Functiones et Approximatio
XXVIII (2000), 35–43

To Professor Włodzimierz Staś on his 75th birthday

TAME KERNELS OF QUADRATIC NUMBER FIELDS: NUMERICAL HEURISTICS

JERZY BROWKIN

Abstract: Basing on conjectures given by H. Cohen, H.W. Lenstra, Jr. and J. Martinet [2], [3], [4], [5] concerning the heuristics on class groups of number fields we deduce some quantitative conjectures on the statistical behaviour of orders of the tame kernel $K_2\mathcal{O}_F$ of the ring $\mathcal{O}_F$ of integers of quadratic number fields F of discriminants D, $|D| \le x$.

We investigate the number of D's such that for $F = \mathbb{Q}(\sqrt{D})$ the order of $K_2\mathcal{O}_F$ is divisible by 3.
Keywords: Tame kernel, quadratic fields, numerical heuristics.

1. Introduction

First we prove an asymptotic formula for the number of fundamental discriminants D of quadratic number fields satisfying $|D| \le x$ and belonging to a fixed arithmetic progression $D \equiv l \pmod{k}$.

Let k, l be integers, $k > 0$. Denote by $D(x,k,l)$, (resp. by $Q(x,k,l)$) the number of fundamental discriminants D of quadratic number fields (resp. the number of squarefree integers D) satisfying

$$0 < D \le x \qquad \text{and} \qquad D \equiv l \pmod{k}.$$

Let

$$d = \gcd(k,l),$$
$$p(k) = \prod_{\substack{p|k \\ p-\text{prime}}} (1-p^{-2})^{-1},$$
$$q(k,l) = \frac{1}{dk}\varrho p(k),$$

2000 Mathematics Subject Classification: Primary 11R11; Secondary 11R70, 11Y40

where

$$\varrho = \varrho(k,l) = \frac{d}{P}\varphi(P),$$

and $P = P(k,l)$ is the product of all prime divisors of d, which do not divide k/d.

Lemma 1.1. (E. Landau [6]) *In the above notation we have*

$$Q(x,k,l) = \frac{6}{\pi^2} q(k,l) \cdot x + o(x),$$

if d is squarefree, and

$$Q(x,k.l) = 0$$

otherwise. ■

Example. $q(4,l) = \frac{1}{3}$, for $l = 1,2,3$, and $q(4,0) = 0$.

Let us remark that $q(k,l)$ depends only on k and $d = \gcd(k,l)$, and not on the particular value of l.

Lemma 1.2. *The function $q(k,l)$ is multiplicative with respect to the first argument:*

If

$$l \equiv l_1 \pmod{k_1},$$

$$l \equiv l_2 \pmod{k_2},$$

where $\gcd(k_1,k_2) = 1$, and $k = k_1 \cdot k_2$, then

$$q(k,l) = q(k_1,l_1) \cdot q(k_2,l_2).$$

Proof. Let $d_i = \gcd(k_i,l_i)$, $\varrho_i = \varrho(k_i,l_i)$, $P_i = P(k_i,l_i)$, for $i = 1,2$.

Then evidently

$$d = d_1 \cdot d_2, \text{ with } \gcd(d_1,d_2) = 1,$$

$$p(k) = p(k_1) \cdot p(k_2),$$

$$P = P_1 \cdot P_2, \quad \varrho = \varrho_1 \cdot \varrho_2,$$

and the lemma follows. ■

Thus it is sufficient to determine $q(k,l)$, for k being a prime power.

Example. From the definition of $q(k,l)$ we get

1) $k=p$: $q(p,1)=p/(p^2-1)$, $q(p,0)=1/(p+1)$.
2) $k=p^2$: $q(p^2,1)=q(p^2,p)=1/(p^2-1)$, $q(p^2,p^2)=0$.
3) $k=p^n$, $(n\geq 3)$: $q(p^n,1)=q(p^n,p)=1/p^{n-2}(p^2-1)$, $q(p^n,p^r)=0$, for $r\geq 2$.

Theorem 1.3. *Let* $k=2^\alpha k'$, *where* $\alpha\geq 0$, *and* $2\nmid k'$.
Then

$$D(x,k,l)=q(k',l)\cdot\delta(2^\alpha,l)\cdot\frac{6}{\pi^2}x+o(x),$$

where $\delta(2^\alpha,l)$ *is equal respectively*

$$\begin{array}{rl}
\frac{1}{2} & \text{if}\quad \alpha=0,\\
\frac{1}{3} & \text{if}\quad \alpha=1,\quad l \text{ is odd},\\
\frac{1}{3\cdot 2^{\alpha-2}} & \text{if}\quad \alpha\geq 2,\quad l\equiv 1 \pmod 4,\\
\frac{1}{6} & \text{if}\quad \alpha=1 \text{ or } 2, \text{ and } l\equiv 0 \pmod{2^\alpha},\\
\frac{1}{12} & \text{if}\quad \alpha=3,\quad l\equiv 0,4 \pmod 8,\\
\frac{1}{3\cdot 2^{\alpha-2}} & \text{if}\quad \alpha\geq 4,\quad l\equiv 8,12 \pmod{16},\\
0 & \text{otherwise.}
\end{array}$$

Proof. We look for fundamental discriminants D satisfying

(0) $D\equiv l \pmod k$, or equivalently

(1) $D\equiv l \pmod{2^\alpha}$,

(2) $D\equiv l \pmod{k'}$.

Fundamental discriminants of quadratic number fields are characterized by the condition:

(3) $D\equiv 1 \pmod 4$, D squarefree,

or

(4) $D\equiv 0 \pmod 4$, $D/4\equiv 2,3 \pmod 4$, $D/4$ squarefree.

Therefore from (1) we get a necessary condition for l :

(5) $l\equiv 1 \pmod{2^{\min(2,\alpha)}}$,

or

(6) $l\equiv 8,12 \pmod{2^{\min(4,\alpha)}}$.

If this condition is not satisfied, then there are no fundamental discriminants satisfying (0).

We consider separately several cases.

1) $\alpha=0$.

Then the number of fundamental discriminants D satisfying (1), (2), and (3) or (4) is

$$q(k',l)\cdot q(4,1)\cdot\frac{6}{\pi^2}x+q(k',l)\cdot(q(4,2)+q(4,3))\frac{6}{\pi^2}\cdot\frac{x}{4}+o(x)=q(k,l)\cdot\frac{3}{\pi^2}\cdot x+o(x).$$

2) l odd, $\alpha\geq 1$.

$\underline{\alpha=1.}$

Then (1) implies that $D\equiv 1 \pmod 4$, D squarefree. Consequently the number of fundamental discriminants in question is

$$q(k',l)\cdot q(4,1)\cdot\frac{6}{\pi^2}x+o(x)=q(k',l)\cdot\frac{2}{\pi^2}x+o(x).$$

$\underline{\alpha\geq 2.}$

From the necessary condition we get $l\equiv 1 \pmod 4$. Hence $D\equiv l \pmod{2^\alpha}$, and the number of fundamental discriminants is

$$q(k',l)\cdot q(2^\alpha,1)\cdot\frac{6}{\pi^2}x+o(x)=q(k',l)\cdot\frac{1}{2^{\alpha-3}\pi^2}x+o(x).$$

3) l is even, $\alpha\geq 1$.

$\underline{\alpha=1.}$

From (6) we get $l\equiv 0 \pmod 2$. Hence (4) holds, and the number of fundamental discriminants is

$$q(k',l)\cdot\big(q(4,2)+q(4,3)\big)\cdot\frac{6}{\pi^2}\cdot\frac{x}{4}+o(x)=q(k',l)\cdot\frac{1}{\pi^2}x+o(x).$$

$\underline{\alpha=2.}$

Now (6) implies $l\equiv 0 \pmod 4$, and we get the same number of discriminants as in the latter case.

$\underline{\alpha=3.}$

From (6) we get $l\equiv 0,\ 4 \pmod 8$. Hence $D/4\equiv 2,\ 3 \pmod 8$ respectively, and in both cases the number of fundamental discriminants is

$$q(k',l)\cdot q(4,2)\cdot\frac{6}{\pi^2}\cdot\frac{x}{4}+o(x)=q(k',l)\cdot\frac{1}{2\pi^2}x+o(x),$$

since $q(4,2)=q(4,3)$.

$\underline{\alpha\geq 4.}$

Now (6) implies $l\equiv 8,\ 12 \pmod{16}$, hence $D/4\equiv l/4 \pmod{2^{\alpha-2}}$, i.e. $D/4\equiv 2,\ 3 \pmod{2^{\alpha-2}}$, $D/4$ squarefree. The number of such D's in both cases is

$$q(k',l)\cdot q(2^{\alpha-2},2)\cdot\frac{6}{\pi^2}\cdot\frac{x}{4}+o(x)=q(k',l)\cdot\frac{1}{2^{\alpha-4}\cdot 3}\cdot\frac{6}{\pi^2}\cdot\frac{x}{4}+o(x)=$$

$$=q(k',l)\cdot\frac{1}{2^{\alpha-3}\pi^2}x+o(x),$$

since $q(2^{\alpha-2},2)=q(2^{\alpha-2},3)$. ∎

Remark. *It is clear that the same formula holds for the number of negative fundamental discriminants D satisfying $|D| \leq x$ and $D \equiv l \pmod{k}$.*

Corollary. *The function*

$$\Delta(x,k,l) := \frac{6}{\pi^2} \cdot \frac{D(x,k,l)}{x}$$

is multiplicative with respect to k asymptotically, i.e. if

$$k = k_1 k_2, \quad \text{and} \quad \gcd(k_1, k_2) = 1,$$

then

$$\Delta(x,k,l) - \Delta(x,k_1,l) \cdot \Delta(x,k_2,l) \to 0, \quad \text{as } x \to \infty.$$

Proof. We can assume that k_2 is odd, then $k = 2^\alpha k'$, $k_1 = 2^\alpha k_1'$, $k_2 = k_2'$, $k' = k_1' k_2'$, and $\gcd(k_1', k_2') = 1$. Consequently

$$\Delta(x,k,l) \to q(k',l) \cdot \delta(2^\alpha, l) = q(k_1', l) \cdot \delta(2^\alpha, l) \cdot q(k_2', l),$$

by Lemma 1.2.

On the other hand

$$\Delta(x,k_1,l) \to q(k_1',l)\delta(2^\alpha . l),$$

$$\Delta(x,k_2,l) \to q(k_2',l).$$

Hence the result. ■

2. The divisibility by 3 of order of $K_2\mathcal{O}_F$

For $a \in \mathbb{Z}$, not a square, let $F = \mathbb{Q}(\sqrt{a})$ and let $\mathcal{O}_F$ be the ring of integers of F. Denote by $K_2\mathcal{O}_F$ the tame kernel of $\mathcal{O}_F$.

Let $h(a)$ be the class number of F, and $k(a)$ the order of $K_2\mathcal{O}_F$.

Assuming a conjecture of Cohen and Martinet we shall determine the number of fundamental discriminants D of quadratic number fields satisfying

(7) $$0 < D \leq x \quad \text{and} \quad 3|k(D).$$

We use the following theorem, which follows from Theorem 5.6 in [1].

Theorem 2.1. *Let $F = \mathbb{Q}(\sqrt{d})$, and $E = \mathbb{Q}(\sqrt{-3d})$, where d is a squarefree integer.*

(*i*) *If $d \not\equiv 6 \pmod{9}$ then*

$$3|k(d) \text{ iff } 3|h(-3d)$$

(*ii*) *If $d \equiv 6 \pmod{9}$, then*

$$3|k(d).$$

■

We assume the following conjecture of H. Cohen and J. Martinet (see [5], p. 330) :

Conjecture. *For fixed natural numbers m and k, the positive (resp. negative) fundamental discriminants D satisfying*

$$m|h(D)$$

are uniformly distributed in arithmetical progressions with the difference k.

More precisely, if for fixed k and l there is a fundamental discriminant D satisfying $D \equiv l \pmod{k}$, and e.g. $D < 0$, then

$$\lim_{x\to\infty} \frac{\#\{|D| \le x : D < 0,\ D - \text{fundamental},\ D \equiv l \pmod{k}, m|h(D)\}}{\#\{|D| \le x : D < 0,\ D - \text{fundamental},\ D \equiv l \pmod{k}\}} =: \gamma_-(m)$$

exists and does not depend on k and l.

The corresponding constant for $D > 0$ we denote by $\gamma_+(m)$.

All pairs k, l satisfying the above assumption are described in Theorem 1.3 above.

The values of conjectural constants $\gamma_\pm(m)$ are given in the paper [3]. E.g.

$$\gamma_-(3) \approx 0.439874 \quad \text{and} \quad \gamma_+(3) \approx 0.159811.$$

Let d be the squarefree kernel of a fundamental discriminant $D > 0$, and let D' be the discriminant of the imaginary field $\mathbb{Q}(\sqrt{-3D})$. Thus

$$D' = \begin{cases} -3D, & \text{if } 3\nmid D, \\ -D/3, & \text{if } 3|D. \end{cases}$$

To estimate the number of D's satisfying (7) we shall consider several cases.

<u>1° $\quad d \equiv 1 \pmod{3}$.</u>

Then

$$D = \begin{cases} d, & \text{if } d \equiv 1 \pmod{12}, \\ 4d, & \text{if } d \equiv 7 \text{ or } 10 \pmod{12} \end{cases} \equiv \begin{cases} 1 & \pmod{12}, \\ 28 \text{ or } 40 & \pmod{48}. \end{cases}$$

and

$$D' = -3D \equiv \begin{cases} -3 & \pmod{36}, \\ -84 \text{ or } -120 & \pmod{144}. \end{cases}$$

Hence (7) is equivalent to

$$|D'| \le 3x \quad \text{and} \quad 3|h(D').$$

Thus the number of D's in question is respectively

$$D(3x, 36, -3)\gamma + o(x) = q(9,3)\cdot\delta(4,1)\cdot\frac{6}{\pi^2}\cdot 3x\gamma + o(x) = \frac{3x}{4\pi^2}\gamma + o(x),$$

$$D(3x, 144, -84)\gamma + o(x) = q(9,3)\cdot\delta(16,12)\cdot\frac{6}{\pi^2}\cdot 3x\gamma + o(x) = \frac{3x}{16\pi^2}\gamma + o(x),$$

$$D(3x, 144, -120)\gamma + o(x) = q(9,3)\cdot\delta(16,8)\cdot\frac{6}{\pi^2}\cdot 3x\gamma + o(x) = \frac{3x}{16\pi^2}\gamma + o(x),$$

where $\gamma = \gamma_-(3)$.

2° $d \equiv 2 \pmod 3$.

Then

$$D = \begin{cases} d, & \text{if } d \equiv 5 \pmod{12}, \\ 4d, & \text{if } d \equiv 2 \text{ or } 11 \pmod{12}, \end{cases} \equiv \begin{cases} 5 & \pmod{12}, \\ 8 \text{ or } 44 & \pmod{48}, \end{cases}$$

and

$$D' = -3D \equiv \begin{cases} -15 & \pmod{36}, \\ -24 \text{ or } -132 & \pmod{144}. \end{cases}$$

Hence (7) is equivalent to

$$|D'| \le 3x \quad \text{and} \quad 3|h(D').$$

Thus the number of D's in question is respectively

$$D(3x, 36, -15)\gamma + o(x) = q(9,3)\cdot\delta(4,1)\cdot\frac{6}{\pi^2}\cdot 3x\gamma + o(x) = \frac{3x}{4\pi^2}\gamma + o(x),$$

$$D(3x, 144, -24)\gamma + o(x) = q(9,3)\cdot\delta(16,8)\cdot\frac{6}{\pi^2}\cdot 3x\gamma + o(x) = \frac{3x}{16\pi^2}\gamma + o(x),$$

$$D(3x, 144, -132)\gamma + o(x) = q(9,3)\cdot\delta(16,12)\cdot\frac{6}{\pi^2}\cdot 3x\gamma + o(x) = \frac{3x}{16\pi^2}\gamma + o(x),$$

where $\gamma = \gamma_-(3)$.

3° $d \equiv 3 \pmod 9$.

Then

$$D = \begin{cases} d, & \text{if } d \equiv 21 \pmod{36}, \\ 4d, & \text{if } d \equiv 3 \text{ or } 30 \pmod{36}, \end{cases} \equiv \begin{cases} 21 & \pmod{36}, \\ 12 \text{ or } 120 & \pmod{144}, \end{cases}$$

and

$$D' = -D/3 \equiv \begin{cases} -7 & \pmod{12}, \\ -4 \text{ or } -40 & \pmod{48}. \end{cases}$$

Hence (7) is equivalent to

$$|D'| \le x/3 \quad \text{and} \quad 3|h(D').$$

Thus the number of D's in question is respectively

$$D(x/3, 12, -7)\gamma + o(x) = q(3,1)\cdot\delta(4,1)\cdot\frac{6}{\pi^2}\cdot\frac{x}{3}\gamma + o(x) = \frac{x}{4\pi^2}\gamma + o(x),$$

$$D(x/3,48,-4)\gamma + o(x) = q(3,1)\cdot\delta(16,12)\cdot\frac{6}{\pi^2}\cdot\frac{x}{3}\gamma + o(x) = \frac{x}{16\pi^2}\gamma + o(x),$$

$$D(x/3,48,-40)\gamma + o(x) = q(3,1)\cdot\delta(16,8)\cdot\frac{6}{\pi^2}\cdot\frac{x}{3}\gamma + o(x) = \frac{x}{16\pi^2}\gamma + o(x),$$

where $\gamma = \gamma_-(3)$.

4° $d \equiv 6 \pmod 9$.

Then

$$D = \begin{cases} d, & \text{if } d \equiv 33 \pmod{36}, \\ 4d, & \text{if } d \equiv 6 \text{ or } 15 \pmod{36} \end{cases} \equiv \begin{cases} 33 & \pmod{36}, \\ 24 \text{ or } 60 & \pmod{144}, \end{cases}$$

and

$$D' = -D/3 \equiv \begin{cases} -11 & \pmod{12}, \\ -8 \text{ or } -20 & \pmod{48}. \end{cases}$$

Hence (7) is equivalent to

$$|D'| \le x/3.$$

Thus the number of D's in question is respectively

$$D(x/3,12,-11) = q(3,1)\cdot\delta(4,1)\cdot\frac{6}{\pi^2}\cdot\frac{x}{3} + o(x) = \frac{x}{4\pi^2} + o(x),$$

$$D(x/3,48,-8) = q(3,1)\cdot\delta(16,8)\cdot\frac{6}{\pi^2}\cdot\frac{x}{3} + o(x) = \frac{x}{16\pi^2} + o(x),$$

$$D(x/3,48,-20) = q(3,1)\cdot\delta(16,12)\cdot\frac{6}{\pi^2}\cdot\frac{x}{3} + o(x) = \frac{x}{16\pi^2} + o(x).$$

This leads to the following

Theorem 2.2. *Assume the above conjecture for $m = 3$ and negative fundamental discriminants of quadratic number fields.*

Then

$$\lim_{x\to\infty} \frac{\#\{0 < D \le x : D - \text{ fundamental}, \ 3|k(D)\}}{\#\{0 < D \le x : D - \text{ fundamental}\}}$$

exists and equals

$$\frac{7\gamma_-(3)+1}{8} \approx 0.509890.$$

Proof. Summing up the numbers of D's in $1^\circ - 4^\circ$ above we get

$$\frac{1}{\pi^2}\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{16}\right)\cdot(3\gamma+3\gamma+\gamma+1)\,x + o(x) = \frac{3}{8\pi^2}(7\gamma+1)x + o(x).$$

Since $D(x,1,1) = \frac{3}{\pi^2}x + o(x)$, the result follows. ■

Remark. *One can prove a similar result for D negative. Then assuming the above conjecture for $m = 3$ and positive fundamental discriminants of quadratic number fields we get that the analogous limit*

$$\lim_{x\to\infty} \frac{\#\{-x \le D < 0 : D - \text{ fundamental, } 3|k(D)\}}{\#\{-x \le D < 0 : D - \text{ fundamental}\}}$$

exists and equals

$$\frac{7\gamma_+(3)+1}{8} \approx 0.264834.$$

Acknowledgement The first version of this paper has been presented during the Workshop on K-Theory and Number Fields, Nijmegen 1995. I would like to express my gratitude to organizers for invitation and hospitality.

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Address: Institute of Mathematics, University of Warsaw, ul. Banacha 2, PL–02–097 Warsaw, Poland
E-mail: `bro@mimuw.edu.pl`
Received: 1 Feb 2000