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To Professor Włodzimierz Staś on his 75th birthday

A SIMPLE DERIVATION OF $\zeta(1-K) = -B_K/K$

M. Ram Murty* and Marilyn Reece

1. Introduction

Given a Dirichlet series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which extends to a meromorphic function (say) for all $s \in \mathbb{C}$, we would like to investigate, what we call the Hurwitz series attached to F, namely,

$$F(s,x) = \sum_{n=1}^{\infty} \frac{a_n}{(n+x)^s}$$

for $0 \le x < 1$. In this paper, we derive a meromorphic continuation of $F(s,x)$ by an exceedingly simple method. As a consequence, we can deduce several striking results. For example, if K is an algebraic number field, and

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s},$$

where the summation is over all integral ideals, then the Hurwitz series attached to $\zeta_K(s)$ is

$$\zeta_K(s,x) = \sum_{\mathfrak{a}} \frac{1}{(N(\mathfrak{a})+x)^s}.$$

We will show that $\zeta_K(s,x)$ extends for all $s \in \mathbb{C}$ and is analytic everywhere except at $s = 1$ where it has a simple pole. It turns out that the special values $\zeta_K(1-k,x)$ can be written as a polynomial in x, thus giving a generalization of the Bernoulli polynomials (in the case K is totally real).

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This work had its genesis in looking for simple proofs of the well-known formula $\zeta(1-k) = -B_k/k$, where ζ denotes the Riemann zeta function and B_k is the k-th Bernoulli number. We present two such proofs below. The first proof has direct relevance to what was discussed above. The second, is a justification of a 'naive' argument, and is of independent interest. We include it here for its novelty. At the end, we discuss the merits and demerits of each of the methods.

Recall that the Riemann ζ-function, originally defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s};$$

for $Re(s) > 1$, extends to an analytic function for all $s \in \mathbb{C}$, apart from $s = 1$, where it has a simple pole. This is easily seen as follows. By partial summation,

$$\zeta(s) = s\int_1^{\infty} \frac{[x]}{x^{s+1}}dx = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx,$$

where $\{x\}$ denotes the fractional part of x. This gives the analytic continuation of $\zeta(s)$ for $Re(s) > 0$. We can now proceed inductively. Writing

$$\int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{x-n}{x^{s+1}}dx = \sum_{n=1}^{\infty} \int_0^1 \frac{udu}{(u+n)^{s+1}}$$

and integrating the last integral by parts, we get

$$\sum_{n=1}^{\infty} \int_0^1 \frac{udu}{(u+n)^{s+1}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{(n+1)^{s+1}} + \frac{(s+1)}{2}\int_1^{\infty} \frac{\{x\}^2}{x^{s+2}}dx$$

and the latter integral converges for $Re(s) > -1$. That is,

$$\zeta(s) = \frac{s}{s-1} - \frac{s}{2}(\zeta(s+1)-1) - \frac{s(s+1)}{2}\int_1^{\infty} \frac{\{x\}^2}{x^{s+2}}dx$$

from which we infer that $\zeta(0) = -1/2$. Thus, inductively, we deduce

$$\begin{aligned} \zeta(s) = 1 + \frac{1}{s-1} &- \sum_{r=1}^{m} \frac{s(s+1)\cdots(s+r-1)}{(r+1)!}(\zeta(s+r)-1) \\ &- \frac{s(s+1)\cdots(s+m)}{(m+1)!}\sum_{n=1}^{\infty}\int_0^1 \frac{u^{m+1}du}{(u+n)^{s+m+1}} \end{aligned} \tag{1.1}$$

and the infinite sum on the right hand side converges for $Re(s) > -m$. This derivation, though not well-known, is not new. For example, it is found in [1]. However, the idea of deriving the special values of $\zeta(1-k)$ from it, seems to have not been noticed before. Indeed, if in (1.1), we put $s = 1-m$, and note that for $r = m$ that $\zeta(s+m)$ has a simple pole at $s = 1-m$, we obtain the recurrence

$$\zeta(1-m) \\ = 1 - \frac{1}{m} + \frac{(-1)^m}{m(m+1)} - \sum_{r=1}^{m-1}(-1)^r\binom{m-1}{r}\frac{1}{r+1}\left(\zeta(1-m+r)-1\right). \tag{1.2}$$

By integrating the expression

$$(1-x)^{m-1} = \sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^r x^r \tag{1.3}$$

we deduce

$$\frac{1-(1-x)^m}{m} = \sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^r\frac{x^{r+1}}{r+1}. \tag{1.4}$$

Putting $x = 1$ gives

$$\frac{1}{m} = \sum_{r=0}^{m-1}\binom{m-1}{r}\frac{(-1)^r}{r+1} \tag{1.5}$$

so that we obtain:

Theorem 1.1. *For positive integers m,*

$$m\zeta(1-m) = \frac{(-1)^m}{(m+1)} - m\sum_{r=1}^{m-1}(-1)^r\binom{m-1}{r}\frac{1}{r+1}\zeta(1-m+r). \tag{1.6}$$

Recall that the generating function for the Bernoulli numbers is given by

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty}\frac{B_k t^k}{k!} \tag{1.7}$$

from which we easily deduce the recurrence

$$\sum_{k=0}^{n}\binom{n}{k}\frac{B_{n-k}}{k+1} = 0 \tag{1.8}$$

for $n \geq 1$. Our goal now is to derive the celebrated formula in:

Theorem 1.2.

$$\zeta(1-k) = -\frac{B_k}{k} \tag{1.9}$$

by induction using (1.6). Since

$$\frac{t}{e^t-1} + \frac{t}{2}$$

is an even function, the Bernoulli numbers for odd subscripts ≥ 3 vanish, and we can rewrite (1.9) as

$$\zeta(1-k) = (-1)^{k-1}\frac{B_k}{k}. \tag{1.10}$$

Then, by induction, we see that the right hand side of (1.6) becomes

$$\frac{(-1)^m}{m+1} + (-1)^m \sum_{r=1}^{m-2} \binom{m}{r} \frac{B_{m-r}}{r+1} - (-1)^{m-1}\zeta(0). \tag{1.11}$$

From (1.8), it is immediate that this reduces to

$$\frac{(-1)^m}{m+1} - (-1)^m \left(B_m + B_1 + \frac{B_0}{m+1}\right) - \frac{(-1)^m}{2} = (-1)^{m-1}B_m \tag{1.12}$$

by virtue of the fact that $\zeta(0) = -\frac{1}{2}$. The result is now immediate.

2. Hurwitz zeta functions

The method used for the Riemann zeta function easily generalizes to give the analytic continuation of the Hurwitz zeta function. This is partly due to the fact that the Euler product and functional equation of $\zeta(s)$ were never used. In this aspect, the method is versatile.

Fix $0 < a \leq 1$. Recall that the Hurwitz zeta function $\zeta(s,a)$ is defined by the series

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

which converges absolutely for $Re(s) > 1$. For $Re(s) > 1$, we have by partial summation

$$\zeta(s,a) = s\int_0^\infty \frac{[x]}{(x+a)^{s+1}}dx \tag{2.1}$$

and as before, integration by parts leads to the formula

$$\begin{aligned}\zeta(s,a) = \frac{a^{1-s}}{s-1} &- \sum_{r=1}^{m} \frac{s(s+1)\cdots(s+r-1)}{(r+1)!}\left(\zeta(s+r,a) - a^{-s-r}\right)\\ &+ \frac{s(s+1)\cdots(s+m)}{(m+1)!}\sum_{n=1}^{\infty}\int_0^1 \frac{u^{m+1}du}{(u+n+a)^{s+m+1}}.\end{aligned} \tag{2.2}$$

Again, the infinite sum on the right hand side converges for $Re(s) > -m$ and we can easily derive the following:

Theorem 2.1. *$(s-1)\zeta(s,x)$ extends to an entire function in the complex plane. Moreover, if we let $B_n(x)$ denote the k-th Bernoulli polynomial given by*

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}$$

then,

$$\zeta(1-k,a) = -\frac{B_k(a)}{k}.$$

The method of the previous section carries over mutatis mutandis and we leave the proof to the reader and not derive Theorem 3 in that way. Rather, we will give a swifter derivation of the above theorem which generalizes to a wider context. But before we do that, let us note that from the above, we have the immediate:

Corollary 2.2. *Let χ be a primitive character mod q and $L(s,\chi)$ be the classical Dirichlet L-function attached to χ:*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $Re(s) > 1$. Then $L(s,\chi)$ extends to an entire function. Moreover, if we define the generalized Bernoulli number $B_{n,\chi}$ as

$$B_{n,\chi} = q^{n-1} \sum_{a=1}^{q} \chi(a) B_n(a/q),$$

then, $L(1-n,\chi) = -B_{n,\chi}/n$.

Indeed, we can write $L(s,\chi)$ as

$$q^{-s} \sum_{a=1}^{q} \chi(a)\zeta(s,a/q)$$

and as

$$\sum_{a=1}^{q} \chi(a) = 0$$

the pole disappears and we deduce that $L(s,\chi)$ extends to an entire function.

In a later section, we will show yet another way of deriving both of these results, the method, though not simpler, is instructive and is of intrinsic interest.

Notice that

$$-\frac{1}{x^s} + \zeta(s,x) - \zeta(s) = \sum_{n=1}^{\infty} \left\{ \frac{1}{(n+x)^s} - \frac{1}{n^s} \right\}.$$

Writing the summand as

$$\frac{1}{n^s}\left(\left(1+\frac{x}{n}\right)^{-s} - 1 \right)$$

and using the binomial theorem, we obtain

Theorem 2.3. *For $0 < x < 1$, we have*

$$-\frac{1}{x^s} + \zeta(s,x) - \zeta(s) = \sum_{r=1}^{\infty} \binom{-s}{r} \zeta(s+r) x^r. \tag{2.3}$$

The advantage of (2.3) is that it gives the analytic continuation of $\zeta(s,x)$ from a knowledge of the same for $\zeta(s)$. Moreover, Theorem 2.1 is immediate since putting $s = 1-k$ reduces the sum above into a finite sum and the right hand side becomes $-B_k(x)/k$.

The idea is worth considerable generalization. Suppose

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

extends as a meromorphic function to the entire complex plane. Then, we can define for $0 \le x < 1$, the associated Hurwitz series as

$$F(s,x) = \sum_{n=1}^{\infty} \frac{a_n}{(n+x)^s}$$

and deduce

Theorem 2.4. *$F(s,x)$ extends as a meromorphic function for all s in the complex plane and we have*

$$F(s,x) = F(s) + \sum_{r=1}^{\infty} \binom{-s}{r} F(s+r) x^r.$$

In particular,

$$F(1-k,x) = F(1-k) + \sum_{r=1}^{k-1} \binom{k-1}{r} F(1-k+r) x^r.$$

This simple theorem has remarkable consequences. Firstly, it says that if the special values of $F(1-k)$ are all rational numbers, then so are $F(1-k,x)$ for x rational. In the special case that $F(s)$ is the Dedekind zeta function of a totally real number field K, we have, by a result of Siegel [5] that $\zeta_K(1-k) \in \mathbb{Q}$ for k positive. We deduce immediately that $\zeta_K(1-k,x)$ is also rational for x rational. In fact, we have:

Theorem 2.5. *Let K be a totally real algebraic number field and $\zeta_K(s,x)$ the Hurwitz zeta function attached to $\zeta_K(s)$. Then, for positive integers k, $\zeta_K(1-k,x) \in \mathbb{Q}[x]$.*

Let us remark that if K is not totally real, then $\zeta_K(1-k) = 0$ for all positive integers $k \ge 2$. In addition $\zeta_K(0) = 0$ unless K is an imaginary quadratic field.

in which case it is $-h/w$ where h is the class number of K and w is the number of roots of unity in K.

3. Another derivation

There is a "naive" way of deriving Theorem 1.2 which has very much the spirit of Euler. Observe that, ignoring questions of convergence, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k \zeta(-k) x^k}{k!} = \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} n^k \right) \frac{(-1)^k x^k}{k!}$$
$$= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-nx)^k}{k!} \right)$$

so that we obtain the identity

$$\sum_{k=0}^{\infty} \frac{(-1)^k \zeta(-k) x^k}{k!} = \sum_{n=1}^{\infty} e^{-nx}. \tag{3.1}$$

As the right hand side of (3.1) converges to

$$\frac{1}{e^x - 1},$$

we immediately deduce

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}$$

which is essentially (1.9). Observe that (3.1) is actually false since the right hand side has a pole at $x = 0$ and the left hand side is regular there. However, we will indicate how a modified form of (3.1) can be rigorously deduced by considering the vertical line integral

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} x^{-s} \Gamma(s) \zeta(s) ds.$$

The justification comes from the use of Stirling's formula (which we recall below) and contour integration. Again, the method also works for deriving Corollary 2.2 for classical Dirichlet L-functions and Theorem 2.1 for Hurwitz ζ-functions.

Recall that $\zeta(s)$ satisfies the functional equation

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \tag{3.2}$$

where $\Gamma(s)$ denotes the Γ-function. Since the Γ-function is regular everywhere except at $s = 0, -1, -2, \ldots$, where it has simple poles, we see that $\zeta(s)$ has 'trivial zeros' at $s = -2k$ for k a positive integer. One can use the functional equation and Euler's formula for $\zeta(2k)$ to deduce (1.9).

In 1734, Euler derived the beautiful formula

$$\zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}B_{2k}}{(2k)!}, \tag{3.3}$$

by taking the logarithmic derivatives of the identity

$$\sin(\pi t) = \pi t\prod_{n=1}^{\infty}\left(1-\frac{t^2}{n^2}\right) \tag{3.4}$$

to deduce

$$\frac{\pi t\cos \pi t}{\sin \pi t} = 1 - 2\sum_{k=1}^{\infty}\zeta(2k)t^{2k}. \tag{3.5}$$

Observing that the left hand side of (3.5) is

$$\frac{\pi t i(e^{i\pi t}+e^{-i\pi t})}{e^{i\pi t}-e^{-i\pi t}} = \pi t i + \frac{2\pi i t}{e^{2\pi i t}-1} \tag{3.6}$$

and using (1.7) to expand the right hand side of (3.6), we easily deduce (3.3).

It seems that Euler (see Ayoub [2]) preferred this derivation of (3.3) to his original one which employed the use of divergent series. Indeed, Euler's 'divergent series' proof considers the related series

$$\sum_{n=1}^{\infty}(-1)^n n^k$$

and shows that it is Abel-summable. That is, he shows the limit

$$\lim_{x\to 1^-}\sum_{n=1}^{\infty}(-1)^n n^k x^n$$

exits by considering

$$f(x) = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$$

and noticing that the limit in question is

$$\left[\left(x\frac{d}{dx}\right)^k f(x)\right]_{x=1}.$$

Using this derivation, he conjectured in 1749 a relationship between $\zeta(1-s)$ and $\zeta(s)$, a full century before Riemann actually proves the functional equation in 1859. Weil suggests that Riemann was aware of Euler's work (see [2, p. 1083]).

Given the functional equation (3.2), it is then an easy matter to deduce (1.9) from (3.3) simply by using the fact

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

and the functional equation $s\Gamma(s) = \Gamma(s+1)$, as well as the identity $\Gamma(1/2) = \sqrt{\pi}$.

We recall Stirling's formula (see Titchmarsh [7, p. 151] for example):

$$\log \Gamma(z) = (z - 1/2)\log z - z + \frac{1}{2}\log 2\pi + O(1/|z|) \tag{3.7}$$

uniformly for z satisfying $-\pi + \delta \leq \arg z \leq \pi - \delta$ for any fixed δ with $0 < \delta < \pi$. As a consequence, we have

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi} e^{-\frac{\pi}{2}|t|} |t|^{\sigma - \frac{1}{2}}, \tag{3.8}$$

for any fixed real value σ, and $|t| \to \infty$. Let us recall that $\Gamma(s)$ has simple poles at $s = -k$ with k a non-negative integer. Moreover, the residue of $\Gamma(s)$ at $s = -k$ is $(-1)^k/k!$. Using this, as well as the calculus of residues, one derives the familiar formula

$$e^{-x} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^{-s}\Gamma(s)ds.$$

Thus, if

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is a Dirichlet series absolutely convergent in $Re(s) > 1$, we have

$$\sum_{n=1}^{\infty} a_n e^{-nx} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^{-s}\Gamma(s)F(s)ds. \tag{3.9}$$

We also need the fact that $\zeta(\sigma + it)$ has polynomial growth in $|t|$ for fixed σ. That is, we need some bound of the form

$$\zeta(\sigma + it) = O(|t|^A)$$

for some constant A (that may depend on σ) as $|t| \to \infty$. This can be deduced in one of two ways. One could use the Phragmen-Lindelöf theorem and the functional equation, or one can proceed more elementarily by using (1.1). Thus, in this derivation, we can dispense with the use of the functional equation by using (1.1) to deduce the necessary polynomial growth to justify the moving of the contours below.

By (3.6), we have

$$\sum_{n=1}^{\infty} e^{-nx} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^{-s}\Gamma(s)\zeta(s)ds.$$

The left hand side is convergent for $e^x > 1$, and easily summed to be

$$\frac{1}{e^x - 1}.$$

Since $\zeta(s)$ has polynomial growth in $|t|$ for any fixed σ, we can move the line of integration to the left and pick up the contribution first from the pole at $s = 1$ of $\zeta(s)$ and then from the poles at $s = -k$, for $k = 0, 1, 2, \ldots$ of $\Gamma(s)$. We obtain

$$\sum_{n=1}^{\infty} e^{-nx} = x^{-1} + \sum_{k=0}^{\infty} \frac{(-1)^k x^k \zeta(-k)}{k!}. \tag{3.10}$$

Again by the functional equation and Stirling's formula,

$$|\zeta(-k)| \leq Ce^{k \log k},$$

for some constant C.

Also, from (3.7), we have $\log k! = k \log k - k + O(\log k)$, we deduce that (3.10) converges for $0 < x < \delta$ with δ sufficiently small. Multiplying through (3.10) by x gives

$$\frac{x}{e^x - 1} = 1 + \sum_{k=0}^{\infty} \frac{(-1)^k \zeta(-k) x^k}{k!}$$

and we immediately deduce (1.9).

A similar method can be used for Dirichlet L-functions. If χ is a primitive character $(\bmod\, q)$, the Dirichlet L-series $L(s, \chi)$ defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

converges for $Re(s) > 0$ (if χ is not the trivial character) and extends to an entire function. Moreover, it satisfies a functional equation similar to (3.2) (see Davenport [3]).

By the partial summation method outlined above for $\zeta(s)$, one can deduce that $L(s, \chi)$ has polynomial growth in $|t|$ for any fixed σ. By (3.9), we have

$$\sum_{n=1}^{\infty} \chi(n) e^{-nx} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} x^{-s} \Gamma(s) L(s, \chi) ds. \tag{3.11}$$

Moving the line of integration to the left, and picking up the contribution from the poles of $\Gamma(s)$, we deduce that the right hand side of the above equation is

$$\sum_{k=0}^{\infty} \frac{(-1)^k L(-k, \chi) x^k}{k!}.$$

The left hand side of (3.11) can be evaluated as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty} \chi(n) e^{-nx} &= \sum_{b \pmod q} \chi(b) \Bigg(\sum_{n \equiv b \pmod q} e^{-nx} \Bigg) \\
&= \sum_{b=1}^{q} \chi(b) \left(\sum_{r=0}^{\infty} e^{-(qr+b)x} \right) \\
&= \sum_{b=1}^{q} \chi(b) \frac{e^{-bx}}{1-e^{-qx}} \\
&= \sum_{b=1}^{q} \chi(b) \frac{e^{(q-b)x}}{e^{qx}-1}.
\end{aligned}$$

Recall that the Bernoulli polynomials $B_k(x)$ defined in Theorem 2.1 are given by the generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k(x)t^k}{k!}.$$

Thus, writing

$$\frac{e^{(q-b)x}}{e^{qx}-1} = \frac{e^{(1-b/q)xq}}{e^{qx}-1},$$

we see that

$$\sum_{n=1}^{\infty} \chi(n) e^{-nx} = \sum_{k=0}^{\infty} \left(\sum_{b=1}^{q} \chi(b) B_k \left(1 - \frac{b}{q}\right) \right) \frac{(qx)^{k-1}}{k!}.$$

Since

$$\sum_{b=1}^{q} \chi(b) = 0,$$

the polar term then disappears, and we deduce

$$L(-k,\chi) = \frac{(-q)^k}{k+1} \sum_{b=1}^{q} \chi(b) B_{k+1}\left(1 - \frac{b}{q}\right). \tag{3.12}$$

Note that

$$\frac{te^{(1-x)t}}{e^t-1} = \frac{(-t)e^{x(-t)}}{e^{-t}-1}$$

so that

$$B_k(1-x) = (-1)^k B_k(x). \tag{3.13}$$

Thus, (3.12) becomes

$$L(1-k,\chi) = -\frac{B_{k,\chi}}{k} \tag{3.14}$$

where

$$B_{k,\chi} = q^{k-1}\sum_{b=1}^{q}\chi(b)B_k\left(\frac{b}{q}\right).$$

This method of contour integration can also be applied to the Hurwitz zeta function. This function has a functional equation of the following form: for $\sigma > 1$,

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\left\{e^{-\pi i s/2}F(a,s) + e^{\pi i s/2}F(-a,s)\right\} \tag{3.15}$$

where

$$F(a,s) = \sum_{n=1}^{\infty}\frac{e^{2\pi i n a}}{n^s}.$$

If $a \neq 1$, this representation is also valid for $\sigma > 0$. (See Apostol [1, p. 257])

We don't need the functional equation (3.15) to derive our result. Again, all we need is to know that it has polynomial growth in any bounded vertical strips. This can be deduced from (2.2) as indicated earlier for the Riemann ζ case. In addition, one has $|\zeta(-k,a)| \leq Ce^{k\log k}$.

Thus, we deduce as before

$$\sum_{n=0}^{\infty} e^{-(n+a)x} = x^{-1} + \sum_{k=0}^{\infty}\frac{(-1)^k\zeta(-k,a)x^k}{k!}.$$

The left hand side is

$$\frac{e^{(1-a)x}}{e^x - 1} = \sum_{k=0}^{\infty}\frac{B_k(1-a)x^{k-1}}{k!}$$

so that comparing coefficients, we obtain

$$\zeta(1-k,a) = -\frac{B_k(a)}{k}, \tag{3.16}$$

by an application of (3.13).

4. Concluding remarks

Certainly the treatment given in sections 1 and 2 is elementary requiring only a knowledge of basic calculus. It seems to not have been noticed before. Most books (see for example [8]) that derive (1.9) do so using elaborate contour integration. In [1] for example, (1.1) is given but its application to deriving (1.9) is unnoticed and the author chooses the method of contour integration. There is also a paper of Stark [6] and Ramaswami [4] that overlap in parts with section 1 in their treatment of the zeta function. Neither of these papers discuss the applicability of the method to general Dirichlet series. On the other hand, the treatment in section 3, though it demands a knowledge of contour integration and some basic knowledge of the growth of the Γ-function, also seems to be new.

However, we must remark that this is essentially the limitation of this method. Indeed, for the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k F(-k) x^k}{k!}$$

to converge, we are forced to take a function F satisfying

$$|F(-k)| \leq Ce^{k \log k}.$$

Such an estimate is not satisfied for Dirichlet series attached to modular forms (or L-functions attached to higher GL_n for that matter).

As a historical note, we remark that while the analytic continuation and functional equation of the ζ-function were proved by Riemann in 1860, it was not until 1882 that Hurwitz introduced his ζ function and proved the analogue of Riemann's theorem. His goal was to extend $L(s,\chi)$ as an entire function and derive its functional equation.

The interest in (1.9), and its analogues for the Dirichlet L-functions and Hurwitz zeta functions lies in the key role they play in the p-adic continuations of the Riemann ζ-function, Dirichlet L-functions and the Hurwitz ζ-functions, respectively.

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Addresses: M. Ram Murty, Dept. of Mathematics, Queen's University, Kingston K7L 3N6 Ontario Canada; Marilyn Reece, Dept. of Mathematics, Brown University, Providence, Rhode Island, RI 02912, U.S.A.
E-mail: murty@mast.queensu.ca, mjreece@math.brown.edu
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