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To Professor Włodzimierz Staś on his 75th birthday

A NEW IDENTITY - II

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Abstract: A new identity is proved. From this we deduce as corollaries: 2.1—an identity involving the Lionville's function $\lambda(n)$, 2.2—an identity involving the von-Mongoldt's function $\lambda(n)$ and 2.3—an identity involving the generating function of Waring's problem. Some other identities are also mentioned.
Keywords: Higher powers of partial sums, re-arrangement of series and identities involving infinite series.

1. The new identity

We start with the identity

$$bx^2 - (a+b)(x+a)^2 + a(x+a+b)^2 = ab(a+b) \tag{1.1}$$

and deduce our new identity (namely the theorem below). We then deduce some corollaries. Let $f(0), f(1), f(2), \ldots$ be any sequence of complex numbers. Put

$$H_n = \sum_{m=0}^{n} f(m) \tag{1.2}$$

where $m(\geq 1)$ is any integer. In (1.1) set

$$x = H_n,\ a = f(n+1) \quad \text{and} \quad b = f(n+2), \tag{1.3}$$

where $n(\geq 1)$ is a any integer. We obtain

$$\begin{aligned} f(n+2)H_n^2 - \big(f(n+1) + f(n+2)\big)H_{n+1}^2 + f(n+1)H_{n+2}^2 \\ = f(n+1)f(n+2)\big(f(n+1) + f(n+2)\big). \end{aligned} \tag{1.4}$$

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From this we deduce

$$\begin{aligned}\sum_{n=1}^{\infty} f(n+2)H_n^2 - \sum_{n=1}^{\infty}\big(f(n+1)+f(n+2)\big)H_{n+1}^2 + \sum_{n=1}^{\infty} f(n+1)H_{n+2}^2 \\ = \sum_{n=1}^{\infty} f(n+1)f(n+2)\big(f(n+1)+f(n+2)\big)\end{aligned} \tag{1.5}$$

subject to the convergence of the infinite series involved. Here the LHS is

$$\begin{aligned}\sum_{n=3}^{\infty} f(n+2)H_n^2 + f(3)H_1^2 + f(4)H_2^2 - \sum_{n=3}^{\infty}\big(f(n)+f(n+1)\big)H_n^2 \\ - \big(f(2)+f(3)\big)H_2^2 + \sum_{n=3}^{\infty} f(n-1)H_n^2.\end{aligned}$$

Hence we obtain the following Theorem.

Theorem 1.1. *We have*

$$\begin{aligned}&\sum_{n=3}^{\infty}\big(f(n-1)-f(n)-f(n+1)+f(n+2)\big)H_n^2 \\ &\quad = \sum_{n=1}^{\infty} f(n+1)f(n+2)\big(f(n+1)+f(n+2)\big) - f(3)(f(0)+f(1))^2 \\ &\qquad + \big(f(2)+f(3)-f(4)\big)\big(f(0)+f(1)+f(2)\big)^2\end{aligned} \tag{1.6}$$

provided the two infinite series involved are convergent. Here as stated already

$$H_n = \sum_{m=0}^{n} f(m) \qquad \textit{for} \quad n \geq 1.$$

2. Corollaries

Let $\lambda(n)$ be the Liouville's function defined by

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p (1+p^{-s})^{-1} = \frac{\zeta(2s)}{\zeta(s)} \tag{2.1}$$

where the product is over all primes and $s = \sigma + it$, $\sigma \geq 2$. Putting $f(0) = 0$ and $f(n) = \lambda(n)n^{-1}$ we have

Corollary 2.1.

$$\sum_{n=3}^{\infty}\left(\frac{\lambda(n-1)}{n-1}-\frac{\lambda(n)}{n}-\frac{\lambda(n+1)}{n+1}+\frac{\lambda(n+2)}{n+2}\right)\left(\sum_{m=1}^{n}\frac{\lambda(m)}{m}\right)^2$$
$$=\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\left(\frac{\lambda(n+2)}{n+1}+\frac{\lambda(n+1)}{n+2}\right)+\frac{1}{16}.$$

Remark. The series $\sum_{n=1}^{\infty}\left(\lambda(n)n^{-1}\right)$ is convergent and is equal to zero. Also $\left|\sum_{m=1}^{n}\lambda(m)/m\right|\leq K/(\log n)^2(n\geq 2)$ where K is a constant.

Let $\wedge(n)$ be the von-Mongoldt function defined

$$\sum_{n=1}^{\infty}\frac{\wedge(n)}{n^s}=-\frac{\zeta'(s)}{\zeta(s)}. \tag{2.2}$$

Putting $f(0)=0$ and $f(n)=\wedge(n)z^n(n\geq 1)$, where z is a complex number with $|z|<1$, we have

Corollary 2.2.

$$\sum_{n=3}^{\infty}\left(\wedge(n-1)z^{n-1}-\wedge(n)z^n-\wedge(n+1)z^{n+1}+\wedge(n+2)z^{n+2}\right)\left(\sum_{m=1}^{n}\wedge(m)z^m\right)^2$$
$$=\sum_{n=1}^{\infty}\wedge(n+1)\wedge(n+2)z^{2n+3}\left(\wedge(n+1)z^{n+1}+\wedge(n+2)z^{n+2}\right)$$
$$-(\log 2)^2z^4\left((\log 2)z^2+(\log 3)z^3\right). \tag{2.3}$$

Let $k(\geq 1)$ be any integer and z as before. Putting $f(0)=0$ and $f(n)=z^{n^k}$ we obtain

Corollary 2.3.

$$\sum_{n=3}^{\infty}\left(z^{(n-1)^k}-z^{n^k}-z^{(n+1)^k}+z^{(n+2)^k}\right)\left(\sum_{m=1}^{n}z^{m^k}\right)^2$$
$$=\sum_{n=1}^{\infty}z^{(n+1)^k+(n+2)^k}\left(z^{(n+1)^k}+z^{(n+2)^k}\right)-z^{3^k+2} \tag{2.4}$$
$$+\left(z^{2^k}+z^{3^k}-z^{4^k}\right)(z+z^{2^k})^2$$

Remark. We can go on listing some more corollaries. (For example we can take $f(0)=\zeta(3)$ and $f(n)=-1/n^3$ or $f(0)=\zeta(5)$ and $f(n)=-1/n^5$ and so on). But we stop at these three corollaries.

3. Higher powers of H_n

The identity (1.1) is a special case of the following result (see Theorem 1 of [1]). Let $k(\geq 1)$ be any integer and $x, x_1, x_2 \ldots, x_k$ any $k+1$ non-zero complex numbers no two of which are equal. Then (plainly x can be zero).

$$x^k + \left\{ \sum_{m=1}^{k} (x+x_m)^k (-1)^m x_m^{-1} \left(\prod_{m>j\geq 1} (x_m - x_j)^{-1} \right) \left(\prod_{k\geq j>m} (x_j - x_m)^{-1} \right) \right\} \times x_1 x_2 \ldots x_k$$
$$= (-1)^k x_1 x_2 \ldots x_k.$$

Using this we can get the analogue of our theorem above with H_n replaced by H_n^k. (Here k should not be confused with that in Corallary 2.3 above).

References

[1] K. Ramachandra, *On series, integrals and continued fractions.* III. (to appear).

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