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To Professor Włodzimierz Staś on his 75th birthday

ON THE DENSITY OF SOME SETS OF PRIMES p, FOR WHICH $n \mid \operatorname{ord}_p a$

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Abstract: In the present paper we derive an asymptotic formula for prime numbers p for which n exactly devides the order $a \pmod p$.
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1.

Let n, a denote natural numbers, $n > 1$, $a > 1$ and p a prime number not dividing a. The least positive γ such that $a^\gamma \equiv 1 \pmod p$ we denote by $\operatorname{ord}_p a$.

We proved in [2] some asymptotic formulae for the number of primes p for which $q^r \| \operatorname{ord}_p a$, where q is a fixed prime and $r = 0, 1. 2, \ldots$.

In the present paper we derive an asymptotic formula for prime numbers p for which $n \| \operatorname{ord}_p a$ ($n \| \operatorname{ord}_p a$ means that for each $q^\alpha \| n$ there is $q^\alpha \| \operatorname{ord}_p a$, where q is a prime).

2.

Let us write

$$n = q_1^{\alpha(q_1)} q_2^{\alpha(q_2)} \ldots q_r^{\alpha(q_r)}, \alpha(q_i) \geq 1 \quad \text{for } i = 1, 2, \ldots, r, q_1 < q_2 < \ldots < q_r, \quad (2.1)$$

where q_i are primes, and let

$$\prod_{q \mid n} q = k\,, \qquad \prod_{q \mid a} q = M, \quad (2.2)$$

where q runs over different prime divisors of n. In the following we denote by q a prime number.

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Let $t \geq 1$ be the largest natural number such that a is the t-th power in $\mathbb{Z}$. Denote further

$$H = \prod_{\substack{q|n \\ q^{\gamma(q)} \| t}} q^{\gamma(q)}. \tag{2.3}$$

We shall denote by b a positive integer satisfying the condition

$$a = b^H \ . \tag{2.4}$$

We introduce two parameters δ and s (see [3]) determined as follows

$$b = 2^\delta s v^2 \ , \tag{2.5}$$

where δ is equal to 0 or 1, s denotes the product of different odd primes and v is a positive integer.

In the following $x > 3$, c and m denote natural numbers.

Write further

$$N_1(x, m, \mathrm{c}) = \sum_{\substack{p \leq x, (p.c)=1 \\ (\mathrm{ord}_p\, c, n)=1}} 1 \ , \quad N(x, m, \mathrm{c}) = \sum_{\substack{p \leq x, (p,c)=1 \\ m | \mathrm{ord}_p\, c}} 1 \ ,$$

$$N_2(x, m, c) = \sum_{\substack{p \leq x, (p.c)=1 \\ n \| \mathrm{ord}_p\, c}} 1 \ , \quad \pi(x) = \sum_{p \leq x} 1 \ .$$

The symbols $\mu(l)$, $\varphi(l)$ and (α, β) denote as usual the Möbius function, the Euler function and the greatest common divisor of α, β respectively.

Theorem 1. *If*

$$x \geq \exp M \ , \quad \frac{\log x}{(\log \log x)^2} \geq c_1 k^2 \ ,$$

where c_1 *is a sufficiently large numerical constant, then*

$$\frac{1}{\pi(x)} N_1(x, n. a) = \alpha(k. \delta, s, H) + O\left(\frac{Hrk^3}{\varphi(k) \log^{r-1} q_1} \cdot \frac{(\log \log x)^{r+3}}{\log^2 x} \right) \ , \tag{2.6}$$

where

$$\alpha(k, \delta. s. H) = \beta(k, \delta. s. \gamma(2)) A(k, 2s. H) + A(k, 1, H) \ . \tag{2.7}$$

$$A(k, l, H) = \begin{cases} \mu(l) l / \varphi(l) \prod_{q|l} 1/q^{\gamma(q)} (q+1) \prod_{q|k/l} \left(1 - q^{1-\gamma(q)}/(q^2-1)\right) & \text{for } l \nmid k \\ 0 & \text{for } l \nmid k \end{cases} \tag{2.8}$$

and for an integer γ

$$\beta(k,\delta,s,\gamma) = \begin{cases} -\frac{1}{2} & \text{for } \delta=0,\ s>1,\ 2s \mid k,\ s\equiv 1 \pmod 4 \\ (2-(4^\gamma,4))/4 & \text{for } \delta=0,\ s>1,\ 2s \mid k,\ s\equiv 3 \pmod 4 \\ (2(4^\gamma,4)-(4^\gamma,16))/16 & \text{for } \delta=1,\ s\geq 1,\ 2s \mid k \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

The parameters $\gamma(q)$ are determined as in (2.3), the parameters δ and s as in (2.5).

For the proof see [3]. Let us observe that our formula (2.8) differs from the corresponding formula in [3] (cf. (3.3) there). The new expression gives of course the same value of $A(k,l,H)$, but in a more compact form.

Theorem 2. *If*

$$x \geq \exp M\ , \qquad \frac{\log x}{(\log\log x)^2} \geq c_1 k^2\ ,$$

then

$$\frac{1}{\pi(x)} N(x,n,a) = \frac{1+\beta(k,\delta,s,\alpha(2)+\gamma(2)-1)}{nH\prod_{q\mid k}(1-1/q^2)} + O\left(\frac{nHr2^r k^2}{\varphi(k)\log^{r-1} q_1}\frac{(\log\log x)^{r+3}}{\log^2 x}\right)\ . \tag{2.10}$$

where $\alpha(2)$ *is determined by (2.1) and* H *as in (2.3).*

Proof follows from Theorem 2 and Remark 2 of [3].

Theorem 3. *If*

$$x \geq \exp M\ , \qquad \frac{\log x}{(\log\log x)^2} \geq c_1 k^2\ ,$$

then

$$\frac{1}{\pi(x)} N_2(x,n,a) = \frac{1+\beta'(k,\delta,s,\alpha(2)+\gamma(2))}{nH\prod_{q\mid k}(1+1/q)} + O\left(\frac{Hnr3^r k^3}{\varphi(k)\log^{r-1} q_1}\frac{(\log\log x)^{r+3}}{\log^2 x}\right), \tag{2.11}$$

and for an integer γ

$$\beta'(k,\delta,s,\gamma) = \begin{cases} -1/2 & \text{for } \delta=0,\ s>1,\ 2s \mid k,\ s\equiv 1 \pmod 4, \\ 1 & \text{for } \delta=0,\ s>1,\ 2s \mid k,\ s\equiv 3 \pmod 4,\ \gamma=1 \\ -1/2 & \text{for } \delta=0,\ s>1,\ 2s \mid k,\ s\equiv 3 \pmod 4,\ \gamma\geq 2, \\ -1/8 & \text{for } \delta=1,\ s\geq 1,\ 2s \mid k,\ \gamma=1 \\ 1 & \text{for } \delta=1,\ s\geq 1,\ 2s \mid k,\ \gamma=2, \\ -1/2 & \text{for } \delta=1,\ s\geq 1,\ 2s \mid k,\ \gamma\geq 3 \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Proof. For an even k, we have

$$\frac{1}{\pi(x)}N_2(x,n,a)=\frac{1}{\pi(x)}\sum_{d|k}\mu(d)\sum_{\substack{p\le x\\ nd\mid \operatorname{ord}_p a}}1=\frac{1}{\pi(x)}\sum_{d|k}\mu(d)N(x,nd,a)$$

$$=\frac{1}{\pi(x)}\sum_{d|k/2}\mu(d)N(x,nd,a)+\frac{1}{\pi(x)}\sum_{2d|k}\mu(2d)N(x,2dn,a).$$

Hence, from the Theorem 2 we get

$$\frac{1}{\pi(x)}N_2(x,n,a)=\sum_{d|\frac{k}{2}}\mu(d)\frac{1+\beta(k,\delta,s,\alpha(2)+\gamma(2)-1)}{ndH\prod_{q|k}(1-1/q^2)}$$

$$-\sum_{d|k/2}\mu(d)\frac{1+\beta(k,\delta,s,\alpha(2)+\gamma(2))}{2dnH\prod_{q|k}(1-1/q^2)}+O\left(\frac{Hnr3^rk^3}{\varphi(k)\log^{r-1}q_1}\frac{(\log\log x)^{r+3}}{\log^2 x}\right)$$

$$=\frac{1}{Hn\prod_{q|k}(1+1/q)}(1+2\beta(k.\delta,s,\alpha(2)+\gamma(2)-1)$$

$$-\beta(k,\delta,s,\alpha(2)+\gamma(2))+O\left(\frac{Hnr3^rk^3}{\varphi(k)\log^{r-1}q_1}\frac{(\log\log x)^{r+3}}{\log^2 x}\right). \tag{2.13}$$

Hence, from (2.9) we obtain

$$2\beta(k.\delta.s,\alpha(2)+\gamma(2)-1)-\beta(k,\delta,s,\alpha(2)+\gamma(2))$$

$$=\begin{cases}-1/2 & \text{for } \delta=0,\ s>1,\ 2s\mid k,\ s\equiv 1 \pmod 4\\ 1 & \text{for } \delta=0,\ s>1,\ 2s\mid k,\ s\equiv 3 \pmod 4,\ \alpha(2)+\gamma(2)=1\\ -1/2 & \text{for } \delta=0,\ s>1,\ 2s\mid k,\ s\equiv 3 \pmod 4,\ \alpha(2)+\gamma(2)\ge 2\\ -1/8 & \text{for } \delta=1,\ s\ge 1,\ 2s\mid k,\ \alpha(2)+\gamma(2)=1\\ 1 & \text{for } \delta=1,\ s\ge 1,\ 2s\mid k,\ \alpha(2)+\gamma(2)=2\\ -1/2 & \text{for } \delta=1,\ s\ge 1,\ 2s\mid k,\ \alpha(2)+\gamma(2)\ge 3\\ 0 & \text{otherwise.}\end{cases} \tag{2.14}$$

Owing to (2.13) and (2.14) we get (2.11) for even k.

For an odd k, we have

$$\frac{1}{\pi(x)}N_2(x.n,a)=\frac{1}{\pi(x)}\sum_{d|k}\mu(d)N(x,nd,a)$$

$$=\sum_{d|k}\mu(d)\frac{1}{ndH\prod_{q|k}(1-1/q^2)}+O\left(\frac{Hnr3^rk^3}{\varphi(k)\log^{r-1}q_1}\frac{(\log\log x)^{r+3}}{\log^2 x}\right)$$

$$=\frac{1}{Hn\prod_{q|k}(1+\frac{1}{q})}+O\left(\frac{Hnr3^rk^3}{\varphi(k)\log^{r-1}q_1}\frac{(\log\log x)^{r+3}}{\log^2 x}\right).$$

hence, it follows (2.11) for odd k. ∎

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