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To Professor Włodzimierz Staś on his 75th birthday

ON SUMS OF TWO K-TH POWERS: A MEAN-SQUARE BOUND OVER SHORT INTERVALS

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1. Introduction.

For a fixed integer $k \geq 2$, denote by $r_k(n)$ the number of representations of the positive integer n as a sum of the k-th powers of two integers taken absolutely:

$$r_k(n) = \#\{(u_1, u_2) \in \mathbb{Z}^2 \,:\, |u_1|^k + |u_2|^k = n\,\}.$$

The average order of this arithmetic function is described by the sum

$$R_k(u) = \sum_{1 \leq n \leq u^k} r_k(n),$$

where u is a large real variable[1]. One is interested in precise asymptotic formulas for this summatory function $R_k(u)$.

For $k = 2$, this is the celebrated Gaussian circle problem. (An enlightening account on its history can be found in the monograph of Krätzel [10].) The sharpest published results to date[2] read

$$R_2(u) = \pi u^2 + P_2(u), \tag{1.1}$$

$$P_2(u) = O\big(u^{46/73}(\log u)^{315/146}\big), \tag{1.2}$$

and[3]

$$P_2(u) = \Omega_-\big(u^{1/2}(\log u)^{1/4}(\log\log u)^{\frac{1}{4}\log 2}\exp(-c\sqrt{\log\log\log u})\big) \qquad (c > 0), \tag{1.3}$$

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1 Note that, in part of the relevant literature, $t = u^2$ is used as the basic variable.

2 Actually, M. Huxley has meanwhile improved further this upper bound, essentially replacing the exponent $\frac{46}{73} = 0.6301\ldots$ by $\frac{131}{208} = 0.6298\ldots$. The author is indebted to Professor Huxley for sending him a copy of his unpublished manuscript.

3 We recall that $F_1(u) = \Omega_*(F_2(u))$ means that $\limsup\big(*F_1(u)/F_2(u)\big) > 0$ for $u \to \infty$ where $*$ is either $+$ or $-$, and $F_2(u)$ is positive for u sufficiently large.

$$P_2(u) = \Omega_+\big(u^{1/2}\exp\big(c'(\log\log u)^{1/4}(\log\log\log u)^{-3/4}\big)\big) \qquad (c'>0)\,. \tag{1.4}$$

These are due to Huxley [4], [6], Hafner [3], and Corrádi & Kátai [1], respectively. It is a wide-standing belief that

$$\inf\{\theta\in\mathbb{R}:\ P_2(u)\ll_\theta\ u^\theta\ \} = \frac{1}{2}\,. \tag{1.5}$$

In favour of this conjecture, there is the mean-square asymptotic

$$\int_0^T \big(P_2(u)\big)^2\,\mathrm{d}u = C_2\,T^2 + O\big(T(\log T)^2\big)\,, \qquad C_2 = \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{(r_2(n))^2}{n^{3/2}} \tag{1.6}$$

which has been established (with this precise error term) by Kátai [7].

The proofs of the results (1.3), (1.4), (1.6) were based on the fact that the generating function (Dirichlet series) of $r_2(n)$ is the Epstein zeta- function of the quadratic form $u_1^2+u_2^2$, which satisfies a well-known functional equation and thus makes available the whole toolkit of complex analysis.

The general case, $k\ge 3$, lacks this technical advantage. Nevertheless, the problem concerning the asymptotic behaviour of $R_k(u)$, $k\ge 3$, has attracted a lot of attention, too. It has first been dealt with by Van der Corput [18] and Krätzel [9]. For a thorough account on the history of this problem and the results available until 1988. see again Krätzel's textbook [10]. It turns out that

$$R_k(u) = \frac{2\Gamma^2(1/k)}{k\Gamma(2/k)}u^2 + B_k\Phi_k(u)u^{1-1/k} + P_k(u) \tag{1.7}$$

where

$$B_k = 2^{3-1/k}\,\pi^{-1-1/k}\,k^{1/k}\,\Gamma\big(1+\frac{1}{k}\big)\,,$$

$$\Phi_k(u) = \sum_{n=1}^{\infty} n^{-1-1/k}\sin\big(2\pi nu - \frac{\pi}{2k}\big)\,.$$

and the new error term $P_k(u)$ satisfies an estimate quite analogous to (1.2). i.e..

$$P_k(u) = O\big(u^{46/73}(\log u)^{315/146}\big)\,, \tag{1.8}$$

as was proved by Kuba [11], using Huxley's method [4], [6].

Concerning lower bounds, it was shown by the author [16] that, for any fixed $k\ge 3$,

$$P_k(u) = \Omega_-\big(u^{1/2}(\log u)^{1/4}\big)\,, \tag{1.9}$$

and by Küehleitner, Nowak, Schoißengeier & Wooley [13] that

$$P_3(u) = \Omega_+\big(u^{1/2}(\log\log u)^{1/4}\big)\,. \tag{1.10}$$

The similarity of these results to those for the case $k = 2$ suggested to extend the classic conjecture (1.5) to arbitrary $k \geq 2$. It turned out that this is again true in mean-square: In fact, the author [15] was able to show that, for T large,

$$\frac{1}{T} \int_0^T \left(P_k(u)\right)^2 \mathrm{d}u \ll T \tag{1.11}$$

for any fixed $k \geq 3$. M. Küehleitner [12] refined this estimate, proving an asymptotic formula

$$\frac{1}{T} \int_0^T \left(P_k(u)\right)^2 \mathrm{d}u = C_k\, T + O\left(T^{1-\omega_k+\epsilon}\right), \tag{1.12}$$

with explicite constants C_k and $\omega_k > 0$.

2. Statement of result

In the present note we investigate the question whether the "average moderate size" of this error term $P_k(u)$, as displayed by (1.11), can be observed only "in the long run," i.e., by averaging over an interval of order T, or if a similar estimate is possible for a "short interval mean." In fact, it turns out that it essentially suffices to average over an interval of bounded length—at the cost of a small loss of precision (extra logarithmic factor).

Theorem 2.1. *For T large and arbitrary fixed $k \geq 3$,*

$$\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} (P_k(u))^2 \,\mathrm{d}u \ll T\,(\log T)^2\,,$$

with the $\ll$-constant depending on k.

Remarks. This work is inspired by a paper of Huxley [5] who investigated the corresponding problem for the lattice rest of a convex planar domain (with smooth boundary of finite nonzero curvature throughout), linearly dilated by a large factor u. He obtained the corresponding mean-square bound $O(T \log T)$, thereby including the case of a circle, i.e., that of $k = 2$ in our problem.

In geometric terms, for $k \geq 3$ we are concerned with the number of lattice points in a domain bounded by a Lamé's curve $|\xi|^k + |\eta|^k = u^k$. This has curvature 0 in its points of intersection with the coordinate axes. As a consequence, the expansion of the lattice rest into a trigonometric series, as discovered by Kendall [8] and employed by Huxley [5], is no longer available. Therefore, we use a different approach based on fractional part sums, Vaaler's transition to exponential sums, the Van der Corput transformation ("B-step"), and, in the end, Huxley's trick involving the Féjer kernel.

Catching a word of Huxley [5] (who imagined the dilation factor u as a time variable), we can say that, according to our result, these number-theoretic error terms "*have no memory*," or, a bit more precisely, that their average small size is accomplished "*not by long-term memory, but by short-term memory.*"

3. Proof of the Theorem 2.1

As in our earlier article [15], we start from formulae (3.57), (3.58) (and the asymptotic expansion below) in Krätzel [10], p. 148. In our notation, this reads

$$P_k(u) = -8 \sum_{\alpha u < n \le u} \psi((u^k - n^k)^{1/k}) \; + \; O(1)\,, \tag{3.1}$$

with $\psi(w) = w - [w] - \frac{1}{2}$ throughout, and $\alpha := 2^{-1/k}$. We suppose that T is sufficiently large, $u \in [T - \frac{1}{2}, T + \frac{1}{2}]$, and define q by $1/k + 1/q = 1$, i.e., $q = k/(k-1)$, and thus $1 < q \le \frac{3}{2}$. We break up the range of summation into subintervals $\mathcal{N}_j(u) =]N_j, N_{j+1}]$, where $N_j = u\,(1 + 2^{-jq})^{-1/k}$, $j = 0, 1, \ldots, J$, with J minimal such that $u - N_J < 1$ for all $u \in [T - \frac{1}{2}, T + \frac{1}{2}]$.[4] It follows that the length of any $\mathcal{N}_j(u)$ is equal to $N_{j+1} - N_j \asymp 2^{-jq}T$, and that $w \in \mathcal{N}_j(u)$ implies that $u^k - w^k \asymp 2^{-jq}T^k$. We put

$$I_j(T) := \int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \Bigg(\sum_{n \in \mathcal{N}_j(u)} \psi((u^k - n^k)^{1/k}) \Bigg)^2 \mathrm{d}u$$

and infer from Cauchy's inequality, with some fixed $\epsilon > 0$ sufficiently small, that

$$\begin{aligned} &\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \Bigg(\sum_{j=0}^{J} \sum_{n \in \mathcal{N}_j(u)} \psi((u^k - n^k)^{1/k}) \Bigg)^2 \mathrm{d}u \\ &\qquad\qquad \le \sum_{j=0}^{J} 2^{-j\epsilon} \sum_{j=0}^{J} 2^{j\epsilon}\, I_j(T) \ll \sum_{j=0}^{J} 2^{j\epsilon}\, I_j(T)\,. \end{aligned} \tag{3.2}$$

We now invoke a deep result of Vaaler [17] which connects fractional parts with exponential sums. (See also Graham and Kolesnik [2], p. 116.) For every positive integer D there exists a sequence $\left(\alpha_{h,D}\right)_{h=1}^{D}$ contained in the interval $[0,1]$ such that for all reals w,

$$\left| \psi(w) + \frac{1}{2\pi i} \sum_{1 \le |h| \le D} \frac{\alpha_{|h|,D}}{h}\, e(hw) \right| \le \frac{1}{2D+2} \sum_{h=-D}^{D} (1 - \frac{|h|}{D+1}) e(hw)\,,$$

with $e(w) = e^{2\pi i w}$ as usual. From this it is easy to see that there exists a complex-valued sequence $\left(\beta_{h,D}\right)_{h=1}^{D}$ with

$$\beta_{h,D} \ll \frac{1}{h} \tag{3.3}$$

4 The idea of this special choice of subdivision points is that $\frac{\mathrm{d}}{\mathrm{d}w}\left((u^k - w^k)^{1/k}\right)$ assumes integer values at $w = N_j$. See the application of the Lemma below.

such that

$$I_j(T) \ll \int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \left| \sum_{h=1}^{D} \beta_{h,D} \sum_{n\in\mathcal{N}_j(u)} e\big(-h(u^k - n^k)^{1/k}\big) \right|^2 du \; + \; \big(\frac{2^{-jq}T}{D}\big)^2 . \quad (3.4)$$

We choose $D = \exp\big(\log 2\,[\frac{1}{2}\log T/\log 2]\big)$, i.e., D is a power of 2 and $D \asymp \sqrt{T}$. The last term in (3.4) is thus $\ll 4^{-jq}T$.

We now transform the exponential sums under consideration by a fairly sharp form of the "Van der Corput step."

Lemma 3.1. *Suppose that* f *is a real-valued function which possesses four continuous derivatives on the interval* $[A,B]$. *Let* L *and* U *be real parameters not less than* 1 *such that* $B - A \asymp L$,

$$f^{(j)}(w) \ll UL^{1-j} \qquad \text{for } w \in [A,B],\ j = 1,2,3,4,$$

and, for some $C^* > 0$,

$$f''(w) \geq C^*UL^{-1} \qquad \text{for } w \in [A,B]\,.$$

Suppose further that $f'(A)$ *and* $f'(B)$ *are integers, and denote by* ϕ *the inverse function of* f'. *Then it follows that*

$$\sum_{A\leq k\leq B} e(f(k)) = e\left(\frac{1}{8}\right) \sum_{f'(A)\leq m\leq f'(B)}'' \frac{e\big(f(\phi(m)) - m\phi(m)\big)}{\sqrt{f''(\phi(m))}} + O\big(\log(1+U)\big)\,,$$

where $\sum''$ *means that the terms corresponding to* $m = f'(A)$ *and* $m = f'(B)$ *get a factor* $\frac{1}{2}$. *The* O*-constant depends on* C^* *and on the constants implied in the order symbols in the suppositions.*

Proof. This is Lemma 2 in Kühleitner [12]. For a more general version of the same precision, as well as for comments on the history of this sort of results, see Kühleitner & Nowak [14], Lemma 2.2.

We use this formula to transform each of the sums over n in (3.4), with $[A,B] = [N_j, N_{j+1}]$, and

$$f(w) = -h\big(u^k - w^k\big)^{1/k}.$$

We readily compute the derivatives as[5]

$$f'(w) = hw^{k-1}\big(u^k - w^k\big)^{-1+1/k} \ll h\,2^j\,,$$

$$f''(w) = h(k-1)u^k w^{k-2}\big(u^k - w^k\big)^{-2+1/k} \asymp hT^{-1}\,2^{j-jq}\,,$$

---

5 Recall that $w \in \mathcal{N}_j(u)$ implies that $w \asymp T$ and $u^k - w^k \asymp 2^{-jq}T^k$.

$$\begin{aligned} f'''(w) &= h(k-1)u^k w^{k-3}\left(u^k - w^k\right)^{-3+1/k}\left((k-2)u^k + (k+1)w^k\right) \\ &\ll hT^{-2}\, 2^{j-2jq}\,, \end{aligned}$$

$$\begin{aligned} f^{(4)}(w) = \ & h(k-1)u^k w^{k-4}\left(u^k - w^k\right)^{-4+1/k} \\ & \times \left((k-2)(k-3)u^{2k} + (k+1)(4k-7)u^k w^k + (k+1)(k+2)w^{2k}\right) \\ \ll \ & hT^{-3}\, 2^{j-3jq}\,. \end{aligned}$$

Our Lemma thus applies with $L = N_{j+1} - N_j \asymp 2^{-jq}T$, $U = h\,2^j$, and we obtain by a straightforward calculation, for $u \in [T-\frac{1}{2}, T+\frac{1}{2}]$,

$$\begin{aligned} &\sum_{n\in\mathcal{N}_j(t)} e\left(-h(u^k - n^k)^{1/k}\right) \\ &\quad = \frac{e\left(\frac{1}{8}\right)}{\sqrt{k-1}} h u^{1/2} \sideset{}{''}\sum_{m\in\mathcal{M}_j(h)} (hm)^{-1+q/2}\, \|(h,m)\|_q^{-q+1/2}\, e\left(-u\,\|(h,m)\|_q\right) \\ &\qquad + O(\log T)\,, \end{aligned} \tag{3.5}$$

with

$$\mathcal{M}_j(h) = \left]f'(N_j), f'(N_{j+1})\right] = \left]2^j h, 2^{j+1}h\right]\,,$$

and $\|\cdot\|_q$ denoting the q-norm in $\mathbb{R}^2$, i.e., $\|(u_1,u_2)\|_q = \left(|u_1|^q + |u_2|^q\right)^{1/q}$. With a look back to (3.4), we define

$$S_h(u) := \beta_{h,D}\, h \sideset{}{''}\sum_{m\in\mathcal{M}_j(h)} (hm)^{-1+q/2}\, \|(h,m)\|_q^{-q+1/2}\, e\left(-u\,\|(h,m)\|_q\right)$$

and divide the range $1 \le h \le D = 2^I$ (say) into dyadic subintervals $\mathcal{H}_i =]2^{i-1}, 2^i]$, $i = 1,\dots,I \ll \log T$. Combining (3.4) and (3.5), we conclude by Cauchy's inequality that

$$\begin{aligned} I_j(T) &\ll \int_{T-\frac{1}{2}}^{T+\frac{1}{2}} u\left|\sum_{i=1}^{I}\sum_{h\in\mathcal{H}_i} S_h(u)\right|^2 \mathrm{d}u + (\log T)^2 + 4^{-jq}T \\ &\ll T(\log T)^2 \max_{1\le i\le I} \int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \left|\sum_{h\in\mathcal{H}_i} S_h(u)\right|^2 \mathrm{d}u \ + (\log T)^2 + 4^{-jq}T\,. \end{aligned} \tag{3.6}$$

Following an idea of Huxley [5], we now use the Féjer kernel

$$\varphi(w) := \left(\frac{\sin(\pi w)}{\pi w}\right)^2\,.$$

By Jordan's inequality, $\varphi(w) \ge 4/\pi^2$ for $|w| \le \frac{1}{2}$, and the Fourier transform has the simple shape

$$\widehat{\varphi}(y) = \int_{\mathbb{R}} \varphi(w) e(yw)\,\mathrm{d}w = \max(0, 1-|y|)\,.$$

Therefore,

$$
\begin{aligned}
\frac{4}{\pi^2}\int_{T-\frac12}^{T+\frac12}\Big|\sum_{h\in\mathcal{H}_i} S_h(u)\Big|^2 du &\le \int_{\mathbb{R}} \varphi(u-T)\Big|\sum_{h\in\mathcal{H}_i} S_h(u)\Big|^2 du = \\
&= \sum_{h_1,h_2\in\mathcal{H}_i} (h_1h_2)^{q/2}\beta_{h_1,D}\overline{\beta_{h_2,D}} \sideset{}{''}\sum_{\substack{m_1\in\mathcal{M}_j(h_1),\\ m_2\in\mathcal{M}_j(h_2)}} \frac{\big(\|(h_1,m_1)\|_q\,\|(h_2,m_2)\|_q\big)^{-q+1/2}}{(m_1m_2)^{1-q/2}} \\
&\quad\times e\big(-T(\|(h_1,m_1)\|_q-\|(h_2,m_2)\|_q)\big) \\
&\qquad \int_{\mathbb{R}}\varphi(u)e\big(-u(\|(h_1,m_1)\|_q-\|(h_2,m_2)\|_q)\big)\,du \\
&\ll \sum_{h_1,h_2\in\mathcal{H}_i}(h_1h_2)^{-1+q/2} \sideset{}{''}\sum_{\substack{m_1\in\mathcal{M}_j(h_1),\\ m_2\in\mathcal{M}_j(h_2)}} \frac{\big(\|(h_1,m_1)\|_q\,\|(h_2,m_2)\|_q\big)^{-q+1/2}}{(m_1m_2)^{1-q/2}} \\
&\quad\times \max\big(0,1-\big|\|(h_1,m_1)\|_q-\|(h_2,m_2)\|_q\big|\big),
\end{aligned}
\tag{3.7}
$$

using the bound (3.3) for the β's. We recall that $h\in\mathcal{H}_i$ implies $h\asymp 2^i$ and $m\in\mathcal{M}_j(h)$ implies that $\|(h,m)\|_q\asymp m\asymp 2^j h$. Therefore, the last expression in (3.7) is

$$
\begin{aligned}
&\ll (2^i)^{-2+q}(2^{i+j})^{-1-q}\,\#\big\{(h_1,h_2,m_1,m_2)\in\mathbb{Z}^4:\ h_1,h_2\in\mathcal{H}_i,\\
&\quad m_1\in\mathcal{M}_j(h_1),\ m_2\in\mathcal{M}_j(h_2),\ \big|\|(h_1,m_1)\|_q-\|(h_2,m_2)\|_q\big|<1\big\}.
\end{aligned}
\tag{3.8}
$$

Now denote by $A_q^*(u)$ the number of lattice points $\mathbf{v}\in\mathbb{Z}^2$ with $\|\mathbf{v}\|_q\le u$, then the most elementary estimate

"Number of lattice points = area + O(length of boundary)"

implies, for any fixed (h_1,m_1), $h_1\in\mathcal{H}_i$, $m_1\in\mathcal{M}_j(h_1)$, that

$$
A_q^*\big(\|(h_1,m_1)\|_q+1\big)-A_q^*\big(\|(h_1,m_1)\|_q-1\big)\ll\|(h_1,m_1)\|_q\ll m_1 .
$$

Thus, combining (3.7) and (3.8), it follows that

$$
\int_{T-\frac12}^{T+\frac12}\Big|\sum_{h\in\mathcal{H}_i}S_h(u)\Big|^2du\ll(2^i)^{-2+q}(2^{i+j})^{-1-q}\sum_{h_1\in\mathcal{H}_i,}\ \sum_{m_1\in\mathcal{M}_j(h_1)} m_1\ \ll(2^j)^{1-q},
$$

uniformly in $i=1,\dots,I$. Using this in (3.6), we get

$$
I_j(T)\ll 2^{-j(q-1)}\,T(\log T)^2+(\log T)^2 .
$$

Recalling (3.1), (3.2), and the fact that $q = k/(k-1) > 1$, we complete the proof of the Theorem. ■

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