Functiones et Approximatio
XXVIII (2000), 233–236

To Professor Włodzimierz Staś on his 75th birthday

ON SOME CONNECTIONS BETWEEN ZETA-ZEROS AND 3-FREE INTEGERS

K. M. Bartz

Abstract: A relationship between 3-free integers and zeros of the Riemann zeta-function, which is more explicit than the classical formula is presented.
Keywords: zeros of the Riemann zeta-function, 3-free integers.

As usual, a natural number is called k-free if it is divisible by no integer k-th power other than 1. Denote $\mu_3(n) = 1$ for 3-free n and 0 for remaining n. Following some ideas of my previous works (see [2] and [3] and compare [6]) we will describe the analytic character of some functions $t(z)$ and $T(z)$ defined in the case where there are no multiple zeros ϱ of the Riemann zeta-functions for $\operatorname{Im} z > 0$ as follows

$$t(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<\tau_n}} \frac{\varsigma(\frac{1}{3}\varrho)e^{\frac{1}{3}z\varrho}}{3\varsigma'(\varrho)}$$

and

$$T(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<\tau_n}} \frac{\varsigma(\frac{1}{3}\varrho)e^{\frac{1}{3}\varrho z}}{\varrho\varsigma'(\varrho)}$$

where the summation is over all non-trivial zeros ϱ of $\varsigma(s)$. The sequence τ_n yields a certain grouping of the zeros.

If $\varsigma(s)$ has a multiple zero at $s = \varrho$, the corresponding term in $t(z)$ and $T(z)$ must be replaced by an appropriate residue. In the following we will consider this general case.

First we prove

Acknowledgment. Supported by KBN Grant nr 2 P03A 028 09.
1991 Mathematics Subject Classification: 11N64

Theorem 1. *The function $t(z)$ is holomorphic on the upper half-plane and can be continued analytically to a meromorphic function on the whole complex plane, which satisfies the following functional equation*

$$t(z)+\overline{t(\overline{z})}=-\frac{e^{z}}{\varsigma(3)}-\sum_{l=0}^{\infty}\frac{e^{(-2l-\frac{2}{3})z}(2\pi)^{+4l+\frac{4}{3}}\Gamma(1+2l+\frac{2}{3})\varsigma(2l+\frac{5}{3})}{\pi\sqrt{3}(6l+2)!\varsigma(6l+3)}$$
$$+\sum_{l=0}^{\infty}\frac{e^{(-2l-\frac{4}{3})z}(2\pi)^{4l+\frac{8}{3}}\Gamma(2l+\frac{7}{3})\varsigma(2l+\frac{7}{3})}{\pi\sqrt{3}(6l+4)!\varsigma(6l+5)}$$

where the second term of the right side is an entire function of order $2/3$ of variable $z_1=e^{-z}$ (the Ritt order is equal to $2/3$) and the third term is an entire function of the Ritt order equal to $4/3$.
The only singularities of $t(z)$ are simple poles at the points $z=\log n$ on the real axis, where n is a 3-free number (also $n=1$) with residues

$$\operatorname*{res}_{z=\log n} t(z)=-\frac{\mu_3(n)}{2\pi i}$$

A more difficult problem connected with the analytic character of the function $T(z)$ will be described in

Theorem 2. *The series*

$$\sum_{n=0}^{\infty}T_n(z)=\Bigg(\sum_{\substack{\varrho\\ 0<\operatorname{Im}\varrho<\tau_1}}+\sum_{n=1}^{\infty}\sum_{\tau_n<\operatorname{Im}\varrho<\tau_{n+1}}\Bigg)\frac{\varsigma(\frac{1}{3}\varrho)e^{\frac{1}{3}\varrho z}}{\varrho\varsigma'(\varrho)}$$

where $z=x+iy$ is uniformly convergent for $y\ge\delta>0$ almost uniformly with respect to x. If $y=0$, suppose that, x is not equal to $\log n$, where n is 3-free number, then the series $\sum_{n=0}^{\infty}T_n(x)$ is also convergent to $T(x)$ and the convergence is uniform in every closed interval not containing points of the form $\log n$.

Finally, applying Theorems 1 and 2 we prove an explicit formula for 3-free integers which is also an explicit formula for $\zeta(3)$.

Let $Q_3(x)$ denote the number of 3-free positive integers not exceeding x. Then evidently

$$Q_3(x)=\sum_{n\le x}\mu_3(n)=-2\pi i\sum_{n\le x}\operatorname*{res}_{z=\log n}t(z)$$

Let

$$Q_3^0(x)=\frac{Q_3(x+0)+Q_3(x-0)}{2}=\sum_{n\le x}{}'\mu_3(n)$$

where Σ' indicates that when x is a integer the term corresponding to $n=x$ to have the factor $\frac{1}{2}$. Then we have

Theorem 3. *There is a sequence* τ_n, $2^{n-1}c_0 \le \tau_n < 2^n c_0$, $(n \ge 1)$, *where* c_0 *is an absolute positive constant, such that*

$$Q_3^0(x) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ |\mathrm{Im}\,\varrho|<\tau_n}} \frac{1}{(k_\varrho - 1)!} \frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}} \left[(s-\varrho)^{k_\varrho} \frac{x^{\frac{1}{3}s}\varsigma(\frac{1}{3}s)}{s\varsigma(s)} \right]_{s=\varrho}$$
$$+ \frac{x}{\varsigma(3)} + 1 + \sum_{l=0}^{\infty} \frac{(2\pi)^{4l+\frac{4}{3}}\Gamma(1+2l+\frac{2}{3})\varsigma(2l+\frac{5}{3})}{\pi\sqrt{3}(6l+2)!\varsigma(6l+3)x^{(2l+\frac{2}{3})}}$$
$$- \sum_{l=0}^{\infty} \frac{(2\pi)^{4l+\frac{8}{3}}\Gamma(2l+\frac{7}{3})\varsigma(2l+\frac{7}{3})}{\pi\sqrt{3}(6l+4)!\varsigma(6l+5)x^{2l+\frac{4}{3}}}$$

where k_ϱ *denotes the order of multiplicity of the nontrivial zero* ϱ *of the Riemann zeta-function* $\varsigma(s)$.

For the proof of this theorems it is sufficient to remark that we have to consider for any complex $z = x+iy$ from the upper half-plane $H = \{z \in C; \mathrm{Im}\, z > 0\}$, the integral

$$\int \frac{\varsigma(s)e^{sz}}{\varsigma(3s)} ds$$

taken in the positive sense round the contour with the sides

$$\left[\frac{4}{3}, \frac{4}{3} + i\frac{1}{3}\tau_n\right], \left[\frac{4}{3} + i\frac{1}{3}\tau_n, -\frac{1}{6} + i\frac{1}{3}\tau_n\right], \left[-\frac{1}{6} + i\frac{1}{3}\tau_n, -\frac{1}{6}\right]$$

and by a simple and smooth curve $\tau[0,1] \longrightarrow C$ denoting by $l\left(-\frac{1}{6}, \frac{4}{3}\right)$ such that $\tau(0) = -\frac{1}{6}$, $\tau(1) = \frac{4}{3}$ and $0 < \mathrm{Im}\,\tau < 1$ for $t \in (0,1)$.

The sequence (τ_n) yields a certain grouping of the non-trivial zeros of the Riemann zeta function, implicated by the theorem of Balasubramanian and Ramachandra (see [1]) and independently of Montgomery (see [7]) and compare [5], th.9.4), such that $2^{n-1}c_0 \le \tau_n < 2^n c_0$ for $n \ge 1$ with a suitable chosen constant c_0, such that

$$|\,\varsigma(\sigma + i\tau_n)\,|^{-1} \le c_1(\log \tau_n)^{c_2} \qquad \text{for} \quad \sigma \ge -1$$

where c_1 and c_2 are absolute constants, c_0 depends on c_2.

In the proofs of theorem 1, 2 and 3, using methods presented in [2] and [3], we have to use the Mellin-Barnes integrals (see [4], p.64).

The presence of two last terms in theorem 2 and theorem 3 is easy to explain as follows.

We have by functional equation for $\varsigma(s)$

$$\sum_{l=0}^{\infty} \operatorname*{res}_{s=\left\{\begin{smallmatrix}-2l-2/3\\ -2l-4/3\end{smallmatrix}\right.} \frac{e^{sz}\varsigma(s)}{\varsigma(3s)}$$

$$= \sum_{l=0}^{\infty} \operatorname*{res}_{s=\left\{\begin{smallmatrix}-2l-2/3\\ -2l-4/3\end{smallmatrix}\right.} \frac{e^{sz}\Gamma(1-s)\varsigma(1-s)}{(2\pi)^{2s}(e^{is\pi}+1+e^{-is\pi})\Gamma(1-3s)\varsigma(1-3s)}$$

$$= \sum_{l=0}^{\infty} \frac{e^{(-2l-4/3)z}(2\pi)^{4l+8/3}\Gamma(2l+2+1/3)\varsigma(2l+2+1/3)}{\pi\sqrt{3}(6l+4)!\varsigma(6l+5)}$$

$$- \sum_{l=0}^{\infty} \frac{e^{(-2l-2/3)z}(2\pi)^{4l+4/3}\Gamma(2l+1+2/3)\varsigma(2l+1+2/3)}{\pi\sqrt{3}(6l+2)!\varsigma(6l+3)}$$

since

$$\operatorname*{res}_{s=-2-2/3} \frac{1}{e^{is\pi}+1+e^{-is\pi}} = \frac{1}{\sqrt{3}\pi}$$

and

$$\operatorname*{res}_{s=-2-4/3} \frac{1}{e^{is\pi}+1+e^{-is\pi}} = -\frac{1}{\pi\sqrt{3}} .$$

References

[1] R. Balasubramanian, R. Ramachandra, *On the frequency of Titchmarsh's phenomenon for $\varsigma(s)$*, **III.** Proc. Indian Acad. Sci. 86A (1977) 341–351.

[2] K. M. Bartz, *On some complex explicit formulae connected with the Möbius function*, I, **II** Acta Arithmetica **LVII** (1991), 283–305.

[3] K. M. Bartz, *On some connections between zeta-zeros and square-free integers*, Monatshefte für Mathematik 114 (1992), 15–34.

[4] H. Bateman, A. Erdelyi, *Higher transcendental functions*, Vol.1, Mc Graw-Hill Book Company, 1953 (russian translation, Moskwa 1973).

[5] A. Ivić, *The Riemann zeta function. The theory of the Riemann zeta-function with applications*, Wiley, New York 1985.

[6] J. Kaczorowski, *The k-functions in multiplicative number theory, I. On complex explicit formulae*, Acta Arithmetica **LVI** (1990), 195–211.

[7] H. L. Montgomery, *Extreme values of the Riemann zeta-function*, Comment. Math. Helv. **52** (1977), 511–518.

Address: Krystyna Maria Bartz, Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Matejki 48/49, 60-769 Poznań, Poland

E-mail: `kbartz@math.amu.edu.pl`

Received: 26 Oct 2000