Functiones et Approximatio
XXVIII (2000), 195–200

To Professor Włodzimierz Staś on his 75th birthday

AN ASYMPTOTIC ESTIMATE OF THE NUMBER OF BIFURCATING SOLUTIONS FOR THE EQUATION $-\Delta u = \mu f(u)$

W. Krawcewicz[†] [‡], W. Marzantowicz[‡] [*]

Abstract: In this paper we present a lower estimate on the number of non-zero solutions (u, μ) of the following boundary value problem

$$\begin{cases} -\Delta u = \mu \cdot f(u) & \text{on} \quad \Omega \\ u \equiv 0 & \text{on} \quad \partial\Omega \end{cases} \qquad (\mathcal{P})$$

where $\mu \in \mathbb{R}$, $\Omega = (-\pi/2; \pi/2)^2$ and $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 satisfying some additional requirements. By using the symmetry properties of the problem $(\mathcal{P})$ and classical results from number theory, we show that the numbers $\alpha_\varepsilon(L)$ of all distinct nontrivial solutions (u, μ) of $(\mathcal{P})$ such that $\|u\| < \varepsilon$, for $\varepsilon > 0$, where $0 < \mu < L+1$, satisfy the following inequality

$$\liminf_{\varepsilon \to 0} \alpha_\varepsilon(L) \geq \frac{5}{8}\pi L + O(\sqrt{L}) \qquad \text{as} \quad L \to \infty.$$

Keywords: Boundary value problem, variational problem with symmetries, bifurcation point, asymptotic behaviour.

1. Introduction

This work is a result of our investigation into a possible use of number theory to estimate the multiplicity of solutions of variational problems with symmetries. We consider a nonlinear, parametric Dirichlet problem with the group of symmetries D_4, i.e. the group of symmetries a square. In our earlier paper (see [8]), we have proved that near a value of parameter λ, at which a bifurcation of solutions occurs, the number of solutions is estimated from below by the cardinality of a subset of the integral lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ contained in the disc of radius λ. By using

---

1991 Mathematics Subject Classification: Primary 58E09, Secondary 35J50, 11Lxx
† Research supported by NSERC Grant G121210399
‡ Research supported by the A. von Humboldt fellowship.
* Research supported by KBN grant nr 2 PO3A 03315.

some elementary, and well-known, number theory argument we can evaluate an asymptotic estimate of this cardinality, and consequently estimate from below the number of small amplitude solutions of the studied problem.

The authors are aware that presented here result is not in the most general formulation and it should be considered as an example for such application number theory. However, such generalizations could be obtained using the results presented in [8] for variational problem with the symmetry group G being the isometries of n-cube, where n is an arbitrary natural number. In spite of the fact that there are in the literature (see [1] for an extended list of references) many interesting papers on the multiplicity of solutions to nonlinear problems with symmetries, an asymptotic (with respect to the parameter) formula for the number of solutions has not been studied before. In this paper we would like to demonstrate a possibility of such an approach to studying the multiplicity of solutions for nonlinear problems with symmetry.

2. Main Result

We are interested in the existence of non-trivial solutions to the following boundary value problem:

$$\begin{cases} -\Delta u = \mu \cdot f(u) & \text{on} \quad \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{$\mathcal{P}$}$$

where $\Omega \subset \mathbb{R}^2$ is the square $(-\pi/2, \pi/2)^2$ and $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $f(0) = 0$, $f'(0) = 1$ and

$$|f'(t)| \leq a_1 + a_2|t|^p, \qquad p \geq 1. \tag{2.1}$$

The assumption (2.1) implies that the problem $(\mathcal{P})$ can be expressed as a variational problem, namely $(\mathcal{P})$ is equivalent to the problem of existence of non-zero critical points for the functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) := \frac{1}{2} \int_\Omega \|\nabla u\|^2 - \mu \int_\Omega F(u);$$

where $F(s) = \int_0^s f(t)dt$.

It can be shown that the eigenvalues μ of the Laplace operator $-\Delta$ are the bifurcation points for $(\mathcal{P})$ but the fact that the functional I is invariant with respect to the symmetric group of the square Ω implies that the number of bifurcating branches of solutions can be quite large.

In the paper [8], which generalizes some earlier work of Ekeland and Lasry (cf. [4]), we studied that problem and we obtained lower estimates on the number of the bifurcating branches of solutions from an eigenvalue μ of the Laplacian. This result, which improves some previous attempts in this direction (cf. [2]), can be described as follows:

Let us denote by $|\mathcal{A}_\mu|$ the number of elements of the set

$$\mathcal{A}_\mu = \{(m, n) \in \mathbb{N}^2 : n^2 + m^2 = \mu\}$$

where $\mathbb{N} = \{1, 2, \ldots\}$.

Theorem 1 (cf. [8], Thm. (3.3). *Let $\mu \in \mathbb{N}$ be a number such that $\mathcal{A}_\mu \neq \emptyset$ and suppose that $\varepsilon > 0$ is a sufficiently small real number. Then we have*

i) *If $\mu \equiv 1 \pmod 2$ then μ is a bifurcation point for $(\mathcal{P})$ and the equation $(\mathcal{P})$ has at least $4 \cdot |\mathcal{A}_\mu|$ distinct solutions u such that $\|u\| = \varepsilon$.*

ii) *If $\mu \equiv 0 \pmod 2$ and $\mathcal{A}_\mu$ does not contain a pair of two odd numbers then μ is a bifurcation point for $(\mathcal{P})$ and the equation $(\mathcal{P})$ has at least $2 \cdot |\mathcal{A}_\mu|$ distinct solutions u such that $\|u\| = \varepsilon$.*

Moreover all these solutions are not invariant with respect to the group of symmetry of square.

Using this result we establish an asymptotic lower bound for the number of bifurcating nontrivial solutions to the problem $(\mathcal{P})$. More precisely, let $L > 0$ and $\varepsilon > 0$. By $\alpha_\varepsilon(L)$ we denote the number of all distinct nontrivial solutions (u, μ) the problem $(\mathcal{P})$ such that $\|u\| < \varepsilon$ and $0 < \mu < L + 1$ and we put $\alpha(L) = \liminf_{\varepsilon \to 0} \alpha_\varepsilon(L)$.

Recall that a function $u(x)$, $u : \Omega \to \mathbb{R}$ is not invariant with respect to the group G of symmetries of Ω if there exists $g \in G$ such that $u(gx) \neq u(x)$.

Our main result is the following

Theorem 2. *Under the above assumptions we have the following estimation*

$$\alpha(L) \geq \frac{5}{8}\pi L + O(\sqrt{L}) \qquad \text{as} \quad L \to \infty.$$

Moreover we assess the number of solutions that are not invariant with respect to the group of symmetry of square.

Before we proceed with the proof of Theorem 2 let us introduce the following arithmetical function $r : \mathbb{N} \to \mathbb{N}$

$$r(n) = \sum_{\substack{n=a^2+b^2 \\ a,b \in \mathbb{Z}}} 1$$

expressing the number of representations of an integer $n \geq 1$ as a sum of two integral squares. Put

$$R(N) = \sum_{n=0}^{N} r(n); \quad r(0) = 1.$$

$R(N)$ can be interpreted as the number of lattice points inside and on the circumference of the circle $x^2 + y^2 = N$ and is approximately equal to the area of the circle. More precisely the following classical result of Gauss (cf. [3]) is true:

$$R(N) = \pi N + O(\sqrt{N}). \tag{2.2}$$

Since the set $\mathcal{A}_\mu$ consists only of pairs (m, n) of positive integers we need to restrict our consideration to the arithmetical functions

$$r_+(n) = \sum_{\substack{n=a^2+b^2 \\ a,b \in \mathbb{N}}} 1 \quad \text{and} \quad R_+(N) = \sum_{n=1}^{N} r_+(n).$$

The function $R_+(N)$ expresses the number of lattice points from the first quadrant that are inside and on the circumference of the circle $x^2+y^2 = N$. Using this observation we have that

$$R(N) = 4\,R_+(N) + 4\,[\sqrt{N}] + 1. \tag{2.3}$$

More precisely, if we put $R(N) = \pi N + R_1$, where R_1 denotes the remainder, then $R_+(N) = (\pi/4)\,N - \sqrt{N} + R_1^1$ and the asymptotic behavior of the remainder R_1^1 is the same as those of $\frac{1}{4}\,R_1$.

It should be mentioned that there are many results describing the remainder R_1. For example, one should mention the old one by W. Sierpiński (1906) that $R_1 = O(N^{1/3}\log N)$ and the sharpest known by Iwaniec and Mozzochi ([6]) that $R_1 = O(N^{\frac{7}{22}+\varepsilon})$. The above circle problem and so called the problem of divisors are related each to the other. For a deeper discussion of the problem of divisors we refer the reader to [7] (cf. [7] Ch 13).

Remark 3. Suppose that $\mu \equiv 0 \,(\mathrm{mod}\ 4)$ then it is easy to observe that there is no representation of μ as a sum of squares of two odd integers, i.e. μ is never equal to $(2k+1)^2 + (2\ell+1)^2$ for any $k, \ell \in \mathbb{Z}$. This implies that if $\mathcal{A}_\mu \neq \emptyset$ then the set $\mathcal{A}_\mu$ satisfies the requirement formulated in ii) of Theorem 1.

Proof of Theorem 2. Let $L > 0$ be a real number. We put $R(L) := R([L])$ and $R_+(L) := R([L])$. Let us remark that

$$R_+(L) = \sum_{\substack{n\equiv 0(\mathrm{mod}\,4)\\ n\le L}} r_+(n) + \sum_{\substack{n\equiv 1(\mathrm{mod}\,4)\\ n\le L}} r_+(n) + \sum_{\substack{n\equiv 2(\mathrm{mod}\,4)\\ n\le L}} r_+(n) + \sum_{\substack{n\equiv 3(\mathrm{mod}\,4)\\ n\le L}} r_+(n).$$

We observe that if $n \equiv 0\,(\mathrm{mod}\,2)$ then $r(n) = r(n/2)$. Indeed; it is well known (see [9], [5]) that $r(n) = 4(A-B)$, where A is the number of positive divisors of n of the form $4k+1$ and B is the number of positive divisors of n of the form $4k+3$. It is clear that the number $n/2$ has exactly the same divisors of these types, thus $r(n) = r(n/2)$.

It follows from Theorem 1 and Remark 3 that for all $\varepsilon > 0$

$$\begin{aligned}\alpha_\varepsilon(L) &\ge 4 \sum_{\substack{\mu\equiv 1(\mathrm{mod}\,2)\\ \mu\le L}} |\mathcal{A}_\mu| + 2 \sum_{\substack{\mu\equiv 0(\mathrm{mod}\,4)\\ \mu\le L}} |\mathcal{A}_\mu| \\ &= 4 \sum_{\substack{\mu\equiv 1(\mathrm{mod}\,2)\\ \mu\le L}} r_+(n) + 2 \sum_{\substack{\mu\equiv 0(\mathrm{mod}\,4)\\ \mu\le L}} |\mathcal{A}_\mu|.\end{aligned} \tag{2.4}$$

On the other hand

$$\begin{aligned}R(L) &= \sum_{\substack{n\equiv 1(\mathrm{mod}\,2)\\ n\le L}} r(n) + \sum_{\substack{n\equiv 0(\mathrm{mod}\,2)\\ n\le L}} r(n) \\ &= \sum_{\substack{n\equiv 1(\mathrm{mod}\,2)\\ n\le L}} r(n) + \sum_{n\le\frac{L}{2}} r(n) = \sum_{\substack{n\equiv 1(\mathrm{mod}\,2)\\ n\le L}} r(n) + R\left(\frac{L}{2}\right).\end{aligned}$$

By (2.2) we obtain that

$$\sum_{\substack{n\equiv 1 (\mathrm{mod}\, 2)\\ n\leq L}} r(n) = \pi L + O(\sqrt{L}\,) - \frac{1}{2}\,\pi L - O\Big(\sqrt{\frac{L}{2}}\Big) = \frac{1}{2}\,\pi L + O(\sqrt{L}\,)$$

and it implies that

$$\sum_{\substack{n\equiv 1 (\mathrm{mod}\, 2)\\ n\leq L}} r_+(n) = \frac{1}{8}\,\pi L + O(\sqrt{L}\,). \tag{2.5}$$

We obtain in a similar way that

$$\begin{aligned}\sum_{\substack{n\equiv 0 (\mathrm{mod}\, 2)\\ n\leq L}} r(n) &= \sum_{\substack{n\equiv 0 (\mathrm{mod}\, 4)\\ n\leq L}} r(n) + \sum_{\substack{n\equiv 2 (\mathrm{mod}\, 4)\\ n\leq L}} r(n)\\ &= \sum_{\substack{n\equiv 0 (\mathrm{mod}\, 2)\\ n\leq L}} r(n) + \sum_{\substack{n\equiv 1 (\mathrm{mod}\, 2)\\ n\leq \frac{L}{2}}} r(n)\end{aligned}$$

thus

$$\sum_{\substack{n\equiv 0 (\mathrm{mod}\, 4)\\ n\leq L}} r(n) = \sum_{\substack{n\equiv 0 (\mathrm{mod}\, 2)\\ n\leq \frac{L}{2}}} r(n) = \frac{1}{2}\cdot\pi\cdot\frac{L}{2} + O\Big(\sqrt{\frac{L}{2}}\Big) = \frac{1}{4}\cdot\pi\cdot L + O(\sqrt{L}\,).$$

This leads to

$$\sum_{\substack{n\equiv 0 (\mathrm{mod}\, 4)\\ n\leq L}} r_+(n) = \frac{1}{16}\,\pi L + O(\sqrt{L}\,). \tag{2.6}$$

Now, we substitute (2.5) and (2.6) in (2.4) and we obtain

$$\alpha_\varepsilon \geq 4\cdot\frac{1}{8}\,\pi L + O(\sqrt{L}\,) + 2\cdot\frac{1}{16}\,\pi L + O(\sqrt{L}\,) = \frac{5}{8}\,\pi L + O(\sqrt{L}\,)$$

and therefore

$$\alpha(L) \geq \frac{5}{8}\,\pi L + O(\sqrt{L}\,).$$

This completes the proof. ∎

Remark 4. Using the mentioned approach of [8] and applying directly the result of M. Pinsky (cf. [10], [11]) about the spectrum and eigenspaces of the Laplacian on the equilateral triangle one get the following lower estimate of the distinct nontrivial solutions (u,μ) of $(\mathcal{P})$ in that case.

Remark 5. It is worth of pointing out that the asymptotic estimate of Theorem 2 is not an asymptotic equality in general. It is not an asymptotic equality even if we count the solutions that are not invariant with respect to $G = D_4$, or equivalently do not satisfy $u \notin C^1(\Omega)^G$. The simplest example is the function $f(u) := u$ that fulfills the condition (2.1). Then for a given eigenvalue λ the solutions of $(\mathcal{P})$ are exactly the eigenfunctions of the Laplacian corresponding to λ, which form a subspace, thus an infinite set containing vectors of arbitrary small norms.

References

[1] T. Bartsch *Topological Methods for Variational Problems with symmetries*, Springer-Verlag Lecture Notes in Mathematics, No 1560, 1993.

[2] V. Benci, F. Pacella, *Morse Theory for Symmetric Functionals on the Sphere and Application to a Bifurcation Problem*, Nonlinear Analysis TMA **9**(8) 1985.

[3] K. Chandrasekharan, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1968.

[4] I. Ekeland, J. M. Lasry, *Sur de points critiques de fonctions invariantes par des groupes*, C. R. Acad. Sci. Paris, **282**, Série A, 1976, 841–844

[5] E. Groswald, *Representations of integers as sum of squares*, Springer, 1985.

[6] H. Iwaniec, C. J. Mozzochi, *On the divisor and circle problem*, J. Number Theory, **29** no. 1, 1988, 60–93.

[7] A. Ivič a, *The Riemann zeta function*, Wiley & Sons, 1985.

[8] W. Krawcewicz, W. Marzantowicz, *Lusternik-Schnirelman method for functionals invariant with respect to a finite group action*, J. Diff. Eqn. **85** 1990, 105–124

[9] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, PWN-Polish Scientific Publishers, Warszawa, 1974.

[10] M. Pinsky *The eigenvalues of an equilateral triangle*, SIAM J. Math. Anal. **11** no. 5, 1980, 819–827.

[11] M. Pinsky, *Completeness of the eigenfunctions of the equilateral triangle*, SIAM J. Math. Anal. **16** no. 4, 1985, 848–851.

Address: W. Krawcewicz, Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1;
W. Marzantowicz, Faculty of Mathematics & Computer Sciences, Adam Mickiewicz University, 60-769 Poznań, Poland.
E-mail: `wkrawcew@math.ualberta.ca; marzan@main.amu.edu.pl`
Received: 21 Dec 1999