j-\frac12(\alpha+\beta+1)-ir_j)\Gamma(s_j-\frac12(\alpha+\beta+1)+ir_j)$ and $\Gamma(1-s_j)\Gamma(1+\alpha-s_j)$ to the left and the right, respectively, and it is assumed that the parameters are such that the contour can be drawn. Under the assumption (5.1), one may use the contour $\mathrm{Re}\, s_j = b$ with $\frac12(\alpha+\beta+1) < b < \min(0,\mathrm{Re}\,\alpha)$; then move it appropriately, and get (5.10). Note that if $ir_j = l-\frac12$, $1\le l\in\mathbb{Z}$, then (4.47) has to be taken into account.

If $* = \mathcal{E}_\nu$ or V with $\mathrm{q_V}=(0,0)$, then $\eta_* = \lambda_*$ is a character on the group of units, and

$$\sum_{\epsilon \bmod [\epsilon_0^2]} \eta_*(\epsilon)\prod_{j=1}^{2}\Delta_{e_j}^{\mathrm{sgn}(\epsilon_j)} = \prod_{j=1}^{2}(\Delta_{e_j}^{+} + \lambda_*(\epsilon_j)\Delta_{e_j}^{-}) \tag{5.12}$$

with $\epsilon=\epsilon_0'$ on the right side. We shall show that this can be assumed to hold for $* = $ V with $\mathrm{q_V}\neq(0,0)$ as well. In view of (5.5), we may restrict ourselves to those V with $\eta_{\mathrm{V}}(1)\neq 0$. This implies, by (4.33), that $\lambda_{\mathrm{V}}(\epsilon_{\mathrm{V}}^{-1})\neq 0$. Thus, by (4.30), we see that if $\mathrm{q_V}=(q_1,0)$, $q_1\neq 0$, then the left side of (5.12) is equal to $\eta_{\mathrm{V}}(1)\Delta_1^{+}(\Delta_2^{+}+\lambda_{\mathrm{V}}(\epsilon_0)\Delta_2^{-})$. Also, if $\mathrm{q_V}=(0,q_2)$, $q_2\neq 0$, then by (4.31) it is equal to $\eta_{\mathrm{V}}(1)\Delta_2^{+}(\Delta_1^{+}+\lambda_{\mathrm{V}}(\epsilon_0')\Delta_1^{-})$. Further, if $\mathrm{q_V}=(q_1,q_2)$, $q_1q_2\neq 0$, then by (4.32) it becomes $\eta_{\mathrm{V}}(1)\Delta_1^{+}\Delta_2^{+}$. Hence, we have, as a refinement of (5.12), that for any space $*$ with $\eta_*(1)\neq 0$

$$\sum_{\epsilon \bmod [\epsilon_0^2]} \eta_*(\epsilon)\prod_{j=1}^{2}\Delta_{e_j}^{\mathrm{sgn}(\epsilon_j)} = \eta_*(1)\prod_{j=1}^{2}(\Delta_{e_j}^{+} + \lambda_*(\epsilon_j)\Delta_{e_j}^{-}) \tag{5.13}$$

with $\epsilon=\epsilon_0'$ on the right side.

Lemma 5.1. *Let us assume* (2.35) *and* (3.54). *Let* V, $\kappa_{\mathrm{V}}=(\kappa_1,\kappa_2)$, *and* $\mathrm{q_V}=(q_1,q_2)$ *be as in* (4.2), (4.13), *and* (4.14), *respectively. Then* $\Phi_{\mathrm{V}}(\kappa_{\mathrm{V}};\alpha,\beta;h)$ *is regular, and satisfies*

$$\Phi_{\mathrm{V}}(\kappa_{\mathrm{V}};\alpha,\beta;h) \ll (1+|\kappa_1|+|\kappa_2|)^{-C_0/2}, \tag{5.14}$$

uniformly in V *and* α, β, *provided*

$$|\mathrm{Re}\,\alpha| + \mathrm{Re}\,\beta < 2\min_j \left\{|\mathrm{Im}\,\kappa_j + \tfrac{1}{2}| + \delta_j\right\}. \tag{5.15}$$

Here $\delta_j = 0, 1$, *according as* $q_j = 0, \neq 0$, *respectively. Analogously,* $\Phi_{\mathcal{E}_\nu}(t - \nu\varpi, t + \nu\varpi; \alpha, \beta; h)$ *is regular, and satisfies*

$$\Phi_{\mathcal{E}_\nu}(t - \nu\varpi, t + \nu\varpi; \alpha, \beta; h) \ll (1 + |t| + |\nu|)^{-C_0/2}, \tag{5.16}$$

provided

$$|\mathrm{Re}\,\alpha| + \mathrm{Re}\,\beta < 1 - 2|\mathrm{Im}\,t|. \tag{5.17}$$

Proof. It is enough to prove the assertions on Φ_{V}. If $q_j \neq 0$, then $i\kappa_j = |q_j| - \frac{1}{2}$, and the Γ-factor in (5.10), with $* = \mathrm{V}$ and $(r_1, r_2) = \kappa_{\mathrm{V}}$, is to be modified as indicated in (4.47). After this modification, one may draw contours in (5.10) under (5.15), and the assertion on the regularity follows. The decay property is simply a result of shifting the contours appropriately to the left. This ends the proof.

Now, we may state the first of our explicit formulas:

Theorem 5.2. *Let* $\mathcal{B}_m(\alpha, \beta; h)$ *be defined by* (2.15), *and assume* (2.35). *Let* α, β *be such that*

$$-1 < \mathrm{Re}(\pm\alpha + \beta) < \frac{3}{5}. \tag{5.18}$$

Then we have the spectral decomposition

$$\mathcal{B}_m(\alpha, \beta; h) = \left\{\mathcal{B}_m^{(r)} + \mathcal{B}_m^{(c)} + \mathcal{B}_m^{(e)}\right\}(\alpha, \beta; h), \tag{5.19}$$

where

$$\begin{aligned}\mathcal{B}_m^{(r)}(\alpha, \beta; h) &= \frac{\zeta_{\mathrm{F}}(1-\alpha)\zeta_{\mathrm{F}}(1-\beta)}{\sqrt{\mathrm{D}_{\mathrm{F}}}\,\zeta_{\mathrm{F}}(2-\alpha-\beta)} N(m)\sigma_{\alpha+\beta-1}(m)\ddot{h}(0,0)\\ &+ \frac{\zeta_{\mathrm{F}}(1+\alpha)\zeta_{\mathrm{F}}(1-\beta)}{\sqrt{\mathrm{D}_{\mathrm{F}}}\,\zeta_{\mathrm{F}}(2+\alpha-\beta)} N(m)^{1+\alpha}\sigma_{-\alpha+\beta-1}(m)\ddot{h}(\alpha,0)\\ &+ \frac{\zeta_{\mathrm{F}}(1-\alpha)\zeta_{\mathrm{F}}(1+\beta)}{\sqrt{\mathrm{D}_{\mathrm{F}}}\,\zeta_{\mathrm{F}}(2-\alpha+\beta)} N(m)^{1+\beta}\sigma_{\alpha-\beta-1}(m)\ddot{h}(0,\beta)\\ &+ \frac{\zeta_{\mathrm{F}}(1+\alpha)\zeta_{\mathrm{F}}(1+\beta)}{\sqrt{\mathrm{D}_{\mathrm{F}}}\,\zeta_{\mathrm{F}}(2+\alpha+\beta)} N(m)^{1+\alpha+\beta}\sigma_{-\alpha-\beta-1}(m)\ddot{h}(\alpha,\beta),\end{aligned} \tag{5.20}$$

$$\begin{aligned}\mathcal{B}_m^{(c)}(\alpha, \beta; h) &= (2\pi/\sqrt{\mathrm{D}_{\mathrm{F}}})^{2\beta} N(m)^{\frac{1}{2}(\alpha+\beta+1)}\\ &\times \sum_{\mathrm{V}} a_{\mathrm{V}}\eta_{\mathrm{V}}(1)\mathrm{t}_{\mathrm{V}}((m))H_{\mathrm{V}}(\tfrac{1}{2}(1+\alpha-\beta))\\ &\times H_{\mathrm{V}}(\tfrac{1}{2}(1-\alpha-\beta))\Phi_{\mathrm{V}}(\kappa_{\mathrm{V}}; \alpha, \beta; h),\end{aligned} \tag{5.21}$$

$$\mathcal{B}_m^{(e)}(\alpha,\beta;h) = \frac{(2\pi/\sqrt{D_F})^{2\beta+1}}{2^4\log\epsilon_0} N(m)^{\frac{1}{2}(\alpha+\beta+1)}$$
$$\times \sum_{\nu=-\infty}^{\infty} \left(\frac{m}{m'}\right)^{-\nu\varpi i} \int_{-\infty}^{\infty} \frac{\sigma_{2it}(m,2\nu)}{N(m)^{it}|\zeta_F(1+2it,2\nu)|^2} \zeta_F\left(\tfrac{1}{2}(1+\alpha-\beta)-it,\nu\right)$$
$$\times \zeta_F\left(\tfrac{1}{2}(1+\alpha-\beta)+it,-\nu\right)\zeta_F\left(\tfrac{1}{2}(1-\alpha-\beta)-it,\nu\right)$$
$$\times \zeta_F\left(\tfrac{1}{2}(1-\alpha-\beta)+it,-\nu\right)\Phi_{\mathcal{E}_\nu}(t+\nu\varpi,t-\nu\varpi;\alpha,\beta;h)dt. \tag{5.22}$$

Here V, κ_V, t_V, η_V, a_V, H_V, *are, respectively, defined by* (4.2), (4.13), (4.20), (4.33), (4.36), (4.48); *and* Φ_* *by* (5.10). *Also*

$$\ddot{h}(\eta_1,\eta_2) = \iint_{\mathbb{R}^2} h(u_1,u_2)|u_1u_2|^{\eta_1}(|1+u_1||1+u_2|)^{\eta_2}du_1du_2. \tag{5.23}$$

The expressions on the right sides of (5.20)–(5.22) *are all regular in the domain* (5.18).

Remark. This result should be compared with (3.33) and (3.57) in [9]. We could express Φ_* as linear combinations of products of two integrals of the Mellin–Barnes type. Then the analogy would be made clearer. A special case is treated in the proof of Corollary 5.3 below. In the condition (5.18) the lower bound is to secure the regularity of the Eisenstein term $\mathcal{B}_m^{(e)}$. It could be dropped, but then the residual term $\mathcal{B}_m^{(r)}$ would need a suitable modification.

Proof. As is mentioned after (5.7), a spectral decomposition of $\mathcal{B}_m(\alpha,\beta;h)$ has already been established, provided (5.1). Thus its continuation to (5.18) is to be discussed. The cuspidal contribution has exactly the same form as (5.21). By Lemmas 4.4 and 5.1, the sum converges absolutely and uniformly in the domain

$$|\mathrm{Re}\,\alpha| + \mathrm{Re}\,\beta < 2\min_{V}\min_{j}\left\{|\mathrm{Im}\,\kappa_j + \tfrac{1}{2}| + \delta_j\right\}, \tag{5.24}$$

and there it is regular. According to (4.16), this domain contains (5.18). On the other hand, the contribution of Eisenstein series has the form same as (5.22), but with (α,β) in (5.1); thus it is different from the function defined by (5.22) with (5.18), since the integrand can have singularities in (5.24), say. Those terms with $\nu \neq 0$ have, however, integrands regular in (5.24). The sum over ν converges absolutely and uniformly because of (5.16)–(5.17), and we may exclude this part from consideration. The term remaining to be considered is

$$\frac{\left(2\pi/\sqrt{D_F}\right)^{2\beta+1}}{2^4\log\epsilon_0} N(m)^{\frac{1}{2}(\alpha+\beta+1)} \int_{-\infty}^{\infty} \frac{\sigma_{2it}(m)}{N(m)^{it}\zeta_F(1+2it)\zeta_F(1-2it)}$$
$$\times \zeta_F\left(\tfrac{1}{2}(1+\alpha-\beta)-it\right)\zeta_F\left(\tfrac{1}{2}(1+\alpha-\beta)+it\right)$$
$$\times \zeta_F\left(\tfrac{1}{2}(1-\alpha-\beta)-it\right)\zeta_F\left(\tfrac{1}{2}(1-\alpha-\beta)+it\right)\Psi_t(\alpha,\beta;h)dt, \tag{5.25}$$

where (α, β) in (5.1), and $\Psi_t(\alpha, \beta; h) = \Phi_{\mathcal{E}_0}(t, t; \alpha, \beta; h)$. This is obviously regular when $|\mathrm{Re}\,\alpha| + \mathrm{Re}\,\beta < -1$. Let us consider the subdomain $-\frac{5}{4} < \mathrm{Re}(\pm\alpha + \beta) < -1$. On noting (5.16)–(5.17), we move the contour to $\mathrm{Im}\, t = \frac{1}{8}$. Poles we encounter are $t = -\frac{1}{2}(1 \pm \alpha + \beta)i$, and those from the factor $\zeta_{\mathrm{F}}^{-1}(1 + 2it)$. To avoid having poles on the contour, one could choose an appropriate broken line instead of a vertical line. At any events, the resulting integral can be assumed to be regular for $-\frac{5}{4} < \mathrm{Re}(\pm\alpha + \beta) < -\frac{3}{4}$. Then, restricting ourselves to the domain

$$-1 < \mathrm{Re}(\pm\alpha + \beta) < -\frac{3}{4}, \tag{5.26}$$

we shift the contour back to the original, i.e., the real axis. This time, poles we encounter are $t = \frac{1}{2}(1 \pm \alpha + \beta)i$ and those from the factor $\zeta_{\mathrm{F}}^{-1}(1 + 2it)$. In this way we have obtained the desired continuation to the domain (5.18), since the new integral is regular there. The residual terms arising from this procedure is those from the poles at $t = \pm\frac{1}{2}(1 \pm \alpha + \beta)i$; the other residues cancel out each other. Namely, the continuation, to (5.18), of the contribution of Eisenstein series is the sum of (5.22) and

$$\begin{aligned} &\frac{1}{4}(2\pi/\sqrt{\mathrm{D_F}})^{2(1+\beta)} N(m)^{1+\beta} \sigma_{\alpha-\beta-1}(m) \frac{\zeta_{\mathrm{F}}(-\beta)\zeta_{\mathrm{F}}(1-\alpha)}{\zeta_{\mathrm{F}}(2-\alpha+\beta)} \Psi_{\frac{1}{2}(1-\alpha+\beta)i}(\alpha, \beta; h) \\ &+ \frac{1}{4}(2\pi/\sqrt{\mathrm{D_F}})^{2(1+\beta)} N(m)^{1+\alpha+\beta} \sigma_{-\alpha-\beta-1}(m) \\ &\times \frac{\zeta_{\mathrm{F}}(-\beta)\zeta_{\mathrm{F}}(1+\alpha)}{\zeta_{\mathrm{F}}(2+\alpha+\beta)} \Psi_{\frac{1}{2}(1+\alpha+\beta)i}(\alpha, \beta; h). \end{aligned} \tag{5.27}$$

In deriving this we have used the facts that $\Psi_t(\alpha, \beta; h)$ is an even function of t, as can be seen from (5.10)–(5.11), and that the residue of $\zeta_{\mathrm{F}}(s)$ at $s = 1$ is equal to $2(\log \epsilon_0)/\sqrt{\mathrm{D_F}}$.

Thus we have to compute the last Ψ-factors. By (5.13), we have, for $r_j = \frac{1}{2}(1 - \alpha + \beta)i$, $j = 1, 2$,

$$\begin{aligned} &\sum_{\epsilon \bmod [\epsilon_0^2]} \prod_{j=1}^{2} \Delta_{e_j}^{\mathrm{sgn}(\epsilon_j)}(s_j, r_j; \alpha, \beta) \\ &= 2^4 e_1 e_2 \prod_{j=1}^{2} \left[\sin \pi(s_j - \alpha) \cos \tfrac{1}{2}\pi((1 - e_j)s_j + e_j\beta)\right]. \end{aligned} \tag{5.28}$$

Hence, by (5.10) with $* = \mathcal{E}_0$,

$$\begin{aligned} \Psi_{\frac{1}{2}(1-\alpha+\beta)i}(\alpha, \beta; h) &= -\frac{2}{\pi^4} \sum_{\mathrm{e}} e_1 e_2 \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \widetilde{h}(s_1, s_2; \mathrm{e}) \\ &\times \prod_{j=1}^{2} \left[\cos \tfrac{1}{2}\pi((1 - \mathrm{e}_j)s_j + e_j\beta)\Gamma(s_j - 1 - \beta)\Gamma(1 - s_j)\right] ds_1 ds_2. \end{aligned} \tag{5.29}$$

By definition, the s_j-contour is to separate the poles of $\Gamma(s_j-1-\beta)$ and $\Gamma(1-s_j)$ to the left and the right, respectively. The condition (5.26) implies that the contour can be drawn. In much the same way we have

$$\Psi_{\frac{1}{2}(1+\alpha+\beta)i}(\alpha,\beta;h) = -\frac{2}{\pi^4}\sum_{\mathrm{e}} e_1e_2 \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \widetilde{h}(s_1+\alpha, s_2+\alpha;\mathrm{e})$$

$$\times \prod_{j=1}^{2} \left[\cos\tfrac{1}{2}\pi\big((1-e_j)s_j+e_j\beta\big)\Gamma(s_j-1-\beta)\Gamma(1-s_j)\right] ds_1ds_2. \tag{5.30}$$

The double integrals in (5.29) and (5.30) are to be evaluated, but we shall treat only the latter, which is somewhat more complicated. We use (2.31). A rearrangement shows that the integral is equal to

$$\frac{1}{4i}e_1e_2\int_0^\infty\int_0^\infty h_2(e_1u_1, e_2u_2)R(u_1,e_1)R(u_2,e_2)du_1du_2, \tag{5.31}$$

where $h_2(u_1,u_2)=(\partial_{u_1}\partial_{u_2}h)(u_1,u_2)$,

$$R(u_j,e_j) = \int_{-i\infty}^{i\infty} \cos\tfrac{1}{2}\pi\left((1-e_j)s+e_j\beta\right)\Gamma(s-1-\beta)\Gamma(1-s)u_j^{s+\alpha}\frac{ds}{s+\alpha}. \tag{5.32}$$

Note that (5.26) is the same as $|\mathrm{Re}\,\alpha| < 1+\mathrm{Re}\,\beta < \frac{1}{4}-|\mathrm{Re}\,\alpha|$. Thus we may take, for instance, $\mathrm{Re}\,s=\frac{1}{2}$ as the contour; in particular, the pole $s=-\alpha$ can be assumed to be on the left of the contour. Then, shifting the contour in (5.32) to $\mathrm{Re}\,s=+\infty$ and to $\mathrm{Re}\,s=-\infty$, according as $u_j<1$ and $u_j>1$, respectively, we find that if $u_j<1$ then

$$R(u_j,e_j) = 2\pi i e_j\cos\tfrac{1}{2}\pi\beta\,\Gamma(-\beta)\int_0^{u_j} u^\alpha(1+e_ju)^\beta du, \tag{5.33}$$

and that if $u_j>1$ then

$$R(u_j,e_j) = -2\pi i e_j\cos\tfrac{1}{2}\pi\beta\,\Gamma(-\beta)\int_{u_j}^{\infty} u^\alpha(|1+e_ju|^\beta-u^\beta)du$$

$$+2\pi i e_j\cos\tfrac{1}{2}\pi\beta\,\Gamma(-\beta)\frac{u_j^{1+\alpha+\beta}}{1+\alpha+\beta}$$

$$+2\pi i\cos\left(\tfrac{1}{2}\pi((e_j-1)\alpha+e_j\beta)\right)\Gamma(1+\alpha)\Gamma(-1-\alpha-\beta). \tag{5.34}$$

Hence, for $u_j>0$, $u_j\neq 1$,

$$\partial_{u_j}R(u_j,e_j) = 2\pi i e_j\cos\tfrac{1}{2}\pi\beta\,\Gamma(-\beta)u_j^\alpha|1+e_ju_j|^\beta. \tag{5.35}$$

On noting that $R(u_j,e_j)$ is continuous for $u_j>0$ as (5.32) implies, we have, via (5.31),

$$\Psi_{\frac{1}{2}(1+\alpha+\beta)i}(\alpha,\beta;h) = 4\pi^{-2}\left(\cos\tfrac{1}{2}\pi\beta\,\Gamma(-\beta)\right)^2\ddot{h}(\alpha,\beta), \tag{5.36}$$

with $\ddot{h}$ as in (5.23). Analogously we have

$$\Psi_{\frac{1}{2}(1-\alpha+\beta)i}(\alpha,\beta;h) = 4\pi^{-2}\left(\cos\tfrac{1}{2}\pi\beta\,\Gamma(-\beta)\right)^2\ddot{h}(0,\beta), \tag{5.37}$$

We insert these into (5.27), and apply the functional equation for ζ_{F} to transform the factor $\zeta_{\mathrm{F}}(-\beta)$, which gives (5.20). We finish the proof of Theorem 5.2 with analytic continuation.

Corollary 5.3. *Let d_{F} be the ideal divisor function on* F. *Let $h(x)$ be such that its embedding $h(x,x')$ is smooth and compactly supported on $(0,\infty)^2$. Then we have, for any $\mathcal{O} \ni m \succ 0$,*

$$\sum_{n\in\mathcal{O}_*} d_{\mathrm{F}}(n)d_{\mathrm{F}}(n+m)h(n/m) = \left\{\mathfrak{B}_m^{(r)} + \mathfrak{B}_m^{(c)} + \mathfrak{B}_m^{(e)}\right\}(0,0;h), \tag{5.38}$$

where

$$\mathfrak{B}_m^{(r)}(0,0;h) = 4\frac{(\log\epsilon_0)^2}{\mathrm{D}_{\mathrm{F}}^{\frac{3}{2}}\zeta_{\mathrm{F}}(2)}\int_0^\infty\int_0^\infty h(u_1,u_2)M_{\mathrm{F}}(m;u_1,u_2)du_1du_2, \tag{5.39}$$

$$\mathfrak{B}_m^{(c)}(0,0;h) = N(m)^{\frac{1}{2}}\sum_{\mathrm{V}} a_{\mathrm{V}}\eta_{\mathrm{V}}(1)t_{\mathrm{V}}((m))H_{\mathrm{V}}(\tfrac{1}{2})^2\Phi_{\mathrm{V}}(\kappa_{\mathrm{V}};0,0;h), \tag{5.40}$$

$$\mathfrak{B}_m^{(e)}(\alpha,\beta;h) = \frac{\pi N(m)^{\frac{1}{2}}}{2^3\sqrt{\mathrm{D}_{\mathrm{F}}}\log\epsilon_0}\sum_{\nu=-\infty}^{\infty}\left(\frac{m}{m'}\right)^{-\nu\varpi i}$$
$$\times\int_{-\infty}^{\infty}\frac{\sigma_{2it}(m,2\nu)}{N(m)^{it}}\frac{|\zeta_{\mathrm{F}}(\frac{1}{2}+it,\nu)|^4}{|\zeta_{\mathrm{F}}(1+2it,2\nu)|^2}\Phi_{\mathcal{E}_\nu}(t+\nu\varpi,t-\nu\varpi;0,0;h)dt. \tag{5.41}$$

Here

$$\begin{aligned} M_{\mathrm{F}}(m;u_1,u_2) &= \sigma(m)(\log u_1u_2)(\log(u_1+1)(u_2+1)) \\ &+ \{\sigma(m)(c_0-\log N(m)) + 2\sigma'(m)\}\log(u_1u_2(u_1+1)(u_2+1)) \\ &+ \sigma(m)((c_0-\log N(m))^2 + c_1) + 4\sigma'(m)(c_0-\log N(m)) + 4\sigma''(m), \end{aligned} \tag{5.42}$$

where c_0, c_1 are constants that could be made explicit, and

$$\sigma^{(\nu)}(m) = \sum_{\mathfrak{d}|(m)}(\log N(\mathfrak{d}))^\nu N(\mathfrak{d}). \tag{5.43}$$

Also,

$$\begin{aligned} &\Phi_*(r_1,r_2;0,0;h) \\ &= \eta_*(1)\pi^{-2}\int_0^\infty\int_0^\infty h(u_1,u_2)P_*^{\epsilon_0'}(r_1;u_1)P_*^{\epsilon_0}(r_2;u_2)\frac{du_1du_2}{u_1u_2}, \end{aligned} \tag{5.44}$$

where

$$\begin{aligned} &P_*^{\epsilon}(r;u) \\ &= 2\mathrm{Re}\left[\left(\lambda_*(\epsilon) + \frac{i}{\sinh\pi r}\right)\frac{\Gamma(\frac{1}{2}+ir)^2}{\Gamma(1+2ir)}\right. \\ &\left.\times F\left(\tfrac{1}{2}+ir,\tfrac{1}{2}+ir;1+2ir;-1/u\right)u^{\frac{1}{2}-ir}\right]. \end{aligned} \tag{5.45}$$

with the hypergeometric function F.

Remark. This is an extension of Theorem 3 in [9] to the field F. That $h(x, x')$ is supported compactly on $(0,\infty)^2$ makes the situation relatively simple. Otherwise we would have to overcome a greater complexity that is similar to what is experienced in the proof of Theorem 4 of [9] (the *dual* case). Also, observe that the product of two values of the hypergeometric function in (5.44) is closely related to the free-space resolvent kernel of the Casimir operators Ω_j on the quotient space $G/K \cong \mathbb{H}^2$, the direct product of two copies of the hyperbolic upper half plane. This can be regarded as a higher dimensional analogue of a phenomenon noted on p. 179 of [10].

Proof. The assertion (5.42) is the result of taking the limit on the right side of (5.20) as $(\alpha,\beta)\to(0,0)$, with the present choice of h. On the other hand, (5.13) gives

$$\begin{aligned}\sum_{\epsilon \bmod [\epsilon_0^2]} \eta_*(\epsilon)\prod_{j=1}^{2}\Delta_+^{\mathrm{sgn}(\epsilon)}(s_j, r_j; 0, 0)\\ = 2^4\eta_*(1)\prod_{j=1}^{2}\{\sin\pi s_j - \lambda_*(\epsilon_j)\cos\pi s_j\cosh\pi r_j\}\end{aligned} \tag{5.46}$$

with $\epsilon = \epsilon_0'$ on the right side. Inserting this into (5.10) with our current specification on h, we have the expression (5.44) but with

$$\begin{aligned}P_*^{\epsilon}(r;u) = \frac{1}{\pi^2 i}\int_{(\frac34)}\Gamma(s-\tfrac12+ir)\Gamma(s-\tfrac12-ir)\Gamma(1-s)^2\\ \times\{\sin\pi s-\lambda_*(\epsilon)\cos\pi s\cosh\pi r\}u^s ds.\end{aligned} \tag{5.47}$$

If $ir = l-\frac12$ with an integer $l\ge 1$, an obvious transformation is to be applied to the factor $\Gamma(s-\frac12-ir)$. We have

$$\begin{aligned}P_*^{\epsilon}(r;u) = \frac{1}{2\pi i}\left(\lambda_*(\epsilon)+\frac{i}{\sinh\pi r}\right)\int_{(\frac34)}\frac{\Gamma(s-\frac12+ir)}{\Gamma(\frac32-s+ir)}\Gamma(1-s)^2u^s ds\\ + \frac{1}{2\pi i}\left(\lambda_*(\epsilon)-\frac{i}{\sinh\pi r}\right)\int_{(\frac34)}\frac{\Gamma(s-\frac12-ir)}{\Gamma(\frac32-s-ir)}\Gamma(1-s)^2u^s ds.\end{aligned} \tag{5.48}$$

Invoking the Mellin–Barnes formula for the hypergeometric function, we get (5.45). This ends the proof.

6. The fourth moment of ζ_{F}

Having obtained the spectral decomposition of $\mathcal{B}_m(\alpha,\beta;h)$ with (α,β) in a neighbourhood of $(0,0)$, we are at the position to apply it to our principal problem $\mathcal{Z}_2(g,\mathrm{F})$ via (2.16). To this end we have to see if the condition (2.35) is satisfied by $h=g_*$:

Lemma 6.1. *Let* $g_*(x;\gamma,\delta)$ *be defined by* (2.17), *and put*

$$\widetilde{g_*}(s_1,s_2;\mathrm{e};\gamma,\delta)=\int_0^\infty\int_0^\infty g_*(e_1u_1,e_2u_2;\gamma,\delta)u_1^{s_1-1}u_2^{s_2-1}du_1du_2, \tag{6.1}$$

with $\mathrm{e}=(\mathrm{e}_1,\mathrm{e}_2)$, $e_j=\pm1$. *Then the function* $\widetilde{g_*}(s_1,s_2;\mathrm{e};\gamma,\delta)$ *is regular in the domain*

$$\mathrm{Re}\,(s_j-\gamma-\delta)<0,\quad j=1,2. \tag{6.2}$$

An analytic continuation of it is given by the representation

$$\begin{aligned}\widetilde{g_*}(s_1,s_2;\mathrm{e};\gamma,\delta)&=\frac{\log\epsilon_0}{\pi^3}\Gamma(\gamma+\delta-s_1)\Gamma(\gamma+\delta-s_2)\\&\times\int_{-\infty}^\infty\hat{\mathrm{p}}^2(2\xi\log\epsilon_0)\int_{-\infty}^\infty g(t)\prod_{j=1}^2\Big[\sin\tfrac12\pi(\delta-i(t+(-1)^j\xi))\\&\times\cos\tfrac12\pi((1-\mathrm{e}_j)(\gamma-s_j)+\delta-\mathrm{e}_ji(t+(-1)^j\xi))\\&\times\Gamma(1-\delta+i(t+(-1)^j\xi))\Gamma(s_j-\gamma-i(t+(-1)^j\xi))\Big]dtd\xi,\end{aligned} \tag{6.3}$$

where the t*-contour separates the poles of* $\Gamma(1-\delta+i(t+\xi))\Gamma(1-\delta+i(t-\xi))$ *and those of* $\Gamma(s_1-\gamma-i(t-\xi))\Gamma(s_2-\gamma-i(t+\xi))$ *upwards and downwards, respectively; and* s_1,s_2,γ,δ *are assumed to be such that the contour can be drawn. Moreover, if* γ,δ, *and* $\mathrm{Re}\,s_1$, $\mathrm{Re}\,s_2$ *remain bounded, then we have, regardless of* (6.2),

$$\widetilde{g_*}(s_1,s_2;\mathrm{e};\gamma,\delta)\ll(1+|s_1|+|s_2|)^{-C} \tag{6.4}$$

with any fixed $C>0$.

Proof. We have, by (2.14),

$$\begin{aligned}g_*(\mathrm{e}_1u_1,\mathrm{e}_2u_2;\gamma,\delta)&=\frac{\log\epsilon_0}{2\pi}\int_{-\infty}^\infty\hat{\mathrm{p}}^2(2\xi\log\epsilon_0)\\&\times\int_{-\infty}^\infty g(t)\frac{u_1^{-\gamma-i(t+\xi)}u_2^{-\gamma-i(t-\xi)}}{|1+\mathrm{e}_1u_1|^{\delta-i(t+\xi)}|1+\mathrm{e}_2u_2|^{\delta-i(t-\xi)}}dtd\xi\end{aligned} \tag{6.5}$$

with u_1, $u_2>0$. Obviously

$$g_*(e_1u_1,e_2u_2;\gamma,\delta)\ll(u_1u_2)^{-\gamma-\delta} \tag{6.6}$$

as $u_1,u_2\uparrow\infty$. Shift appropriately the contour in the inner integral to see that g_* is of rapid decay as $u_1,u_2\downarrow0$, and also as $u_1\to1$ with $\mathrm{e}_1=-1$ or $u_2\to1$ with $\mathrm{e}_2=-1$, either. These considerations yield the first assertion. Then, assume temporarily that

$$\mathrm{Re}\,\gamma<\mathrm{Re}\,s<\mathrm{Re}(\gamma+\delta)<\mathrm{Re}\,\gamma+1. \tag{6.7}$$

Under this assumption,

$$\begin{aligned}&\widetilde{g_*}(s_1, s_2; \mathrm{e}; \gamma, \delta)\\&= \frac{\log \epsilon_0}{4\pi} \int_{-\infty}^{\infty} \hat{\mathrm{p}}^2(2\xi \log \epsilon_0) \int_{-\infty}^{\infty} g(t) \prod_{j=1}^{2} b_{\mathrm{e}_j}(s_j, t-(-1)^j\xi; \gamma, \delta) dt d\xi\end{aligned} \tag{6.8}$$

with

$$b_\pm(s, \eta; \gamma, \delta) = \int_0^\infty \frac{x^{s-\gamma-i\eta-1}}{|1 \pm x|^{\delta-i\eta}} dx. \tag{6.9}$$

We have, for any $\eta \in \mathbb{R}$, that

$$\int_0^\infty \frac{x^{s-\gamma-i\eta-1}}{(1+x)^{\delta-i\eta}} dx = \frac{\Gamma(s-\gamma-i\eta)\Gamma(\gamma+\delta-s)}{\Gamma(\delta-i\eta)}, \tag{6.10}$$

and

$$\begin{aligned}&\int_0^\infty \frac{x^{s-\gamma-i\eta-1}}{|1-x|^{\delta-i\eta}} dx\\&= \frac{\Gamma(s-\gamma-i\eta)\Gamma(1-\delta+i\eta)}{\Gamma(s-\gamma-\delta+1)} + \frac{\Gamma(\gamma+\delta-s)\Gamma(1-\delta+i\eta)}{\Gamma(\gamma-s+1+i\eta)}.\end{aligned} \tag{6.11}$$

From these equalities, we get

$$\begin{aligned}b_\pm(s, \eta; \gamma, \delta) =& \frac{2}{\pi} \sin \tfrac{1}{2}\pi(\delta - i\eta) \cos \tfrac{1}{2}\pi((1 \mp 1)(\gamma - s_j) + \delta \mp i\eta)\\&\times \Gamma(s-\gamma-i\eta)\Gamma(\gamma+\delta-s)\Gamma(1-\delta+i\eta).\end{aligned} \tag{6.12}$$

Inserting this into (6.8), we have the representation (6.3) under (6.7), with the contour $t \in \mathbb{R}$. Deforming the contour appropriately we may drop the constrain (6.7) and get the second assertion of the lemma. As to the decay property (6.4), push the new contour far down. This ends the proof.

In dealing with (2.16), let us assume initially that

$$|\mathrm{Re}(z_1 - z_2)| < c_0, \ |\mathrm{Re}(z_3 - z_4)| < c_0; \tag{6.13}$$

$$\mathrm{Re}\, z_1, \ \mathrm{Re}\, z_3 > C_0, \tag{6.14}$$

where C_0 and c_0 are, respectively, sufficiently large and small positive constants. Then the last lemma implies that $g_*(x; z_1, z_3)$ satisfies (2.35). Also the spectral decomposition (5.19) can safely be applied to $\mathcal{B}(z_1 - z_2, z_3 - z_4; g_*(\cdot; z_1, z_3))$. Thus (5.20)–(5.23) yield the decomposition

$$\mathcal{J}_+(z_1, z_2, z_3, z_4; g) = \{\mathcal{J}_+^{(r)} + \mathcal{J}_+^{(c)} + \mathcal{J}_+^{(e)}\}(z_1, z_2, z_3, z_4; g). \tag{6.15}$$

Here we have

$$
\begin{aligned}
&\mathcal{J}_+^{(r)}(z_1,z_2,z_3,z_4;g)\\
&=\frac{\zeta_F(1-z_1+z_2)\zeta_F(1-z_3+z_4)}{\sqrt{D_F}\,\zeta_F(2-z_1+z_2-z_3+z_4)}\zeta_F(z_1+z_3-1)\zeta_F(z_2+z_4)\ddot{g}_*(0,0;z_1,z_3)\\
&+\frac{\zeta_F(1+z_1-z_2)\zeta_F(1-z_3+z_4)}{\sqrt{D_F}\,\zeta_F(2+z_1-z_2-z_3+z_4)}\zeta_F(z_2+z_3-1)\zeta_F(z_1+z_4)\ddot{g}_*(z_1-z_2,0;z_1,z_3)\\
&+\frac{\zeta_F(1-z_1+z_2)\zeta_F(1+z_3-z_4)}{\sqrt{D_F}\,\zeta_F(2-z_1+z_2+z_3-z_4)}\zeta_F(z_1+z_4-1)\zeta_F(z_2+z_3)\ddot{g}_*(0,z_3-z_4;z_1,z_3)\\
&+\frac{\zeta_F(1+z_1-z_2)\zeta_F(1+z_3-z_4)}{\sqrt{D_F}\,\zeta_F(2+z_1-z_2+z_3-z_4)}\\
&\times\zeta_F(z_2+z_4-1)\zeta_F(z_1+z_3)\ddot{g}_*(z_1-z_3,z_2-z_4;z_1,z_3), \qquad (6.16)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_+^{(c)}(z_1,z_2,z_3,z_4;g)&=(2\pi/\sqrt{D_F})^{2(z_3-z_4)}\\
&\times\sum_{V} a_V\eta_V(1)H_V(\tfrac12(z_1+z_2+z_3+z_4-1))H_V(\tfrac12(1+z_1-z_2-z_3+z_4))\\
&\times H_V(\tfrac12(1-z_1+z_2-z_3+z_4))\Phi_V(\kappa_V;z_1-z_2,z_3-z_4;g_*(\cdot;z_1,z_3)), \quad (6.17)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_+^{(e)}(z_1,z_2,z_3,z_4;g)=\frac{(2\pi/\sqrt{D_F})^{2(z_3-z_4)+1}}{2^4\log\epsilon_0}\sum_{\nu=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{Z_F(z_1,z_2,z_3,z_4;t,\nu)}{|\zeta_F(1+2it,2\nu)|^2}\\
\times\Phi_{\mathcal{E}_\nu}(t+\nu\varpi,t-\nu\varpi;z_1-z_2,z_3-z_4;g_*(\cdot;z_1,z_3))dt, \quad (6.18)
\end{aligned}
$$

with

$$
\begin{aligned}
&Z_F(z_1,z_2,z_3,z_4;t,\nu)\\
&=\zeta_F(\tfrac12(z_1+z_2+z_3+z_4-1)+it,-\nu)\zeta_F(\tfrac12(z_1+z_2+z_3+z_4-1)-it,\nu)\\
&\times\zeta_F(\tfrac12(1+z_1-z_2-z_3+z_4)+it,-\nu)\zeta_F(\tfrac12(1+z_1-z_2-z_3+z_4)-it,\nu)\\
&\times\zeta_F(\tfrac12(1-z_1+z_2-z_3+z_4)+it,-\nu)\zeta_F(\tfrac12(1-z_1+z_2-z_3+z_4)-it,\nu). \quad (6.19)
\end{aligned}
$$

The absolute convergence that is necessary to deduce (6.17) and (6.18) is amply secured by (4.37), (4.38), (4.51), (5.14), and (5.16).

We have to continue the expansion (6.15) to a neighbourhood of the central point $p_{\frac12}=(\frac12,\frac12,\frac12,\frac12)$. We shall consider first the contribution of the cuspidal subspaces, i.e., (6.17). We need to examine the function $\phi=\Phi_V(\kappa_V;z_1-z_2,z_3-z_4;g_*(\cdot;z_1,z_3))$. This has to be well-defined in the domain (6.13)–(6.14). In fact, on noting (4.16) and (6.2), we may take the vertical lines $\mathrm{Re}\,s_j=\frac34$ as the contours in (5.10) with the current specification. Then, it follows readily that ϕ is regular in the domain

$$\left\{\mathrm{Re}(z_1+z_3)>\tfrac34 \text{ and } (6.13) \text{ hold}\right\}. \quad (6.20)$$

Also, shifting both contours to the left sufficiently far, we see, in view of (6.4), that ϕ is of fast decay with respect to κ_V uniformly with respect to bounded (z_1, z_2, z_3, z_4) in (6.20). Thus we find that $\mathcal{I}_+^{(c)}(z_1, z_2, z_3, z_4; g)$ continues to (6.20). In particular, it is regular at $p_{\frac{1}{2}}$; that is, $(z_1, z_2, z_3, z_4) = p_{\frac{1}{2}}$, the right side of (6.17) converges and represents $\mathcal{I}_+^c(p_{\frac{1}{2}}; g)$.

As to the continuation of (6.18), the part corresponding to $\nu \neq 0$ is analogous to the cuspidal contribution, and it is regular in (6.20). Thus, we need to consider only the term with $\nu = 0$:

$$\frac{(2\pi/\sqrt{D_F})^{2(z_3-z_4)+1}}{2^4 \log \epsilon_0} \int_{-\infty}^{\infty} \frac{Z_F(z_1, z_2, z_3, z_4; t, 0)}{\zeta_F(1+2it)\zeta_F(1-2it)} \Psi_t(z_1 - z_2, z_3 - z_4; g_*(\cdot; z_1, z_3))dt, \tag{6.21}$$

where Ψ_t is as in (5.25), and (z_1, z_2, z_3, z_4) still satisfies (6.13)–(6.14). Obviously this continues to the domain where $\mathrm{Re}(z_1 + z_2 + z_3 + z_4) > 3$ and (6.13) hold. Let us consider its subdomain where $3 < \mathrm{Re}(z_1 + z_2 + z_3 + z_4) < \frac{13}{4}$ and (6.13) hold, and move the contour in (6.21) to $\mathrm{Im}\, t = \frac{1}{8}$. Poles we encounter are $t = -\frac{1}{2}(z_1+z_2+z_3+z_4-3)i$, and those from the factor $\zeta_F^{-1}(1+2it)$. Here the argument is analogous to that following (5.25). Thus, as before, we may suppose that the resulting integral is regular in the domain where $\frac{11}{4} < \mathrm{Re}(z_1+z_2+z_3+z_4) < \frac{13}{4}$ and (6.13) hold. Restricting ourselves to the domain where $\frac{11}{4} < \mathrm{Re}(z_1+z_2+z_3+z_4) <$ 3 and (6.13) hold, we shift the contour back to $\mathbb{R}$. This time the poles we encounter are $t = \frac{1}{2}(z_1 + z_2 + z_3 + z_4 - 3)i$ and those from the factor $\zeta_F^{-1}(1 + 2it)$. In this way we obtain the desired continuation of (6.21), since the new integral over $\mathbb{R}$ is regular in the domain

$$\{\mathrm{Re}(z_1 + z_2 + z_3 + z_4) < 3 \text{ and (6.20) hold}\}, \tag{6.22}$$

which contains $p_{\frac{1}{2}}$. More precisely, this continuation of (6.21) has the expression that is the sum of the same expression as (6.21) but with (z_1, z_2, z_3, z_4) in (6.22) and the residual correction

$$\begin{aligned} &\frac{1}{4}\left(\frac{2\pi}{\sqrt{D_F}}\right)^{2(z_3-z_4+1)} \zeta_F(2-z_2-z_3)\zeta_F(z_1+z_4-1)\zeta_F(2-z_1-z_3) \\ &\times \zeta_F(z_2+z_4-1)(\zeta_F(4-z_1-z_2-z_3-z_4))^{-1} \\ &\times \Psi_{\frac{1}{2}(z_1+z_2+z_3+z_4-3)i}(z_1-z_2, z_3-z_4; g_*(\cdot; z_1, z_3)). \end{aligned} \tag{6.23}$$

Let $\mathcal{I}^{(r)}$ be the sum of (2.10), (6.16) and (6.23). This has to be regular at $p_{\frac{1}{2}}$, since we have, in a neighbourhood of $p_{\frac{1}{2}}$,

$$\mathcal{I} = \mathcal{I}^{(r)} + \mathcal{I}_+^{(c)} + \mathcal{I}_+^{(e)}, \tag{6.24}$$

and have seen already that $\mathcal{I}$ itself and $\mathcal{I}_+^{(c)}$, $\mathcal{I}_+^{(e)}$ are all regular at $p_{\frac{1}{2}}$. We write $M_V(g) = \mathcal{I}^{(r)}(p_{\frac{1}{2}}) + b_0 g(-\frac{1}{2}i) + a_1 g'(\frac{1}{2}i) + b_1 g'(-\frac{1}{2}i)$, which is in fact a transform of g. Also, we put $\Lambda_V(g) = \Phi_V(\kappa_V; 0, 0; g_*(\cdot; \frac{1}{2}, \frac{1}{2}))$ and $\Xi_\nu(t; g) = \Phi_{\mathcal{E}_\nu}(t - \nu\varpi, t + \nu\varpi; 0, 0; g_*(\cdot; \frac{1}{2}, \frac{1}{2}))$, which are integral transforms of g.

In this way we have established an explicit formula for the fourth power moment of the Dedekind zeta-function ζ_F of a real quadratic number field F:

Theorem 6.2. *Let* F *be of class number one, and have the fundamental unit* $\epsilon_0 > 1$ *with norm equal to* -1. *Let* g *be entire and of rapid decay in any fixed horizontal strip. Then we have, with transforms* $M_{\mathrm{F}}(g)$, $\Lambda_{\mathrm{V}}(g)$, *and* $\Xi_\nu(t;g)$ *as above,*

$$\int_{-\infty}^{\infty} |\zeta_{\mathrm{F}}(\tfrac{1}{2}+it)|^4 g(t) = M_{\mathrm{F}}(g) + \sum_{\mathrm{V}} a_{\mathrm{V}} \eta_{\mathrm{V}}(1) H_{\mathrm{V}}(\tfrac{1}{2})^3 \Lambda_{\mathrm{V}}(g)$$

$$+ \frac{\pi}{2^3\sqrt{\mathrm{D_F}}\log\epsilon_0} \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\zeta_{\mathrm{F}}(\frac{1}{2}+it,\nu)|^6}{|\zeta_{\mathrm{F}}(1+2it,2\nu)|^2} \Xi_\nu(t;g)dt, \quad (6.25)$$

where $\mathrm{D_F}$ *is the fundamental discriminant* of F. *Here* $\zeta_{\mathrm{F}}(\cdot,\nu)$, t_{V}, η_{V}, a_{V}, H_{V}, *are, respectively, defined by* (3.15), (4.20), (4.33), (4.36), (4.48); *and* V *runs over an orthonormal system of Hecke invariant irreducible subspaces of* $L^2(\Gamma\backslash\mathrm{PSL}_2(\mathbb{R})^2)$ *with* Γ *being the Hilbert modular group over* F.

Remark. This is an extension, to the field F, of Theorem 4.2 of [10] which asserts a spectral expansion of

$$\mathcal{Z}_2(g,\mathbb{Q}) = \int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it)|^4 g(t)dt \quad (6.26)$$

in terms of the spectral theory of $L^2(\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R}))$. The corresponding extension to the Gaussian number field is obtained in Theorem 14.1 of [2], where $\mathcal{Z}_2(g,\mathbb{Q}(i))$ is decomposed in terms of the spectral theory of $L^2(\mathrm{PSL}_2(\mathbb{Z}[i]) \backslash\mathrm{PSL}_2(\mathbb{C}))$. A common feature is the appearance of cubic powers of central values of Hecke series. This peculiar rôle of cubic powers of Hecke series was first found in [8], where (6.26) is dealt with. It is even possible to show that any single non-zero Hecke series contributes non-trivially to the formation of values of respective zeta-functions (see Section 5.4 of [10] for the modular case). It should be stressed that as is done for $\mathcal{Z}_2(g,\mathbb{Q})$ in [10] we could give precise expressions for the transforms $M_{\mathrm{F}}(g)$, $\Lambda_{\mathrm{V}}(g)$, and $\Xi_\nu(t;g)$.

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