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WEYL'S INEQUALITY AND EXPONENTIAL SUMS OVER BINARY FORMS

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1. Introduction

Weyl's estimates for exponential sums have played an influential role in the subsequent development of analytic number theory throughout this century, especially so far as the theory and application of the Hardy-Littlewood method is concerned. Roughly speaking, Weyl [9] shows that exponential sums over polynomials in a single variable are small whenever the leading coefficients of these polynomials are badly approximated by rational numbers with small denominators. While Weyl's methods are by now rather well understood for exponential sums over polynomials in a single variable, the corresponding body of knowledge for polynomials in many variables remains primitive. General methods of Birch [3], Schmidt [7] and Arkhipov, Karatsuba and Chubarikov [1] provide estimates substantially weaker than might be expected by comparison with the situation for a single variable, and until recently it was only for cubic forms satisfying suitable conditions (see Chowla and Davenport [5] and Heath-Brown [6]) and diagonalisable forms (see Birch and Davenport [4]) that one had estimates of quality matching those available for a single variable. The author has very recently obtained a version of Weyl's inequality for exponential sums of the type

$$\sum_{1\le x\le P}\sum_{1\le y\le Q} e(\alpha\Phi(x,y)),$$

in which $\Phi(x,y)$ is a non-degenerate binary form with integral coefficients, and as usual, we write $e(z)$ to denote $e^{2\pi iz}$ (see Wooley [10]). Although the quality

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of this new bound is of strength similar to that of Weyl for a polynomial in a single variable, our conclusions are restricted exclusively to exponential sums over binary *homogeneous* polynomials. Our purpose in this paper is to remedy this technical defect, and thereby permit applications going beyond those described in our earlier work [10].

In order to describe our version of Weyl's inequality, we require a little notational discussion. Suppose that $\Phi(x,y) \in \mathbb{Z}[x,y]$ is a binary form of degree d exceeding 1. Then we say that Φ is *degenerate* if there exist complex numbers α and β such that $\Phi(x,y)$ is identically equal to $(\alpha x+\beta y)^d$. It is easily verified that when $\Phi(x,y)$ is degenerate, then there exist integers a, b and c with $\Phi(x,y) = a(bx+cy)^d$.

Theorem. *Suppose that $\Phi(x,y) \in \mathbb{Z}[x,y]$ is a non-degenerate form of degree $d \geq 3$. Let $\phi(x,y) \in \mathbb{R}[x,y]$ be any polynomial of total degree at most $d-1$. Let $\alpha \in \mathbb{R}$, and suppose that there exist $r \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(r,q)=1$ and $|\alpha - r/q| \leq q^{-2}$. Finally, suppose that P and Q are real numbers sufficiently large in terms of the coefficients of Φ, and satisfying $P \asymp Q$. Then for each $\varepsilon > 0$, one has*

$$\sum_{1\leq x\leq P}\sum_{1\leq y\leq Q} e(\alpha\Phi(x,y)+\phi(x,y)) \ll P^{2+\varepsilon}(q^{-1}+P^{-1}+qP^{-d})^{2^{2-d}}.$$

The conclusion of this theorem is identical with that of [10, Theorem 1] in the special case in which $\phi(x,y)$ is identically zero, and was established in the case $d=3$ by Chowla and Davenport [5] using rather different methods. We note that when d is larger than 12 or thereabouts, a trivial variant of Vinogradov's methods yields an estimate superior to that provided above (see [10, §8] for the relevant ideas).

Our basic strategy for the proof of the above theorem remains the same as that which we wrought to establish [10, Theorem 1], and is described in detail in §§2 and 4 below. By a suitable change of variables and deft use of Hölder's inequality, we are able to estimate the exponential sum occurring in the statement of the theorem in terms of a simpler one amenable to a more efficient Weyl differencing process than would ordinarily be the case. The presence of the term $\phi(x,y)$ offers the possibility that the product of this differencing process will no longer be directly accessible via reciprocal sum technology familiar to practitioners of the circle method. We instead develop, in §3, estimates stemming from the theory of uniform distribution, which ensure either that suitable estimates do indeed hold, or else that the parameter α has good rational approximations. In the latter circumstance one is able to make use of familiar reciprocal sum technology to derive the desired bound.

Throughout this paper, implicit constants occurring in Vinogradov's notation $\ll$ and $\gg$ will depend at most on the coefficients of the implicit binary forms, a small positive number ε, exponents d and k, and quantities occurring as subscripts to the latter notations, unless otherwise indicated. We write $f \asymp g$ when $f \ll g$ and $g \ll f$. Also, we use vector notation for brevity. Thus, for example.

the j-tuple $(h_1, \dots, h_j)$ will be abbreviated simply to $\mathbf{h}$. In an effort to simplify our exposition, we adopt the convention that whenever ε appears in a statement, we are implicitly asserting that the statement holds for each $\varepsilon > 0$. Note that the "value" of ε may consequently change from statement to statement. Finally, we write $\|\alpha\|$ to denote $\min_{y\in\mathbb{Z}} |\alpha - y|$, and adopt the convention that whenever $\|\alpha\|$ is zero, then $\min\{N, \|\alpha\|^{-1}\} = N$.

2. Preliminary reductions

Let k be an integer with $k \geq 3$ and let $\Phi(x,y) \in \mathbb{Z}[x,y]$ be a non-degenerate homogeneous polynomial of degree k. Also, let $\phi(x,y) \in \mathbb{R}[x,y]$ be any polynomial of total degree at most $k-1$. Let P and Q be large real numbers with $P \asymp Q$, and define the exponential sum $F(\alpha) = F(\alpha; P, Q)$ by

$$F(\alpha; P, Q) = \sum_{1\leq x\leq P} \sum_{1\leq y\leq Q} e(\alpha\Phi(x,y) + \phi(x,y)).$$

We aim initially to transform $F(\alpha)$ into a related exponential sum amenable to our modified differencing procedure, and to this end we follow closely the argument of [10, §2]. At this stage our argument is sufficiently close to the latter that we may sacrifice detail in the interests of concision.

When $\Phi(x,y) \in \mathbb{Z}[x,y]$ and $\phi(x,y) \in \mathbb{R}[x,y]$, we describe the pair of polynomials (Ψ, ψ) as being a *condensation* of (Φ, ϕ) when the following condition $(\mathcal{C}^*)$ is satisfied.

$(\mathcal{C}^*)$ We have $\Psi(u,v) \in \mathbb{Z}[u,v]$ and $\psi(u,v) \in \mathbb{R}[u,v]$, the coefficients of Ψ depend at most on those of Φ, and the coefficients of ψ depend at most on those of Φ and ϕ. Further, the polynomial $\Psi(u,v)$ has the same degree as $\Phi(x,y)$, likewise $\psi(u,v)$ has the same degree as $\phi(x,y)$, and $\Psi(u,v)$ takes the shape

$$\Psi(u,v) = Au^k + Bu^{k-t}v^t + \sum_{j=t+1}^{k} C_j u^{k-j} v^j, \tag{2.1}$$

with $AB \neq 0$ and $2 \leq t \leq k$.

Lemma 2.1. *There is a condensation (Ψ, ψ) of (Φ, ϕ), a positive integer D depending at most on the coefficients of Φ, and a positive real number X with $X \asymp P$, satisfying the property that for every real number α one has*

$$|F(\alpha; P, Q)| \ll (\log X)^2 \sup_{\beta,\gamma\in\mathbb{R}} |H(\alpha/D; \beta, \gamma; X)|,$$

where

$$H(\theta; \beta, \gamma; X) = \sum_{|u|\leq X} \sum_{|v|\leq X} e(\theta\Psi(u,v) + \psi(u,v) + \beta u + \gamma v). \tag{2.2}$$

Proof. Following the argument of the proof of [10, Lemma 2.2], it is a consequence of Lemma 2.1 of that paper that there exist integers a, b, c, d, with $ad - bc \neq 0$, for which the polynomial $\Psi(u, v)$ defined by

$$\Psi(u, v) = \Phi(au + bv, cu + dv)$$

satisfies the conditions on Ψ imposed by $(\mathcal{C}^*)$. Write $\Delta = ad - bc$, and define also

$$\psi(u. v) = \phi\Big(\frac{au + bv}{\Delta}, \frac{cu + dv}{\Delta}\Big).$$

Then we find that (Ψ, ψ) is a condensation of (Φ, ϕ). Next observe that

$$F(\alpha; P, Q) = \sum_{1 \leq x \leq P} \sum_{1 \leq y \leq Q} e\left(\alpha\Psi\left(\frac{dx - by}{\Delta}, \frac{ay - cx}{\Delta}\right) + \psi(dx - by, ay - cx)\right).$$

Then on following the argument concluding the proof of [10, Lemma 2.2], we find that

$$|F(\alpha; P, Q)| \ll (\log P)(\log Q) \sup_{\beta, \gamma \in \mathbb{R}} |H(\alpha/\Delta^k; \beta, \gamma; X)|,$$

where

$$X = \max\{|d|P + |b|Q, |c|P + |a|Q\},$$

and the conclusion of the lemma now follows immediately. ∎

We require a modified reciprocal sums lemma similar to that supplied by Lemma 2.2 of Vaughan [8].

Lemma 2.2. *Suppose that* X, Y *and* α *are real numbers with* $X \geq 1$ *and* $Y \geq 1$. *Suppose also that* D *is a positive integer, and that* $a \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *satisfy* $|\alpha - a/q| \leq q^{-2}$ *and* $(a, q) = 1$. *Then*

$$\sum_{1 \leq x \leq X} \min\{XYx^{-1}, \|\alpha x/D\|^{-1}\} \ll XY(Dq^{-1} + Y^{-1} + Dq(XY)^{-1}) \log(2DqX).$$

Proof. This is immediate from the argument of the proof of [10, Lemma 3.1]. ∎

We next recall the Weyl differencing lemma. Let Δ_j denote the jth iterate of the forward differencing operator, so that for any function Ω of a real variable α, one has

$$\Delta_1(\Omega(\alpha); \beta) = \Omega(\alpha + \beta) - \Omega(\alpha).$$

and when j is a natural number,

$$\Delta_{j+1}(\Omega(\alpha); \beta_1, \ldots, \beta_{j+1}) = \Delta_1(\Delta_j(\Omega(\alpha); \beta_1, \ldots, \beta_j); \beta_{j+1}).$$

We adopt the convention that $\Delta_0(\Omega(\alpha); \beta) = \Omega(\alpha)$.

Lemma 2.3. *Let X be a positive real number, and let $\Omega(x)$ be an arbitrary arithmetical function. Write*

$$T(\Omega) = \sum_{|x| \le X} e(\Omega(x)).$$

Then for each natural number j there exist intervals $I_i = I_i(\mathbf{h})$ $(1 \le i \le j)$, possibly empty, satisfying

$$I_1(h_1) \subseteq [-X, X] \quad \text{and} \quad I_i(h_1, \dots, h_i) \subseteq I_{i-1}(h_1, \dots, h_{i-1}) \quad (2 \le i \le j),$$

with the property that

$$|T(\Omega)|^{2^j} \le (4X+1)^{2^j - j - 1} \sum_{|h_1| \le 2X} \dots \sum_{|h_j| \le 2X} T_j,$$

and here we write

$$T_j = \sum_{x \in I_j \cap \mathbb{Z}} e(\Delta_j(\Omega(x); h_1, \dots, h_j)).$$

Proof. This trivial variant of Lemma 2.3 of Vaughan [8] is recorded as Lemma 3.2 of [10]. ■

In what follows, it is convenient to define also a two dimensional forward differencing operator $\Delta_{i,j}$ as follows. When $\Omega(x, y)$ is a function of the real variables x and y, one defines

$$\Delta_{1,0}(\Omega(x, y); \beta) = \Omega(x + \beta, y) - \Omega(x, y)$$

and

$$\Delta_{0,1}(\Omega(x, y); \gamma) = \Omega(x, y + \gamma) - \Omega(x, y).$$

When i and j are non-negative integers, one then defines

$$\Delta_{i,j}(\Omega(x, y); \beta_1, \dots, \beta_i; \gamma_1, \dots, \gamma_j)$$

by taking $\Delta_{0,0}(\Omega(x, y); \beta; \gamma) = \Omega(x, y)$, and in general by means of the relations

$$\begin{aligned} &\Delta_{i+1,j}(\Omega(x, y); \beta_1, \dots, \beta_{i+1}; \gamma_1, \dots, \gamma_j) \\ &\qquad = \Delta_{1,0}(\Delta_{i,j}(\Omega(x, y); \beta_1, \dots, \beta_i; \gamma_1, \dots, \gamma_j); \beta_{i+1}) \end{aligned}$$

and

$$\begin{aligned} &\Delta_{i,j+1}(\Omega(x, y); \beta_1, \dots, \beta_i; \gamma_1, \dots, \gamma_{j+1}) \\ &\qquad = \Delta_{0,1}(\Delta_{i,j}(\Omega(x, y); \beta_1, \dots, \beta_i; \gamma_1, \dots, \gamma_j); \gamma_{j+1}). \end{aligned}$$

Further, we define a second two-dimensional forward differencing operator Δ_i^* as follows. When $\Omega(x,y)$ is a function of the real variables x and y, one defines

$$\Delta_1^*(\Omega(x,y);\beta;\gamma) = \Omega(x+\beta, y+\gamma) - \Omega(x,y).$$

When i is a non-negative integer, one then defines

$$\Delta_i^*(\Omega(x,y);\beta_1,\dots,\beta_i;\gamma_1,\dots,\gamma_i)$$

by taking $\Delta_0^*(\Omega(x,y);\beta;\gamma) = \Omega(x,y)$, and in general by means of the relation

$$\Delta_{i+1}^*(\Omega(x,y);\beta_1,\dots,\beta_{i+1};\gamma_1,\dots,\gamma_{i+1})$$
$$= \Delta_1^*(\Delta_i^*(\Omega(x,y);\beta_1,\dots,\beta_i;\gamma_1,\dots,\gamma_i);\beta_{i+1};\gamma_{i+1}).$$

Plainly, if one specialises variables suitably in the operator Δ_{i+j}^*, then one obtains the operator $\Delta_{i,j}$. It is convenient, however, to distinguish the two operators as above.

An obvious variant of the argument leading to Lemma 2.3 yields the following Weyl differencing lemma.

Lemma 2.4. *Let X be a positive real number, and let $\Omega(x,y)$ be an arbitrary arithmetical function. Write*

$$T(\Omega) = \sum_{|x|\le X}\sum_{|y|\le X} e(\Omega(x,y)).$$

Then for each natural number j there exist rectangles $I_i = I_i(\mathbf{g};\mathbf{h})$ $(1\le i\le j)$, possibly empty, satisfying

$$I_1(g_1;h_1) \subseteq [-X,X]^2$$

and

$$I_i(g_1,\dots,g_i;h_1,\dots,h_i) \subseteq I_{i-1}(g_1,\dots,g_{i-1};h_1,\dots,h_{i-1}) \quad (2\le i\le j),$$

with the property that

$$|T(\Omega)|^{2^j} \le \left((4X+1)^2\right)^{2^j-j-1} \sum_{|g_1|,|h_1|\le 2X} \dots \sum_{|g_j|,|h_j|\le 2X} T_j,$$

and here we write

$$T_j = \sum_{(x,y)\in I_j\cap\mathbb{Z}^2} e(\Delta_j^*(\Omega(x,y);g_1,\dots,g_j;h_1,\dots,h_j)).$$

3. Reciprocal sum estimates

In order to bring our differencing argument to a successful conclusion, we require a reciprocal sum estimate not immediately available from the literature. Fortunately, the estimate that we seek is readily extracted from §3.2 of Baker [2]. We begin with a reciprocal sum lemma which yields diophantine approximations.

Lemma 3.1. *Suppose that δ is a positive number, and that α and β are real numbers. Let N, R and B be positive real numbers with $B \gg N^{1+\delta} + R^{1+\delta}$. Suppose further that*

$$\sum_{1 \le z \le R} \min\{N, \|z\alpha + \beta\|^{-1}\} \gg B.$$

Then there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$(a, q) = 1, \quad 1 \le q \ll NRB^{-1} \quad \text{and} \quad |q\alpha - a| < N^\delta B^{-1}. \tag{3.1}$$

Proof. This is Lemma 3.3 of [2]. ■

We also require a modification of Lemma 3.2 of [2], this following from a standard transference principle. First we recall the latter lemma.

Lemma 3.2. *Suppose that α and β are real numbers. Let N and R be positive real numbers, and write $A = N + R$. Suppose further that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ and $|\alpha - a/q| \le q^{-2}$. Then one has*

$$\sum_{1 \le z \le R} \min\{N, \|z\alpha + \beta\|^{-1}\} \ll (\log A)(A + q + NR/q).$$

Proof. Under the hypotheses of the lemma, but subject instead to $|\alpha - a/q| < q^{-2}$, it follows from [2, Lemma 3.2] that

$$\sum_{1 \le z \le R} \min\{N, \|z\alpha + \beta\|^{-1}\} \ll (N + q \log q)(R/q + 1).$$

The aforementioned condition may be replaced by $|\alpha - a/q| \le q^{-2}$ in Baker's argument without loss. Moreover, since the conclusion of the lemma is trivial for $q > NR$, we may suppose instead that $q \le NR$, and hence that $\log q \ll \log A$. The desired conclusion follows immediately. ■

Lemma 3.3. *Suppose that α and β are real numbers. Let N and R be positive real numbers, and write $A = N + R$. Suppose further that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ and $0 < |\alpha - a/q| \le q^{-2}$. Then*

$$\sum_{1 \le z \le R} \min\{N, \|z\alpha + \beta\|^{-1}\} \ll (\log A)\left(A + \|q\alpha\|^{-1} + NR|q\alpha - a|\right).$$

Proof. By Dirichlet's theorem on diophantine approximation, there exist $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ with

$$(b, r) = 1, \quad 1 \le r \le 2|q\alpha - a|^{-1} \quad \text{and} \quad |r\alpha - b| \le \tfrac{1}{2}|q\alpha - a|. \tag{3.2}$$

Notice, in particular, that the latter inequality ensures that $r \neq q$, whence $a/q \neq b/r$. The triangle inequality therefore yields

$$\frac{1}{qr} \leq \left| \frac{b}{r} - \frac{a}{q} \right| \leq \left| \alpha - \frac{b}{r} \right| + \left| \alpha - \frac{a}{q} \right| \leq \left(\frac{1}{2r} + \frac{1}{q} \right) |q\alpha - a|.$$

But $|\alpha - a/q| \leq q^{-2}$, so that

$$1 \leq \left(r + \tfrac{1}{2} q \right) |q\alpha - a| \leq \tfrac{1}{2} + r|q\alpha - a|.$$

Thus we deduce that

$$r \geq (2|q\alpha - a|)^{-1}. \tag{3.3}$$

Next we observe that $|\alpha - b/r| \leq r^{-2}$ and $(b, r) = 1$, so that in view of (3.2) and (3.3), one finds that Lemma 3.2 yields

$$\begin{aligned} \sum_{1 \leq z \leq R} \min\{N, \|z\alpha + \beta\|^{-1}\} &\ll (\log A)\,(A + r + NR/r) \\ &\ll (\log A)\left(A + |q\alpha - a|^{-1} + NR|q\alpha - a|\right). \end{aligned}$$

But $|q\alpha - a| \geq \|q\alpha\|$, and hence the conclusion of the lemma follows immediately.■

Either Lemma 3.1 provides a satisfactory estimate for the reciprocal sum of interest to us, or else it provides a diophantine approximation which may be converted into a satisfactory estimate via Lemma 3.3.

Lemma 3.4. *Suppose that δ is a positive number, and that α and β are real numbers. Let N and R be large real numbers, and write $B = N^{1+\delta} + R^{1+\delta}$. Then*

$$\sum_{1 \leq z \leq R} \min\{N, \|z\alpha + \beta\|^{-1}\} \ll B + (\log B) \sum_{1 \leq q \leq BN^{-\delta}} \min\{NR/q, \|q\alpha\|^{-1}\}. \tag{3.4}$$

Proof. If the left hand side of (3.4) is at most B, then the conclusion of the lemma is immediate. Then we may suppose that

$$\sum_{1 \leq z \leq R} \min\{N, \|z\alpha + \beta\|^{-1}\} \gg B,$$

whence Lemma 3.1 shows that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying (3.1). In particular, it is apparent that $|\alpha - a/q| \leq q^{-2}$ and $(a, q) = 1$, and so it follows from Lemma 3.2 that

$$\sum_{1 \leq z \leq R} \min\{N, \|z\alpha + \beta\|^{-1}\} \ll B + (\log B)(q + NR/q).$$

Furthermore, when $\alpha \neq a/q$, we deduce from Lemma 3.3 that

$$\sum_{1\le z\le R} \min\{N, \|z\alpha+\beta\|^{-1}\} \ll B + (\log B)\left(\|q\alpha\|^{-1} + NR|q\alpha - a|\right).$$

Since (3.1) implies that

$$q \le \min\{B(\log B)^{-1}, BN^{-\delta}\} \quad \text{and} \quad |q\alpha - a| < N^{\delta}B^{-1},$$

we deduce that in any case,

$$\begin{aligned}\sum_{1\le z\le R} \min\{N, \|z\alpha+\beta\|^{-1}\} &\ll B + (\log B)\min\{NR/q, \|q\alpha\|^{-1}\}\\ &\le B + (\log B)\sum_{1\le q\le BN^{-\delta}} \min\{NR/q, \|q\alpha\|^{-1}\}.\end{aligned}$$

This completes the proof of the lemma. ∎

4. Differencing the auxiliary exponential sum

Our primary apparatus now assembled in the previous sections, we begin our differencing process in earnest. We start by considering the situation in which $\Psi(u,v)$ takes the shape (2.1) with $t = k$, so that for fixed integers A and B depending at most on the coefficients of Φ, one has $\Psi(u,v) = Au^k + Bv^k$. In this case the argument is classical and routine. On recalling the definition (2.2) and applying Lemma 2.4, we obtain

$$|H(\alpha/D;\beta,\gamma;X)|^{2^{k-1}} \ll X^{2^k-2k} \sum_{\mathbf{g}\in[-2X,2X]^{k-1}} \sum_{\mathbf{h}\in[-2X,2X]^{k-1}} \sum_{(x,y)\in I_{k-1}} e(\Upsilon_{k-1}). \tag{4.1}$$

where $I_{k-1} = I_{k-1}(\mathbf{g},\mathbf{h})$ is a rectangular set of integer pairs contained in $[-X,X]^2$, and

$$\Upsilon_{k-1} = \Delta^*_{k-1}\left(\frac{\alpha}{D}(Ax^k + By^k) + \psi(x,y) + \beta x + \gamma y; \mathbf{g}; \mathbf{h}\right).$$

Since $\psi(x,y)$ has degree $k-1$, it is apparent from the definition of Δ^*_j that

$$\Delta^*_{k-1}(\psi(x,y) + \beta x + \gamma y; \mathbf{g}; \mathbf{h})$$

is independent of x and y. A simple calculation, moreover, reveals that

$$\Delta^*_{k-1}(Ax^k + By^k; \mathbf{g}; \mathbf{h}) = Ap_{k-1}(x;\mathbf{g}) + Bp_{k-1}(y;\mathbf{h}),$$

where

$$p_{k-1}(z;\mathbf{m}) = \tfrac{1}{2}k!m_1\dots m_{k-1}(2z + m_1 + \dots + m_{k-1}).$$

The number of terms $x, \mathbf{g}$ counted by the summation in (4.1) with $g_1 \dots g_{k-1}$ equal to zero is $O(X^{k-1})$, and similarly for $y, \mathbf{h}$. Thus, on applying a familiar bound to estimate the sums over x and y in (4.1), and making use of a simple estimate for the divisor function, we obtain

$$|H(\alpha/D;\beta,\gamma;X)|^{2^{k-1}} \ll X^{2^k-2k}\Big(X^{k-1} + X^{\varepsilon}\sum_{1\le m\le M}\min\{X, \|m\alpha/D\|^{-1}\}\Big)^2, \tag{4.2}$$

where $M = \max\{|A|,|B|\}k!(2X)^{k-1}$. Suppose that α satisfies the hypotheses of the statement of the theorem. Then on recalling that D depends at most on the coefficients of Φ, an application of Lemma 2.2 to (4.2) reveals that when $1 \le q < X^k$, one has

$$|H(\alpha/D;\beta,\gamma;X)| \ll X^{2+\varepsilon}(q^{-1} + X^{-1} + qX^{-k})^{2^{2-k}}. \tag{4.3}$$

When $q \ge X^k$, meanwhile, we may appeal to the trivial estimate

$$|H(\alpha/D;\beta,\gamma;X)| \le (2X+1)^2,$$

and hence the estimate (4.3) holds no matter how large q may be.

Consider next the situation in which $\Psi(u,v)$ takes the shape (2.1) with $2 \le t \le k-1$. We first view the exponential sum $H(\theta;\beta,\gamma;X)$ as a sum over v, so that on applying Hölder's inequality, and then making use of Lemma 2.3, we deduce that

$$\begin{aligned}|H(\theta;\beta,\gamma;X)|^{2^{t-1}} &\ll X^{2^{t-1}-1}\sum_{|u|\le X}\Big|\sum_{|v|\le X} e(\theta\Psi(u,v) + \psi(u,v) + \gamma v)\Big|^{2^{t-1}} \\ &\ll X^{2^t-t-1}\sum_{\mathbf{h}\in[-2X,2X]^{t-1}}\sum_{v\in I(\mathbf{h})}|K(\theta;X;\mathbf{h};v)|,\end{aligned} \tag{4.4}$$

where $I(h_1,\dots,h_{t-1})$ is an interval of integers contained in $[-X,X]$, and

$$K(\theta;X;\mathbf{h};v) = \sum_{|u|\le X} e\left(\Delta_{0,t-1}(\theta\Psi(u,v) + \psi(u,v);\mathbf{h})\right).$$

But now applying Lemma 2.3 to the latter exponential sum, we obtain

$$|K(\theta;X;\mathbf{h};v)|^{2^{k-t-1}} \ll X^{2^{k-t-1}-k+t}\sum_{\mathbf{g}\in[-2X,2X]^{k-t-1}}|L(\theta;X;\mathbf{g},\mathbf{h};v)|, \tag{4.5}$$

where

$$L(\theta;X;\mathbf{g},\mathbf{h};v) = \sum_{u\in J(\mathbf{g})} e\left(\Delta_{k-t-1,t-1}(\theta\Psi(u,v) + \psi(u,v);\mathbf{g};\mathbf{h})\right). \tag{4.6}$$

and $J(g_1, \dots, g_{k-t-1})$ is an interval of integers contained in $[-X, X]$. On combining (4.4) and (4.5) through the medium of Hölder's inequality, we conclude thus far that

$$|H(\theta;\beta,\gamma;X)|^{2^{k-2}} \ll X^{2^{k-1}-k} \sum_{\mathbf{g}\in[-2X,2X]^{k-t-1}} \sum_{\mathbf{h}\in[-2X,2X]^{t-1}} \sum_{v\in I(\mathbf{h})} |L(\theta;X;\mathbf{g},\mathbf{h};v)|. \tag{4.7}$$

We next examine the argument of the exponential sum L resulting from our differencing procedure. Suppose that the coefficient of $u^{k-t-1}v^t$ in $\psi(u,v)$ is λ, that the corresponding coefficient of $u^{k-t}v^{t-1}$ is μ, and that the corresponding coefficient of $u^{k-t-1}v^{t-1}$ is κ. Then in view of (2.1), a modicum of computation reveals that

$$\begin{aligned}
&\Delta_{k-t-1,t-1}(\theta\Psi(u,v)+\psi(u,v);\mathbf{g};\mathbf{h})\\
&\quad=\theta\left(B\Delta_{k-t-1}(u^{k-t};\mathbf{g})\Delta_{t-1}(v^t;\mathbf{h})+C_{t+1}\Delta_{k-t-1}(u^{k-t-1};\mathbf{g})\Delta_{t-1}(v^{t+1};\mathbf{h})\right)\\
&\qquad+\lambda\Delta_{k-t-1}(u^{k-t-1};\mathbf{g})\Delta_{t-1}(v^t;\mathbf{h})+\mu\Delta_{k-t-1}(u^{k-t};\mathbf{g})\Delta_{t-1}(v^{t-1};\mathbf{h})\\
&\qquad+\kappa\Delta_{k-t-1}(u^{k-t-1};\mathbf{g})\Delta_{t-1}(v^{t-1};\mathbf{h}).
\end{aligned}$$

Moreover, a simple calculation reveals that $\Delta_{k-t-1}(u^{k-t-1};\mathbf{g})$ is independent of u, and further that

$$\Delta_{k-t-1}(u^{k-t};\mathbf{g})=\tfrac{1}{2}(k-t)!g_1\dots g_{k-t-1}(2u+g_1+\dots+g_{k-t-1}).$$

Also, one has

$$\Delta_{t-1}(v^{t-1};\mathbf{h})=(t-1)!h_1\dots h_{t-1}.$$

and

$$\Delta_{t-1}(v^t;\mathbf{h})=\tfrac{1}{2}t!h_1\dots h_{t-1}(2v+h_1+\dots+h_{t-1}).$$

Write

$$\begin{aligned}
&\nu(\theta,\mu;\mathbf{g},\mathbf{h};v)\\
&\quad=(k-t)!(t-1)!h_1\dots h_{t-1}g_1\dots g_{k-t-1}\left(\tfrac{1}{2}Bt(2v+h_1+\dots+h_{t-1})\theta+\mu\right).
\end{aligned}$$

Then it follows from (4.6) that whenever $\nu(\theta,\mu;\mathbf{g},\mathbf{h};v)$ is non-zero, one has

$$|L(\theta;X;\mathbf{g},\mathbf{h};v)|\ll\min\{X,\|\nu(\theta,\mu;\mathbf{g},\mathbf{h};v)\|^{-1}\}.$$

But as a consequence of Lemma 3.4, one has

$$\begin{aligned}
&\sum_{v\in I(\mathbf{h})}\min\{X,\|\nu(\theta,\mu;\mathbf{g},\mathbf{h};v)\|^{-1}\}\\
&\quad\ll X^{1+\varepsilon}+X^{\varepsilon}\sum_{1\le r\le X}\min\{X^2/r,\|rB(k-t)!t!h_1\dots h_{t-1}g_1\dots g_{k-t-1}\theta\|^{-1}\}.
\end{aligned}$$

whence an elementary divisor function estimate leads to the upper bound

$$\sum_{\mathbf{g}\in[-2X,2X]^{k-t-1}} \sum_{\mathbf{h}\in[-2X,2X]^{t-1}} \sum_{v\in I(\mathbf{h})} |L(\theta; X; \mathbf{g}, \mathbf{h}; v)| \\ \ll X^{k-1+\varepsilon} + X^{\varepsilon} \sum_{1\le m\le M} \min\{X^k/m, \|m\theta\|^{-1}\},$$

where $M = t!(k-t)!|B|(2X)^{k-1}$. Thus we deduce from (4.7) that

$$|H(\alpha/D; \beta, \gamma; X)|^{2^{k-2}} \\ \ll X^{2^{k-1}-1+\varepsilon} + X^{2^{k-1}-k+\varepsilon} \sum_{1\le m\le M} \min\{X^k/m, \|m\alpha/D\|^{-1}\}.$$

But by Lemma 2.2, under the hypotheses of the statement of the theorem, one has

$$\sum_{1\le m\le M} \min\{X^k/m, \|m\alpha/D\|^{-1}\} \ll X^k(q^{-1} + X^{-1} + qX^{-k}) \log(2qX),$$

and thus, when $1 \le q \le X^k$, we arrive at the upper bound

$$|H(\alpha/D; \beta, \gamma; X)| \ll X^{2+\varepsilon}(q^{-1} + X^{-1} + qX^{-k})^{2^{2-k}}.$$

When $q > X^k$, meanwhile, the latter estimate follows from a trivial upper bound for $H(\alpha/D; \beta, \gamma; X)$.

On recalling the treatment of the diagonal case concluding with (4.3), we have now only to apply Lemma 2.1 in order to deduce that under the hypotheses of the statement of the theorem, one has

$$|F(\alpha; P, Q)| \ll X^{2+\varepsilon}(q^{-1} + X^{-1} + qX^{-k})^{2^{2-k}},$$

which provides the desired conclusion.

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