

ON THE INTEGRAL OF THE ERROR TERM IN THE FOURTH MOMENT OF THE RIEMANN ZETA-FUNCTION

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1. Introduction

The aim of this note is to provide an asymptotic formula for $\int_0^T E_2(t) dt$, where $E_2(T)$ is the error term in the asymptotic formula for the fourth moment of $|\zeta(\frac{1}{2} + it)|$. The asymptotic formula for the fourth moment of the Riemann zeta-function $\zeta(s)$ on the critical line is customarily written as

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt = TP_4(\log T) + E_2(T), \quad (1.1)$$

where

$$P_4(x) = \sum_{j=0}^4 a_j x^j. \quad (1.2)$$

It is classically known that $a_4 = 1/(2\pi^2)$, and it was proved by D. R. Heath-Brown [1] that

$$a_3 = 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2})\pi^{-2}.$$

He also produced more complicated expressions for a_0, a_1 and a_2 in (1.2) ($\gamma = 0.577\dots$ is Euler's constant). For an explicit evaluation of the a_j 's the reader is referred to [4].

In recent years, due primarily to the application of powerful methods of spectral theory (see Y. Motohashi's monograph [13] for a comprehensive account), much advance has been made in connection with $E_2(T)$. We refer the reader to the works [5]–[9], [11]–[13] and [16]. It is known now that

$$E_2(T) = O(T^{2/3} \log^{C_1} T), \quad E_2(T) = \Omega(T^{1/2}), \quad (1.3)$$

$$\int_0^T E_2(t) dt = O(T^{3/2}), \quad \int_0^T E_2^2(t) dt = O(T^2 \log^{C_2} T), \quad (1.4)$$

with effective constants $C_1, C_2 > 0$ (the values $C_1 = 8, C_2 = 22$ are worked out in [13]). The above results were proved by Y. Motohashi and the author: (1.3) and the first bound in (1.4) in [3], [8], [13] and the second upper bound in (1.4) in [7]. The omega-result in (1.3) ($f = \Omega(g)$ means that $f = o(g)$ does not hold, $f = \Omega_{\pm}(g)$ means that $\limsup f/g > 0$ and that $\liminf f/g < 0$) was improved to $E_2(T) = \Omega_{\pm}(T^{1/2})$ by Y. Motohashi [12]. Recently the author [6] made further progress in this problem by proving the following quantitative omega-result: there exist two constants $A > 0, B > 1$ such that for $T \geq T_0 > 0$ every interval $[T, BT]$ contains points T_1, T_2 for which

$$E_2(T_1) > AT_1^{1/2}, \quad E_2(T_2) < -AT_2^{1/2}. \tag{1.5}$$

There is an obvious discrepancy between the O -result and Ω -result in (1.3), and it may be well conjectured that $E_2(T) = O_{\varepsilon}(T^{1/2+\varepsilon})$ for any given $\varepsilon > 0$ (ε will denote arbitrarily small constants, not necessarily the same ones at each occurrence). This bound, if true, is very strong, since it would imply (e.g., by Lemma 7.1 of [3]) the hitherto unproved bound $\zeta(\frac{1}{2} + it) \ll_{\varepsilon} t^{1/8+\varepsilon}$. The upper bound in (1.3) seems to be the limit of the existing methods, since the only way to estimate the relevant exponential sum in this problem, namely (see [3],[8] and [13])

$$\sum_{K < \kappa_j \leq 2K} \alpha_j H_j^3\left(\frac{1}{2}\right) \exp\left(ik_j \log\left(\frac{T}{\kappa_j}\right)\right) \quad (1 \ll K \leq T^{1/2}) \tag{1.6}$$

appears to be trivial estimation, coming from the bound

$$\sum_{K < \kappa_j \leq 2K} \alpha_j \left|H_j^3\left(\frac{1}{2}\right)\right| \ll K^2 \log^C K \quad (C > 0). \tag{1.7}$$

This follows by the Cauchy-Schwarz inequality from the bounds (see [13])

$$\sum_{\kappa_j \leq K} \alpha_j H_j^2\left(\frac{1}{2}\right) \ll K^2 \log K, \quad \sum_{\kappa_j \leq K} \alpha_j H_j^4\left(\frac{1}{2}\right) \ll K^2 \log^{15} K \tag{1.8}$$

with $C = 8$ in (1.7). Here as usual $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$ denotes the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2, \mathbb{Z})$ -automorphic forms, and $\alpha_j = |\rho_j(1)|^2 (\cosh \pi \kappa_j)^{-1}$, where $\rho_j(1)$ is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue λ_j to which the Hecke series $H_j(s)$ is attached. It is precisely the presence of $H_j^3(\frac{1}{2})$ in (1.6) which makes the sum in question very hard to deal with, and any decrease of the exponent $2/3$ in the upper bound for $E_2(T)$ in (1.3) will likely involve the application of genuine new ideas.

In [6] the author proved that there exist constants $A > 0$ and $B > 1$ such that, for $T \geq T_0 > 0$, every interval $[T, BT]$ contains points t_1, t_2 for which

$$\int_0^{t_1} E_2(t) dt > At_1^{3/2}, \quad \int_0^{t_2} E_2(t) dt < -At_2^{3/2}. \tag{1.9}$$

This result, of course, implies that $\int_0^T E_2(t) dt = \Omega_{\pm}(T^{3/2})$. It was also used in [6] to prove a lower bound result, whose special case $a = 2$ gives

$$\int_0^T E_2^2(t) dt \gg T^2, \quad (1.10)$$

thus sharpening (1.8) and showing that the upper bound in (1.4) is very close to the true order of magnitude of the mean square integral of $E_2(T)$.

The main aim of this paper is to prove a result, which gives an asymptotic formula for the integral of $E_2(t)$, thereby sharpening the first bound in (1.4). This is the following

Theorem 1.1. *Let*

$$\eta(T) := (\log T)^{3/5} (\log \log T)^{-1/5}, \quad (1.11)$$

$$R_1(\kappa_h) := \sqrt{\frac{\pi}{2}} \left(2^{-i\kappa_h} \frac{\Gamma(\frac{1}{4} - \frac{1}{2}i\kappa_h)}{\Gamma(\frac{1}{4} + \frac{1}{2}i\kappa_h)} \right)^3 \Gamma(2i\kappa_h) \cosh(\pi\kappa_h). \quad (1.12)$$

Then there exists a constant $C > 0$ such that

$$\int_0^T E_2(t) dt = 2T^{\frac{3}{2}} \Re \left\{ \sum_{j=1}^{\infty} \alpha_j H_j^3\left(\frac{1}{2}\right) \frac{T^{i\kappa_j}}{\left(\frac{1}{2} + i\kappa_j\right)\left(\frac{3}{2} + i\kappa_j\right)} R_1(\kappa_j) \right\} + O(T^{\frac{3}{2}} e^{-C\eta(T)}). \quad (1.13)$$

From Stirling's formula for the gamma-function it follows that $R_1(\kappa_j) \ll \kappa_j^{-1/2}$, hence by (1.7) and partial summation it follows that the series on the right-hand side of (1.13) is absolutely convergent, and it can be also shown (see [3], [5], [6]) that $\Re \{ \dots \}$ is also $\Omega_{\pm}(1)$. Thus from Theorem 1.1 we can easily deduce all previously known Ω -results for $E_2(T)$. The error term in (1.13) is similar to the error term in the strongest known form of the prime number theorem (see e.g., [2, Chapter 12]). This is by no means a coincidence, and the reason for such a shape of the error term in (1.13) will transpire from the proof of Theorem 1.1, which will be given in Section 3.

2. A mean square result

We shall deduce the proof of Theorem 1.1 from a mean square result for the function

$$\mathcal{Z}_2(s) := \int_1^{\infty} |\zeta(\frac{1}{2} + ix)|^4 x^{-s} dx \quad (\Re s = \sigma > 1). \quad (2.1)$$

It was introduced and studied in [12], [13, Chapter 5], and then further used and studied in [5], [6] and [9]. Y. Motohashi [12] has shown that $\mathcal{Z}_2(s)$ has meromorphic continuation over \mathbb{C} . In the half-plane $\Re s > 0$ it has the following

singularities: the pole $s = 1$ of order five, simple poles at $s = \frac{1}{2} \pm i\kappa_j$ ($\kappa_j = \sqrt{\lambda_j - 1/4}$) and poles at $s = \frac{1}{2}\rho$, where ρ denotes complex zeros of $\zeta(s)$. The residue of $\mathcal{Z}_2(s)$ at $s = \frac{1}{2} + i\kappa_h$ equals

$$R(\kappa_h) := \sqrt{\frac{\pi}{2}} \left(2^{-i\kappa_h} \frac{\Gamma(\frac{1}{4} - \frac{1}{2}i\kappa_h)}{\Gamma(\frac{1}{4} + \frac{1}{2}i\kappa_h)} \right)^3 \Gamma(2i\kappa_h) \cosh(\pi\kappa_h) \sum_{\kappa_j = \kappa_h} \alpha_j H_j^3(\frac{1}{2}),$$

and the residue at $s = \frac{1}{2} - i\kappa_h$ equals $\overline{R(\kappa_h)}$. The function $\mathcal{Z}_2(s)$ is a natural tool for investigations involving $E_2(T)$ (see (3.3) and (3.4)). Its spectral decomposition (see [12] and [13, Chapter 5]) enables one to connect problems with $E_2(T)$ to results from spectral theory. We shall prove the following

Theorem 2.1. *Let*

$$\sigma = \frac{1}{2} - C\delta(V), \quad \delta(V) := (\log V)^{-2/3}(\log \log V)^{-1/3}, \quad (2.2)$$

where $C > 0$ is a suitable constant. Then

$$\int_V^{2V} |\mathcal{Z}_2(\sigma + iv)|^2 dv \ll_\varepsilon V^{2+\varepsilon}. \quad (2.3)$$

Proof. We note that in [9] the bound (2.3) was shown to hold for $\frac{1}{2} < \sigma < 1$, but it is the region $\sigma < \frac{1}{2}$ that is more difficult to deal with. As in [9] we write

$$\begin{aligned} \mathcal{Z}_2(s) &= \int_1^\infty I(T, \Delta) T^{-s} dT + \int_1^\infty (|\zeta(\frac{1}{2} + iT)|^4 - I(T, \Delta)) T^{-s} dT \\ &= \mathcal{Z}_{21}(s) + \mathcal{Z}_{22}(s), \end{aligned} \quad (2.4)$$

say, where

$$I(T, \Delta) = \frac{1}{\sqrt{\pi\Delta}} \int_{-\infty}^\infty |\zeta(\frac{1}{2} + i(T+t))|^4 \exp\left(-\left(\frac{t}{\Delta}\right)^2\right) dt \quad (\Delta = T^\xi, \frac{1}{3} \leq \xi \leq \frac{1}{2}). \quad (2.5)$$

Before we pass to specific bounds, we shall discuss the method that will be used. Let us suppose that we want to obtain an upper bound for

$$I := \int_T^{2T} \left| \int_a^b g(x) x^{-s} dx \right|^2 dt \quad (s = \sigma + it, T \geq T_0 > 0), \quad (2.6)$$

where $g(x)$ is a real-valued, integrable function on $[a, b]$, a subinterval of $[1, \infty)$ (which is not necessarily finite), and which satisfies $g(x) \ll x^C$ for some $C > 0$. Let $\varphi(x) \in C^\infty(0, \infty)$ be a test function such that $\varphi(x) \geq 0$, $\varphi(x) = 1$ for

$T \leq x \leq 2T$, $\varphi(x) = 0$ for $x < \frac{1}{2}T$ or $x > \frac{5}{2}T$ ($T \geq T_0 > 0$), $\varphi(x)$ is increasing in $[\frac{1}{2}T, T]$ and decreasing in $[2T, \frac{5}{2}T]$. Then we have, by r integrations by parts,

$$\int_{T/2}^{5T/2} \varphi(t) \left(\frac{y}{x}\right)^{it} dt = (-1)^r \int_{T/2}^{5T/2} \varphi^{(r)}(t) \frac{(y/x)^{it}}{(i \log(y/x))^r} dt \tag{2.7}$$

$$\ll_r T^{1-r} \left| \log \frac{y}{x} \right|^{-r} \ll T^{-A}$$

for any fixed $A > 0$ and any given $\varepsilon > 0$, provided that $|y - x| \geq xT^{\varepsilon-1}$ and $r = r(A, \varepsilon)$ is large enough. Recalling that $g(x) \ll x^C$ and using (2.7) it follows that

$$I \leq \int_{T/2}^{5T/2} \varphi(t) \left| \int_a^b g(x)x^{-s} dx \right|^2 dt$$

$$= \int_a^b \int_a^b g(x)g(y)(xy)^{-\sigma} \int_{T/2}^{5T/2} \varphi(t) \left(\frac{y}{x}\right)^{it} dt dx dy \tag{2.8}$$

$$\ll 1 + \int_{T/2}^{5T/2} \varphi(t) \int_a^b |g(x)|x^{-\sigma} \int_{x-xT^{\varepsilon-1}}^{x-xT^{\varepsilon-1}} |g(y)|y^{-\sigma} dy dx dt.$$

and the problem is reduced to the estimation of the integral of $g(x)$ over short intervals; here actually $g(x)$ does not have to be real-valued. In (2.8) we may further use the elementary inequality $|g(x)g(y)| \leq \frac{1}{2}(g^2(x) + g^2(y))$, and thus reduce the problem to mean square estimates.

In the expression for $\mathcal{Z}_{22}(s)$ in (2.4) we denote by $I_1(s, X)$ the integral in which $T \leq X$, and by $I_2(s, X)$ the remaining integral, where $X (\ll V^C)$ is a parameter to be chosen later. We have ($s = \sigma + it$)

$$\int_V^{2V} |I_1(s, X)|^2 dt \ll \int_V^{2V} \left| \int_1^X |\zeta(\frac{1}{2} + iT)|^4 T^{-s} dT \right|^2 dt$$

$$+ \int_V^{2V} \left| \int_{-\log V}^{\log V} \int_1^X |\zeta(\frac{1}{2} + iT + iu)|^4 T^{-s} dT e^{-u^2} du \right|^2 dt$$

$$+ 1.$$

Both mean square integrals above are estimated analogously. The first one is, by using (2.8),

$$\ll_\varepsilon 1 + \int_V^{2V} \int_1^X |\zeta(\frac{1}{2} + ix)|^4 x^{-\sigma} \int_{x-xV^{\varepsilon-1}}^{x+xV^{\varepsilon-1}} |\zeta(\frac{1}{2} + iy)|^4 y^{-\sigma} dy dx$$

$$\ll_\varepsilon 1 + \int_V^{2V} \int_1^X |\zeta(\frac{1}{2} + ix)|^4 x^{-2\sigma} (xV^{\varepsilon-1} + x^{c+\varepsilon}) dx dt \tag{2.9}$$

$$\ll_\varepsilon V^\varepsilon (X^{2-2\sigma} + V + VX^{1+c-2\sigma}) \ll_\varepsilon V^\varepsilon (X + VX^c).$$

Here we used (1.1), (2.2) the weak form of the fourth moment of $|\zeta(\frac{1}{2} + ix)|$ and the bound (see (1.3))

$$E_2(T) \ll_{\varepsilon} T^{c+\varepsilon} \quad \left(\frac{1}{2} \leq c \leq \frac{2}{3}\right). \quad (2.10)$$

To estimate the contribution of $I_2(s, X)$, note that from [9, (4.10)] we have that the relevant part of $I_2(s, X)$ is, on integrating by parts,

$$\begin{aligned} & \int_0^b \int_X^{\infty} E_2'(\tau) f(\tau, \alpha) d\tau d\alpha \\ &= O\left(\sup_{\alpha} |E_2(X) f(X, \alpha)|\right) - \int_0^b \int_X^{\infty} E_2(\tau) \frac{\partial f(\tau, \alpha)}{\partial \tau} d\tau d\alpha. \end{aligned}$$

where $b > 0$ is a small constant, and $f(\tau, \alpha)$ is precisely defined in [9]. It was shown there that, for $0 < \sigma < \frac{1}{2}$, $t \ll V$, we have the estimates

$$f(\tau, \alpha) \ll \tau^{2\xi-2-\sigma} (\log^2 \tau + V \log \tau + V^2) \log^3 \tau$$

and

$$\frac{\partial f(\tau, \alpha)}{\partial \tau} \ll \tau^{2\xi-3-\sigma} V \log^3 \tau (\log^2 \tau + V \log \tau + V^2).$$

We use (2.5), (2.9), (2.10) and the above estimates to obtain, if σ satisfies (2.2).

$$\begin{aligned} \int_V^{2V} |I_1(s, X)|^2 dt &\ll_{\varepsilon} V^5 X^{2c+4\xi-4-2\sigma} + V^6 \int_X^{\infty} E_2'(\tau) \tau^{4\xi-5-2\sigma} d\tau \\ &\ll_{\varepsilon} V^{\varepsilon} (V^5 X^{2c+4\xi-5} + V^6 X^{4\xi-4}). \end{aligned}$$

It follows that

$$\begin{aligned} & \int_V^{2V} |\mathcal{Z}_{22}(\sigma + it)|^2 dt \\ &\ll_{\varepsilon} V^{\varepsilon} (VX^c + X + V^5 X^{2c+4\xi-5} + V^6 X^{4\xi-4}) \\ &\ll_{\varepsilon} V^{\varepsilon} (V^{5/(4+c-4\xi)} + V^{(4+6c-4\xi)/(4+c-4\xi)} + V^{(15c-5)/(4+c-4\xi)}) \\ &\ll_{\varepsilon} V^{(4+6c-4\xi)/(4+c-4\xi)+\varepsilon} \end{aligned} \quad (2.11)$$

with $X = V^{5/(4+c-4\xi)}$, since in view of $\xi \leq \frac{1}{2}$, $\frac{1}{2} \leq c \leq \frac{2}{3}$ we have

$$5 \leq 4 + 6c - 4\xi, \quad 15c - 5 \leq 4 + 6c - 4\xi.$$

Then with $\xi = \frac{1}{3}$, which we henceforth assume, we obtain

$$\int_V^{2V} |\mathcal{Z}_{22}(\sigma + iv)|^2 dv \ll_{\varepsilon} V^{2+\varepsilon},$$

so that (2.3) will follow from

$$\int_V^{2V} |\mathcal{Z}_{21}(\sigma + iv)|^2 dv \ll_\varepsilon V^{2+\varepsilon}. \tag{2.12}$$

It was shown in [9] that the major contribution to $\mathcal{Z}_{22}(s)$ comes from ($s = \sigma + it, V \leq t \leq 2V$ and σ satisfies (2.2))

$$\sum_{t-V^\varepsilon \leq \kappa_j \leq t+V^\varepsilon} \alpha_j H_j^3(\tfrac{1}{2}) \left| \frac{1}{2} + i\kappa_j - s \right|^{-1} \kappa_j^{-\frac{1}{2}} \left| \int_{T(\kappa_j)}^\infty M^*(\kappa_j; T) T^{\frac{1}{2} + i\kappa_j - s} dT \right|, \tag{2.13}$$

where

$$T(r) := r^{\frac{1}{1-\varepsilon}} \log^{-D} r = r^{\frac{1}{2}} \log^{-D} r \quad (D > 0), \tag{2.14}$$

and $M^*(r; T)$ is a precisely defined function from spectral theory which satisfies, for $T \geq T(r)$ (cf. [9, (4.28)]), the bound

$$M^*(r; T) \ll_\varepsilon r T^{-2} + r^{2+\varepsilon} T^{2\varepsilon-3}. \tag{2.15}$$

Thus the major contribution to the integral in (2.13) will therefore be, since $H_j(\frac{1}{2}) \geq 0$ (see Katok-Sarnak [10]),

$$\int_V^{2V} \left| \sum_{t-V^\varepsilon \leq \kappa_j \leq t+V^\varepsilon} \alpha_j H_j^3(\tfrac{1}{2}) \left| \frac{1}{2} + i\kappa_j - s \right|^{-1} V^{-\frac{1}{2}} \times \right. \\ \left. \times \left| \int_{T(V)}^\infty M^*(\kappa_j; T) T^{\frac{1}{2} + i\kappa_j - s} dT \right| \right|^2 dt. \tag{2.16}$$

Recall that σ is given by (2.2), and that by the zero-free region for $\zeta(s)$ we have the bound (see [2, Lemma 12.3] and (2.2))

$$\frac{1}{\zeta(\alpha + it)} \ll (\log t)^{2/3} (\log \log t)^{1/3} \quad (\alpha \geq 1 - \delta(t), t \geq t_0 > 0).$$

This gives $|\frac{1}{2} + i\kappa_j - s|^{-1} \ll \log V$ in (2.16). We use the Cauchy-Schwarz inequality, (1.8) and the asymptotic formula (see [13])

$$\sum_{\kappa_j \leq K} \alpha_j H_j^2(\tfrac{1}{2}) = (A \log K + B) K^2 + O(K \log^6 K) \quad (A > 0)$$

to estimate sums of $\alpha_j H_j^2(\frac{1}{2})$ in short intervals. We obtain then that the expression in (2.16) is, on using (2.9) and the inequality $|g(x)g(y)| \leq \frac{1}{2}(g^2(x) + g^2(y))$, (2.14)

and (2.15),

$$\begin{aligned}
&\ll V^{-1} \log^2 V \int_V^{2V} \sum_{t-V^\varepsilon \leq \kappa_j \leq t+V^\varepsilon} \alpha_j H_j^2\left(\frac{1}{2}\right) \sum_{t-V^\varepsilon \leq \kappa_j \leq t+V^\varepsilon} \alpha_j H_j^4\left(\frac{1}{2}\right) \times \\
&\quad \times \left| \int_{T(V)}^\infty M^*(\kappa_j; T) T^{\frac{1}{2}+i\kappa_j-s} dT \right|^2 dt \\
&\ll_\varepsilon V^\varepsilon \sum_{V-V^\varepsilon \leq \kappa_j \leq 2V+V^\varepsilon} \alpha_j H_j^4\left(\frac{1}{2}\right) \int_{T(V)}^\infty |M^*(\kappa_j; T)|^2 T^{2-2\sigma} dT \\
&\ll_\varepsilon V^\varepsilon \sum_{V-V^\varepsilon \leq \kappa_j \leq 2V+V^\varepsilon} \alpha_j H_j^4\left(\frac{1}{2}\right) \int_{T(V)}^\infty (V^2 T^{-4} + V^4 T^{4\xi-6}) T dT \\
&\ll_\varepsilon V^\varepsilon \sum_{V-V^\varepsilon \leq \kappa_j \leq 2V+V^\varepsilon} \alpha_j H_j^4\left(\frac{1}{2}\right) (V^2 T^{-2}(V) + V^4 T^{4\xi-4}(V)) \\
&\ll V^\varepsilon \sum_{V-V^\varepsilon \leq \kappa_j \leq 2V+V^\varepsilon} \alpha_j H_j^4\left(\frac{1}{2}\right) \ll_\varepsilon V^{2+\varepsilon}.
\end{aligned}$$

This establishes (2.12) and thus finishes the proof of Theorem 2.1. ■

3. The proof of Theorem 1.1

In this section we shall prove Theorem 1.1. The starting point is the inversion formula

$$|\zeta\left(\frac{1}{2} + ix\right)|^4 = \frac{1}{2\pi i} \int_{(1+\varepsilon)} \mathcal{Z}_2(s) x^{s-1} ds, \quad (3.1)$$

where as usual $\int_{(c)} = \lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT}$. Namely, if $F(s) = \int_0^\infty f(x) x^{s-1} dx$ is the Mellin transform of $f(x)$. $y^{\sigma-1} f(y) \in L^1(0, \infty)$ and $f(y)$ is of bounded variation in a neighbourhood of $y = x$, then one has the Mellin inversion formula (see [14])

$$\frac{f(x+0) + f(x-0)}{2} = \frac{1}{2\pi i} \int_{(\sigma)} F(s) x^{-s} ds.$$

We use this formula with $f(x) = \frac{1}{x} |\zeta(\frac{1}{2} + \frac{i}{x})|^4$ for $0 < x \leq 1$ and $f(x) = 0$ for $x > 1$, and then change x to $1/x$ to obtain (3.1).

Now we replace the line of integration in (3.1) by the contour \mathcal{L} , consisting of the same straight line from which the segment $[1 + \varepsilon - i, 1 + \varepsilon + i]$ is removed and replaced by a circular arc of unit radius, lying to the left of the line, which passes over the pole $s = 1$ of the integrand. By the residue theorem we have

$$|\zeta\left(\frac{1}{2} + ix\right)|^4 = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{Z}_2(s) x^{s-1} ds + Q_4(\log x) \quad (x > 1), \quad (3.2)$$

where we have, since the coefficients of $P_4(z)$ are naturally connected to the principal part of the Laurent expansion of $\mathcal{Z}_2(s)$ at $s = 1$ (see [3] and [13]),

$$Q_4(\log x) = P_4(\log x) + P'_4(\log x)$$

and $P_4(y)$ is given by (1.1) and (1.2). If we integrate (3.2) from $x = 1$ to $x = T$ and take into account the defining relation (1.1) of $E_2(T)$, we shall obtain

$$E_2(T) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{Z}_2(s) \frac{T^s}{s} ds + O(1) \quad (T > 1). \tag{3.3}$$

A further integration, coupled with the deformation of the contour, enables one to deduce from (3.3) the formula

$$\int_0^T E_2(t) dt = \frac{1}{2\pi i} \int_{(c)} \mathcal{Z}_2(s) \frac{T^{s+1}}{s(s+1)} ds + O(T) \quad \left(\frac{1}{2} < c < 1, T > 1\right), \tag{3.4}$$

since in view of the bound (see [9])

$$\int_0^T |\mathcal{Z}_2(\sigma + it)|^2 dt \ll_{\epsilon} T^{2+\epsilon} \quad \left(\frac{1}{2} < \sigma < 1\right) \tag{3.5}$$

we may take $\frac{1}{2} < c < 1$ as the range for c in (3.4). The formula (3.4) is the key one in the proof of Theorem 1.1. We replace the line of integration in the integral on the right-hand side of (3.4) by the contour consisting of the segment $[\sigma_0 - it_0, \sigma_0 + it_0]$, and the curve

$$\sigma = \frac{1}{2} - C\delta(|t|), \quad \delta(x) := (\log x)^{-2/3}(\log \log x)^{-1/3}, \quad |t| \geq t_0, \quad \sigma_0 = \frac{1}{2} - C\delta(t_0), \tag{3.6}$$

where C denotes positive, possibly different constants. Since $\mathcal{Z}_2(s)$ has poles at complex zeros of $\zeta(2s)$ it follows, by the strongest known zero-free region for $\zeta(s)$ (see [6, Chapter 6]), that the function $\mathcal{Z}_2(s)$ is regular on the new contour. The residue theorem yields

$$\begin{aligned} \int_0^T E_2(t) dt &= 2\Re \left\{ \sum_{j=1}^{\infty} \frac{T^{\frac{3}{2}+i\kappa_j}}{(\frac{1}{2} + i\kappa_j)(\frac{3}{2} + i\kappa_j)} \alpha_j H_j^3\left(\frac{1}{2}\right) R_1(\kappa_j) \right\} \\ &\quad + O(T^{\sigma_0+1}) \\ &\quad + O\left(\int_{t_0}^{\infty} T^{\frac{3}{2}-C\delta(t)} t^{-2} |\mathcal{Z}_2(\frac{1}{2} - C\delta(t) + it)| dt \right) \end{aligned} \tag{3.7}$$

with $R_1(\kappa_j)$ given by (1.12). Let $\eta(T)$ be defined by (1.11) and put

$$U = U(T) := e^{C\eta(T)} = e^{C \log^{3/5} T (\log \log T)^{-1/5}}.$$

Then

$$\int_{t_0}^{\infty} = \int_{t_0}^U + \int_U^{\infty} \ll T^{3/2} e^{-C\delta(U) \log T} + T^{3/2} U^{\varepsilon - \frac{1}{2}} \ll T^{3/2} e^{-C\eta(T)}, \quad (3.8)$$

since by Theorem 2.1 we have

$$\int_V^{2V} |\mathcal{Z}_2(\tfrac{1}{2} - C\delta(v) + iv)|^2 dv \ll_{\varepsilon} V^{2+\varepsilon}. \quad (3.9)$$

Namely we split the integral in the O -term in (3.7) into subintegrals over $[V, 2V]$. The contour $\sigma = \frac{1}{2} - C\delta(v)$ is replaced by $\sigma = \frac{1}{2} - C\delta(V)$, which is technically easier. In this process we obtain integrals over horizontal segments whose contributions will be $\ll_{\varepsilon} V^{2+\varepsilon}$, since by (5.10) and (5.24) of [9] (with $\xi = \frac{1}{3}$) we have the bound

$$\mathcal{Z}_2(\tfrac{1}{2} - C\delta(v) + iv) \ll_{\varepsilon} v^{1+\varepsilon}.$$

Finally by the Cauchy-Schwarz inequality for integrals and (3.9) we obtain

$$\begin{aligned} \int_1^{\infty} |\mathcal{Z}_2(\tfrac{1}{2} - C\delta(v) + iv)| v^{-2} dv &\ll 1, \\ \int_V^{\infty} |\mathcal{Z}_2(\tfrac{1}{2} - C\delta(v) + iv)| v^{-2} dv &\ll_{\varepsilon} V^{\varepsilon - \frac{1}{2}}. \end{aligned}$$

thereby establishing (3.8) and completing the proof of Theorem 1.1.

In concluding it may be remarked that, similarly as in [5], one may obtain quickly from (3.3) the bound (see (1.3))

$$E_2(T) \ll_{\varepsilon} T^{\frac{2}{3} + \varepsilon}, \quad (3.10)$$

which is (up to “ ε ”) the strongest one known. Namely by [5, (5.3)] we have

$$\begin{aligned} E_2(T) &\leq C_1 H^{-1} \int_T^{T+H} E_2(x) f(x) dx + C_2 H \log^4 T \\ &\quad (C_1, C_2 > 0, 1 \ll H \leq \tfrac{1}{4}T), \end{aligned} \quad (3.11)$$

where $f(x)$ (> 0) is a smooth function supported in $[T, T+H]$, such that $f(x) = 1$ for $T + \frac{1}{4}H \leq x \leq T + \frac{3}{4}H$. Then from (3.3) we have ($\frac{1}{2} < c < 1$)

$$E_2(T) \leq \frac{C_1}{2\pi i H} \int_{(c)} \frac{\mathcal{Z}_2(s)}{s} \int_T^{T+H} f(x) x^s dx ds + C_2 H \log^4 T.$$

We take $c = \frac{1}{2} + \varepsilon$, use (3.5), the Cauchy-Schwarz inequality, and the fact that by r integrations by parts it follows that

$$\begin{aligned} \int_T^{T+H} f(x) x^s dx &= (-1)^r \int_T^{T+H} \frac{x^{s+r}}{(s+1) \dots (s+r)} f^{(r)}(x) dx \\ &\ll_{\sigma, r} T^{\sigma+r} H^{1-r} |t|^{-r}. \end{aligned}$$

Hence the above integral over s may be truncated at $|\Im m s| = T^{1+\varepsilon}H^{-1}$ with a negligible error, and we obtain

$$\begin{aligned} E_2(T) &\ll_{\varepsilon} T^{\frac{1}{2}+\varepsilon} \int_1^{T^{1+\varepsilon}H^{-1}} |\mathcal{Z}_2(\tfrac{1}{2} + \varepsilon + it)| \frac{dt}{t} + H \log^4 T \\ &\ll_{\varepsilon} T^{\varepsilon} (TH^{-\frac{1}{2}} + H) \ll T^{\frac{2}{3}+\varepsilon} \end{aligned}$$

with $H = T^{2/3}$. A lower bound for $E_2(T)$, similar to (3.11), also holds, and therefore (3.10) follows as asserted.

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