Functiones et Approximatio
XXVIII (2000), 253–257

To Professor Włodzimierz Staś on his 75th birthday

DIRICHLET WEAK UNIFORM DISTRIBUTION OF MULTIPLICATIVE FUNCTIONS

W. NARKIEWICZ

Keywords: Weak uniform distribution, multiplicative function, Dirichlet series.

1. Let f be an integer-valued arithmetical function and let $N > 2$ be a given integer. The function f is said to be *weakly uniformly distributed* $(\bmod\ N)$ (WUD $(\bmod\ N)$), provided it assumes infinitely many values prime to N and for all i, j prime to N the ratio

$$\frac{\#\{n \le x : f(n) \equiv i \pmod N\}}{\#\{n \le x : f(n) \equiv j \pmod N\}}$$

tends to unity, when $x \to \infty$. In the case when f is a polynomial-like multiplicative function (i.e. for $k = 1, 2, \dots$ one has $f(p^k) = V_k(p)$ for all primes p with suitable $V_k \in Z[X]$) a necessary and sufficient condition for WUD $(\bmod\ N)$ has been obtained in [2]. (See also [4] and the literature quoted there). This condition can be restated in a form making sense for multiplicative functions which are not necessarily polynomial-like and in [5] an attempt has been made to reveal its analytical meaning. In that paper the notion of *Dirichlet weak uniform distribution* $(\bmod\ N)$ (*Dirichlet-WUD* $(\bmod\ N)$) has been considered and it turned out that for a class of multiplicative functions (encompassing all polynomial-like functions) the said condition is both necessary and sufficient for Dirichlet-WUD $(\bmod N)$ to hold. However, as noted in the corrigendum to [5] the class of functions to which this condition can be applied is in reality smaller than originally asserted.

The aim of this note is to modify slightly the previous definition of Dirichlet-WUD $(\bmod\ N)$ so that it will be applicable to all integer-valued multiplicative functions which assume sufficiently many values prime to N and obtain a necessary

1991 Mathematics Subject Classification: 2000 Mathematics Subject Classification: 11A25, 11B50,11N64.

and sufficient condition for Dirichlet-WUD (mod N) to hold for such a function. This new condition coincides with conditions considered in [2] and [5] for classes of functions to which they were applicable.

2. Let $N > 2$ be an integer and denote by $\mathcal{C}_N$ the set of all integer-valued functions for which the abscissa α of absolute convergence of the Dirichlet series

$$F(s) = \sum_{\substack{n \\ (f(n),N)=1}} \frac{1}{n^s} \tag{1}$$

is positive. (Obviously we have $\alpha \leq 1$). It is not difficult to show that if a function f does not belong to $\mathcal{C}_N$, then for every $\epsilon > 0$ one has

$$\#\{n \leq x : (f(n), N) = 1\} = O(x^\epsilon).$$

We put also for $(k, N) = 1$

$$F_k(s) = \sum_{\substack{n \\ f(n)\equiv k \pmod N}} \frac{1}{n^s}, \tag{2}$$

the series being absolutely convergent for $s > \alpha$.

We shall say that a function $f \in \mathcal{C}_N$ is *Dirichlet-WUD* (mod N) provided for all i, j prime to N one has

$$\lim_{s\to\alpha+0} \frac{F_i(s)}{F_j(s)} = 1. \tag{3}$$

It follows from an old result of Dedekind (see e.g. [4], Lemma 5.4) that if f is WUD (mod N) and the series $F(s)$ diverges at $s = \alpha$, then f is also Dirichlet-WUD (mod N). It has been shown in [3] that if f is multiplicative and the series

$$\sum_{\substack{p \text{ prime} \\ (f(p),N)>1}} \frac{1}{p}$$

converges, then Dirichlet-WUD (mod N) and WUD (mod N) coincide. Cf. [1], where a simple criterion for WUD (mod N) has been obtained for this class of functions.

It should be noted that if the series $F(s)$ converges at $s = \alpha$, then it may happen that f is WUD (mod N) without being Dirichlet-WUD (mod N), as the following example shows:

If we put $b_m = [m \log^2 m + 1]$ for $m = 1, 2, 3, \ldots$ then the series

$$\sum_{m=2}^{\infty} \frac{1}{b_m^c}$$

converges for $c = 1$ and diverges for all $c < 1$. Define f by

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is a member of the sequence } \{a_m\} \\ -1 & \text{if } n \text{ is a member of the sequence } 1 + \{a_m\} \\ 0 & \text{otherwise} \end{cases}.$$

Then f is WUD (mod 3) but on the other hand we have $\alpha = 1$ and for $i = 1$, $j = 2$ the ratio (3) tends to a number exceeding 1.

3. Our criterion for Dirichlet WUD (mod N) is contained in the following result:

Theorem. *Let $N > 2$ be an integer, let $f \in C_N$ be multiplicative and let α be the abscissa of absolute convergence of the series (1). For $(k, N) = 1$ denote by A_k the set of all prime-powers $q \equiv k (\mathrm{mod}\ N)$ which satisfy $(f(q), N) = 1$. Let R be the set of all residue classes k (mod N), prime to N, for which the series*

$$\sum_{q \in A_k} \frac{1}{q^\alpha}$$

diverges. Let $G(N)$ be the multiplicative group of residues (mod N) prime to N and denote by H the subgroup of $G(N)$ generated by R.

The function f will be Dirichlet WUD (mod N) if and only if for every non-principal character $\chi (\mathrm{mod} N)$ which is trivial on H there is a prime p for which

$$\sum_{k=0}^{\infty} \frac{\chi(f(p^k))}{p^{k\alpha}} = 0. \tag{4}$$

Proof. Our argument is a modification of that in [5]. For every character χ (mod N) define for $s > \alpha$

$$F(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(f(n))}{n^s}$$

and for prime p put

$$A_p(s, \chi) = \sum_{k=0}^{\infty} \frac{\chi(f(p^k))}{p^{ks}},$$

thus for $s > \alpha$ one gets

$$F(s, \chi) = \prod_p A_p(s, \chi). \tag{5}$$

In view of

$$F(s, \chi) = \sum_{(k,N)=1} \chi(k) F_k(s)$$

and

$$\frac{1}{\varphi(N)}F_k(s) = \sum_{\chi} \overline{\chi(k)}F(s,\chi),$$

(where F_k is given by (2)) one sees that f will be Dirichlet WUD (mod N) if and only if for every non-principal character $\chi(\text{mod } N)$ we would have

$$\lim_{s\to\alpha+0} \frac{F(s,\chi)}{F(s)} = 0. \tag{6}$$

If the series $F(s)$ converges for $s=\alpha$ then R is empty, H consists of the unit element and (6) is equivalent to $F(\alpha,\chi)=0$ for all non-principal χ and as the equality (5) holds in this case also for $s=\alpha$, the assertion follows. So we may assume that $F(s)$ diverges at $s=\alpha$. Put $j=1+[1/\alpha]$, so that $1/j<\alpha\le 1/(j-1)$ holds and split the set of all primes into three parts by putting

$$P_0 = \{p : p > 2^j,\ (f(p^i), N) > 1 \text{ for } i = 1,2,\dots,j-1\},$$

$P_1 = \{p : p \le 2^j\}$ and defining P_2 to be the set of all remaining primes. If for $i=0,1,2$ we put

$$B_i(s,\chi) = \prod_{p\in P_i} A_p(s,\chi),$$

then $B_0(s,\chi)$ and $B_1(s,\chi)$ will be continuous for $s\ge\alpha$ and $B_0(\alpha,\chi)\ne 0$. Moreover none of the factors $A_p(s,\chi)$ of $B_2(s,\chi)$ vaanishes at $s=\alpha$. If $a(\chi)$ denotes the order of the possible zero of $B_1(s,\chi)$ at $s=\alpha$ then we can write for $s>\alpha$

$$F(s,\chi) = (s-\alpha)^{a(\chi)} B_2(s,\chi) g(s,\chi),$$

where $g(s,\chi)$ is a function continuous for $s\ge\alpha$ and not vanishing at $s=\alpha$.

The number $a(\chi)$ is positive if and only if there is a prime $p\le 2^j$ for which $A_p(\alpha,\chi)$ vanishes, and thus for the principal character χ_0 we have $a(\chi_0)=0$. Since for $p>2^j$, $s\ge\alpha$ and every character χ one has $|A_p(s,\chi)-1| < \sum_{k=1}^{\infty} p^{-ks} < 1$, we can write for $s>\alpha$

$$\log B_2(s,\chi) = \sum_{p\in P_2}\sum_{k=1}^{\infty} \frac{\chi(f(p^k))}{p^{ks}} + g_1(s,\chi)$$

with $g_1(s,\chi)$ bounded for $s\ge\alpha$.

For $s>\alpha$ put

$$\begin{aligned} E(s,\chi) &= a(\chi)\log\left(\frac{1}{s-\alpha}\right) + \sum_{p\in P_2}\sum_{k=1}^{\infty} \frac{\chi(f(p^k)) - \chi_0(f(p^k))}{p^{ks}} \\ &= a(\chi)\log\left(\frac{1}{s-\alpha}\right) + \sum_{r \pmod N} (\chi(r)-1) \sum_{q} \frac{1}{q^{ks}}, \end{aligned}$$

where the inner sum is taken over all powers q of primes in P_2 for which $f(q) \equiv r \pmod{N}$. Observe that f is Dirichlet WUD (mod N) if and only if for every non-principal character χ (mod N) the real part of $E(s,\chi)$ tends to $-\infty$ when $s > \alpha$ tends to α. This statement will remain true, if the summation over r in the formula for $E(s,\chi)$ will be restricted by the condition $r \in R$.

If now f is Dirichlet WUD (mod N) but the assertion of the theorem fails, then there exists a non-principal character χ trivial on H for which (6) does not hold for any choice of the prime p. This implies $a(\chi) = 0$ and since for $r \in R$ we have $\chi(r) = 1$, the equality $E(s,\chi) = 0$ follows, giving a contradiction.

Conversely, assume that the condition given in the theorem is satisfied and let χ be non-principal. If $\chi(r) = 1$ holds for all $r \in R$, then $E(s,\chi) = a(\chi)\log(1/(s-1))$ and since (6) implies $a(\chi) > 0$ we get

$$\lim_{s\to\alpha+0} \Re(E(s.\chi)) = -\infty. \tag{7}$$

If however for some $r \in R$ we have $\chi(r) \neq 1$, then $\Re(\chi(r) - 1) > 0$ and since for all k one has $\Re(\chi(k) - 1) \leq 0$ we again obtain (7).

This concludes the proof of the theorem.

References

[1] H. Delange, *Sur les fonctions multiplicatives à valeurs entiers*, Acta Arith., **46** (1986), 165–195.

[2] W. Narkiewicz, *On distribution of values of multiplicative functions in residue classes*, Acta Arith., **12** (1966/67), 269–279.

[3] W. Narkiewicz, *Values of integer-valued multiplicative functions in residue classes*, Acta Arith., **32** (1977), 179–182.

[4] W. Narkiewicz, *Uniform Distribution of Sequences of Integers in Residue Classes*, Lecture Notes in Math., **1087**, Springer 1984.

[5] W. Narkiewicz, J.Śliwa, *On a kind of uniform distribution of values of multiplicative functions in residue classes*, Acta Arith., **31** (1976), 291–294; corr. Lecture Notes in Math., **46** (1986), 301.

Address: Mathematical Institute, Wrocław University, Plac Grunwaldzki 2-4, PL-50-384, Wrocław, Poland

E-mail: narkiew@math.uni.wroc.pl