Functiones et Approximatio
XXX (2002), 37–47

APPLICATION OF THE Q-METHOD TO EQUATIONS RELATED TO FRACTALS

S.N. DACHKOVSKI

Abstract: We consider the question of the regularity of solutions of one model equation related to a fractal set in the scale of Besov spaces. The theorem of existence of solution with the optimal regularity is proved.
Keywords: Semi-linear equations, Besov spaces, fractals, regularity of solutions.

1. Introduction

To solve the question of the optimal regularity of solutions of some model semi-linear equations in Besov and Bessel potential spaces, the Q-method was introduced by H. Triebel in [8] (see also [9, Section 27]). This method is based on the quarkonial decomposition of functions and the supersolution technique. It enables to find solutions with the best possible smoothness which satisfy equations in $\mathbb{R}^n$. We formulate the main result proved there. Consider the following two equations

$$u(x) = \int_{\mathbb{R}^n} K(y)(T^+u)(x-y)\,dy + h(y), \quad x \in \mathbb{R}^n, \tag{1.1}$$

and

$$(\mathrm{id} - \Delta)u(x) = \varepsilon T^+ u(x) + h(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\varepsilon > 0$ is a small positive constant and T^+ is the semi-linear operator defined on real functions by

$$T^+ : \quad f(x) \mapsto f_+(x) := \max(f(x), 0), \quad x \in \mathbb{R}^n. \tag{1.3}$$

By $\mathbb{B}^s_{pq}(\mathbb{R}^n)$ and $\mathbb{H}^s_p(\mathbb{R}^n)$ we denote the real parts of $B^s_{pq}(\mathbb{R}^n)$ and $H^s_p(\mathbb{R}^n)$ respectively. Here $B^s_{pq}(\mathbb{R}^n)$ are the well-known Besov spaces and $H^s_p(\mathbb{R}^n)$ are the Bessel-potential spaces. For details and definitions see for example [1], [4], [10].

2000 Mathematics Subject Classification: 46E35

Theorem 1.1. *Let $n \in \mathbb{N}$.*

(i) Let $K(y) \geq 0$ in $\mathbb{R}^n$ with $K \in L_1(\mathbb{R}^n)$,

$$1 \leq p \leq \infty, \quad 1 \leq q \leq \infty, \quad 0 < s < 1 + \frac{1}{p}. \tag{1.4}$$

There is a positive number ε such that if $\|K \mid L_1(\mathbb{R}^n)\| \leq \varepsilon$, then for any $h \in \mathbb{B}^s_{pq}(\mathbb{R}^n)$ the equation (1.1) has a uniquely determined solution $u \in \mathbb{B}^s_{pq}(\mathbb{R}^n)$. (It was proved even for some more general kernels $K(x,y)$.)

(ii) Let for some $\lambda \in [0,2]$

$$0 < p \leq \infty, \quad 0 < q \leq \infty, \quad n\left(\frac{1}{p} - 1\right)_+ < s + \lambda < 1 + \frac{1}{p}. \tag{1.5}$$

There is a positive number $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$, then for any $h \in \mathbb{B}^s_{pq}(\mathbb{R}^n)$, the equation (1.2) has a uniquely determined solution $u \in \mathbb{B}^{s+2}_{pq}(\mathbb{R}^n)$.

A similar result was proved also for $\mathbb{H}^s_p(\mathbb{R}^n)$ spaces. Only these two model equations were considered in [8]. Some generalizations of this result may be found in [12].

Remark 1.1. The Q-method used to prove this theorem provides us with the solution $u(x) \in \mathbb{H}^s_p(\mathbb{R}^n)$ such that the equations (1.1) and (1.2) are satisfied almost everywhere. One of the necessary conditions is $K(y) \geq 0$ in (1.1) (the same holds for the Green's function of $(\mathrm{id} - \Delta)^{-1}$ in (1.2)).

If we are interested in the validity of these equations only almost everywhere, then one can do this without the Q-method, as we did in the anisotropic case in [13]. One can get even the refined validity (see [11]) of (1.1), which enables to consider this equation on some sets of the Lebesgue measure zero like hyperplanes or even d-sets (see the definition below). The refinement depends on p and s. Moreover the condition $K(y) \geq 0$ (as in the theorem above) was not used in [11]. Under the refined validity we understand the following. By Theorem 1.1 we have that there is a unique solution $u \in H^s_p(\mathbb{R}^n)$ of (1.1) such that the equality makes sense almost everywhere (a.e.) with respect to the Lebesgue measure μ_L. Let Γ be a compact subset of $\mathbb{R}^n$ with $\mu_L(\Gamma) = 0$ and μ be a Radon measure such that

$$supp\, \mu = \Gamma \quad \text{and} \quad 0 < \mu(\mathbb{R}^n) = \mu(\Gamma) < \infty. \tag{1.6}$$

We ask, whether the equation (1.1) makes sense μ-a.e. for some Radon measure μ which is more "sensitive" as μ_L? The answer depends on s and p. In the next section we are going to answer this question.

2. Refined validity

We introduce some notions and facts following [1]. To measure the lack of continuity of functions in $H^s_p(\mathbb{R}^n)$, when $sp < n$, we introduce the so called (s,p)-capacity.

Definition 2.1. Let $\Gamma \subset \mathbb{R}^n$ be compact, $s > 0$ and $1 < p < \infty$. Then

$$C_{s,p}(\Gamma) = \inf \left\{ \|\varphi \mid H_p^s(\mathbb{R}^n)\|^p : \quad \varphi \in S(\mathbb{R}^n), \quad \varphi \geq 1 \quad \text{on} \quad \Gamma \right\}.$$

This definition can be extended to arbitrary sets as in §2.2, [1]. We say that a property holds (s,p)-quasieverywhere (abbreviated (s,p)-q.e.) if it holds true for all $x \in \mathbb{R}^n$ except those belonging to a set E with $C_{s,p}(E) = 0$.

Definition 2.2. If $f \in L_1^{loc}(\mathbb{R}^n)$, then a point $x \in \mathbb{R}^n$ is called a Lebesgue point for f if

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0. \tag{2.1}$$

By the theorem of Lebesgue almost every point is a Lebesgue point. But if $f \in H_p^s(\mathbb{R}^n) \subset L_1^{loc}(\mathbb{R}^n)$ with $s > 0$, one can say more. By [1], Theorem 6.2.1, for such f

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy = \widetilde{f}(x) \tag{2.2}$$

exists (s,p)-q.e., where $\widetilde{f}(x)$ is an (s,p)-quasicontinuous distinguished representative for f, which is unique by [1], Theorem 6.1.4. It enables us to choose in each equivalence class $[f] \in H_p^s(\mathbb{R}^n)$ a uniquely determined representative f such that (2.1) and (2.2) hold (s,p)-q.e. Sometimes $\widetilde{f}(x)$ is called strictly defined function.

We introduce a qualitative characteristics of Radon measures following [9, 9.25]. Let $1 < v < \infty$ and $t \geq 0$, then

$$\mu_v^t = \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{tjv} \mu(2Q_{jm})^v \right)^{\frac{1}{v}}. \tag{2.3}$$

For more information of these characteristic numbers see [9, 9.26]. By definition a Radon measure μ in $\mathbb{R}^n$ is locally finite, i.e. $\mu(B) < \infty$ for any ball B in $\mathbb{R}^n$. The restriction $\mu|B$ of a Radon measure μ to B is a Radon measure again. The collection of all Radon measures such that $\mu|B \in M_v^t$ for any ball B in $\mathbb{R}^n$ we denote by $M_v^{t,loc}$, where, by definition, M_v^t is the collection of all those Radon measures μ in $\mathbb{R}^n$ for which $\mu_v^t < \infty$. The following was proved in [11].

Theorem 2.1. *(i) Let $n \in \mathbb{N}$, $K \in L_1(\mathbb{R}^n)$ be a real function and*

$$1 < p < \infty, \quad \frac{n}{p} < s < 1 + \frac{1}{p}. \tag{2.4}$$

There is a positive number ε_0 such that for any ε with $0 < \varepsilon \leq \varepsilon_0$ and any $h \in \mathbb{H}_p^s(\mathbb{R}^n)$, if $\|K \mid L_p(\mathbb{R}^n)\| \leq \varepsilon$, than the equation (1.1) has a unique solution $u \in \mathbb{H}_p^s(\mathbb{R}^n)$. Let both u and h be the continuous representatives. Then (1.1) is valid for all $x \in \mathbb{R}^n$.

(ii) Let $n \in \mathbb{N}$, $K \in L_1(\mathbb{R}^n)$ *be a real function and*

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 0 < s \leq \frac{n}{p}, \quad s < 1 + \frac{1}{p}. \tag{2.5}$$

There is a positive number ε_0 *such that for any* ε *with* $0 < \varepsilon \leq \varepsilon_0$ *and any* $h \in \mathbb{H}^s_p(\mathbb{R}^n)$, *if* $\|K \mid L_p(\mathbb{R}^n)\| \leq \varepsilon$, *then the equation (1.1) has a unique solution* $u \in \mathbb{H}^s_p(\mathbb{R}^n)$. *Let* $\mu \in M_{p'}^{\frac{n}{p}-s,loc}$, u *and* h *be the distinguished representatives as discussed above. Then (1.1) is valid* μ-a.e.

Remark 2.1. This theorem can be generalized to some more general kernels $K(x, y)$ as it is done in [11]. In the same way we can get a similar assertion as in Theorem 1.1 (ii) also for the equation (1.2). But we do not need the Q-method neither to prove the theorem nor to get this generalizations.

So far we can work without the Q-method. One can ask, why the Q-method is useful? And what are the applications? In the following sections we are going to show that this method is not only elegant but also useful.

3. d-sets and traces

Of course if we are interested in the validity of such equations only up to sets of Lebesgue measure zero, then we do not need the Q-method. But if we are going to consider some equations on sets which have Lebesgue measure zero, for example on a d-set Γ with $d < n$, then the proof of Theorem 2.1 does not work.

d-sets have been introduced by A. Jonsson and H. Wallin in [3], where an extensive study of these sets in connection to function spaces is developed.

Definition 3.1. Let $n \in \mathbb{N}$ and Γ be a compact set in $\mathbb{R}^n$ and let $0 \leq d \leq n$. Then Γ is called a d-set if there exists a Borel measure μ in $\mathbb{R}^n$ such that $\operatorname{supp} \mu = \Gamma$ and there are two positive constants c_1, c_2 such that for all $\gamma \in \Gamma$ and all r with $0 < r < 1$,

$$c_1 r^d \leq \mu(B(\gamma, r) \cap \Gamma) \leq c_2 r^d. \tag{3.1}$$

If $0 \leq d < n$, then by [6], Corollary 3.6, $|\Gamma| = 0$, where $|\Gamma|$ stands for the Lebesgue measure, and μ in (3.1) is a Radon measure. By definition, $L_p(\Gamma)$ is the set of all functions for which the following quasinorm

$$\|f \mid L_p(\Gamma)\| = \left(\int_\Gamma |f(\gamma)|^p \mu(d\gamma) \right)^{\frac{1}{p}}, \quad 0 < p < \infty \tag{3.2}$$

or

$$\|f \mid L_\infty(\Gamma)\| = ess \sup_{\gamma \in \Gamma} |f(\gamma)|, \quad \text{if} \quad p = \infty \tag{3.3}$$

is finite.

The trace $tr_\Gamma f$ of a function $f \in B^s_{pq}(\mathbb{R}^n)$ on Γ can be defined as it is done in [6], §18.5-18.12. The idea is the following. For any smooth function $\varphi(x) \in S(\mathbb{R}^n)$ one can define $tr_\Gamma \varphi = \varphi|_\Gamma$ as a pointwise restriction of φ on Γ. Let us suppose, that for some $s > 0, 0 < p < \infty, 0 < q < \infty$ the inequality

$$\|tr_\Gamma \varphi \mid L_p(\Gamma)\| \leq c\|\varphi \mid B^s_{pq}(\mathbb{R}^n)\| \tag{3.4}$$

holds for all $\varphi \in S(\mathbb{R}^n)$ (which is dense in $B^s_{pq}(\mathbb{R}^n)$ for $0 < p < \infty$ and $0 < q < \infty$), where $c > 0$ does not depend on φ. Then, by completion, one can also define the trace of any function of $B^s_{pq}(\mathbb{R}^n)$ and it is denoted by $tr_\Gamma f$. It was proved in [6] that if $0 < p < \infty, \quad 0 < q \leq \min(1, p)$, then

$$tr_\Gamma B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma). \tag{3.5}$$

This is not the only way to define traces. Another approach, which uses so called strictly defined functions (see the previous section) can be found for example in [2], §6, but only for $p > 1$ and $s > \frac{n-d}{p}$.

The above optimal assertion (3.5) allows us to give the following definition (see [6], §20.2), if $s > 0, \ 0 < p \leq \infty$ and $0 < q < \infty$, then

$$B^s_{pq}(\Gamma) = tr_\Gamma B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \tag{3.6}$$

quasinormed by

$$\|f \mid B^s_{pq}(\Gamma)\| = \inf \left\| g \mid B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \right\|, \tag{3.7}$$

where the infimum is taken over all $g \in B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ with $tr_\Gamma g = f$. If $p < 1$ and $d \in \mathbb{N}$ it may happen that $B^s_{pq}(\mathbb{R}^d)$ defined by (3.6) do not coincide with $B^s_{pq}(\mathbb{R}^n)$ with $n = d$, defined in the usual way, but we will always suppose $p > 1$. By [6], Theorem 20.6, $B^s_{pq}(\Gamma)$ is embedded in $L_p(\Gamma)$ if $s > 0$. If $p > 1$, then by [6], Theorem 18.2, any $f^\Gamma \in L_p(\Gamma)$ can be identified with the singular distribution

$$f(\varphi) = \int_\Gamma f^\Gamma(\gamma)(\varphi|\Gamma)(\gamma)\mu(d\gamma), \quad \varphi \in S(\mathbb{R}^n), \tag{3.8}$$

which belongs to $B^{-\frac{n-d}{p'}}_{p,\infty}(\mathbb{R}^n)$ if $1 < p < \infty$, where $\frac{1}{p} + \frac{1}{p'} = 1$. We denote this identification by

$$id_\Gamma : \quad L_p(\Gamma) \mapsto B^{-\frac{n-d}{p'}}_{p,\infty}(\mathbb{R}^n). \tag{3.9}$$

The operator tr^Γ is defined as (see [6], Chapter IV)

$$tr^\Gamma = id_\Gamma \circ tr_\Gamma, \tag{3.10}$$

which maps

$$B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \mapsto B^{-\frac{n-d}{p'}}_{p,\infty}(\mathbb{R}^n), \quad 1 < p < \infty, \quad 0 < q \leq \infty, \quad s > 0. \tag{3.11}$$

4. Q-operator

Let $s > \sigma_p$, we define the Q-operator introduced by H. Triebel in [8] (see also [9, Ch. 26]). Recall that by [6], §14, one can expand any function f of $B^s_{pq}(\mathbb{R}^n)$ with $s > \sigma_p$ in the series by the β-quarks

$$(\beta qu)_{jm}(x) = 2^{-j(s-\frac{n}{p})}\psi^\beta(2^j x - m), \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \quad \beta \in \mathbb{N}_0^n, \tag{4.1}$$

where $\psi^\beta(x) = x_1^{\beta_1} \cdot \ldots \cdot x_n^{\beta_n}\psi(x)$ and ψ is a non-negative C^∞ function with compact support near the origin such that

$$\sum_{m\in\mathbb{Z}^n} \psi(x-m) = 1, \quad x \in \mathbb{R}^n. \tag{4.2}$$

It turns out that a distribution f belongs to $B^s_{pq}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{\beta\in\mathbb{N}_0^n}\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda^\beta_{jm}(f)(\beta qu)_{jm}(x), \quad x \in \mathbb{R}^n \tag{4.3}$$

(absolute and unconditional convergence in $S'(\mathbb{R}^n)$ or in $L_p(\mathbb{R}^n)$ for $p \geq 1$), where the coefficients $\lambda^\beta_{jm}(f)$ depend linearly on f and the sequences

$$\lambda^\beta = (\lambda^\beta_{jm} : \quad j \in \mathbb{N}_0, m \in \mathbb{Z}^n) \tag{4.4}$$

belong to the sequence space b_{pq} with

$$\sup_\beta 2^{\nu|\beta|}(\|\lambda^\beta \mid b_{pq}\|) < \infty \tag{4.5}$$

for some positive number ν. It is always possible to chose a positive function $\psi(x)$ in (4.1), (4.2) such that $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_n \geq 0\}$. Then all quarks $(\beta qu)_{jm}(x)$ are non-negative and one can define the operator Q by the following

$$(Qf)(x) = \sum_{\beta\in\mathbb{N}_0^n}\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \left|\lambda^\beta_{jm}(f)\right| (\beta qu)_{jm}(x). \tag{4.6}$$

It follows immediately from the definition that for any real function $f(x)$ it holds

$$Qf(x) \geq |f(x)| \geq T^+ f(x). \tag{4.7}$$

5. The equation

Let $p \geq 1$, Γ be a compact d-set as above and μ is a corresponding Radon measure. Let tr_Γ and id_Γ be the operators defined as above. The truncation operator can be considered also on $L_p(\Gamma)$:

$$(T_\Gamma^+ f^\Gamma)(\gamma) = \max(f^\Gamma(\gamma), 0), \quad \gamma \in \Gamma. \tag{5.1}$$

Remark 5.1. We preserve the same notation $T_\Gamma^+ f = f_+$ also in this case. Obviously T_Γ^+ is bounded and Lipschitz-continuous operator on $L_p(\Gamma)$.

Consider the equation

$$(-\Delta + \mathrm{id})u(x) = \varepsilon\, \mathrm{id}_\Gamma\, T_\Gamma^+ tr_\Gamma u(x) + h(x), \quad x \in \mathbb{R}^n, \tag{5.2}$$

in some $B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$ with $s > 0$. It is known, that the Green's function $G(x)$ of the operator $-\Delta + \mathrm{id}$ is a positive and integrable function on $\mathbb{R}^n$. With the help of the Green's function we can rewrite (5.2) as follows

$$\begin{aligned} u(x) &= \varepsilon(-\Delta + \mathrm{id})^{-1}(\mathrm{id}_\Gamma\, T_\Gamma^+ tr_\Gamma u(x) + h(x)) && (5.3)\\ &= \varepsilon(G * \mathrm{id}_\Gamma\, (tr_\Gamma u)_+)(x) + H(x), \quad x \in \mathbb{R}^n, && (5.4)\end{aligned}$$

where $H(x) = (-\Delta + \mathrm{id})^{-1}h(x)$. Using the interpretation (3.8-3.10) we get

$$u(x) = \varepsilon \int_\Gamma (tr_\Gamma u(\gamma))_+ G(x-\gamma)\, \mu(d\gamma) + H(x), \quad x \in \mathbb{R}^n. \tag{5.5}$$

We generalize this equation inserting an additional volume term

$$u(x) = \varepsilon \int_\Gamma G(x-\gamma)(tr_\Gamma u)_+(\gamma)\, \mu(d\gamma) + \varepsilon \int_{\mathbb{R}^n} K(x-y)u_+(y)\, dy + H(x), \tag{5.6}$$

$x \in \mathbb{R}^n$, where $K(y)$ is supposed to be a non-negative and integrable on $\mathbb{R}^n$.

6. Application of the Q-method

Theorem 6.1. *Let* $1 < p < \infty$ *and* $n\left(\frac{1}{p} - 1\right)_+ < s < 1 + \frac{1}{p}$,

$$0 < s < 2 - n + d, \quad n - 2 < d < n \quad \text{and} \quad 1 < q \leq \infty. \tag{6.1}$$

Let Γ *be a compact* d*-set as above. If* $\varepsilon > 0$ *is small enough, then for any* $H(x) \in \mathbb{B}_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$ *there exists at least one solution of the equation (5.6) belonging to* $\mathbb{B}_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$.

Proof. Let us denote

$$(B^Q u)(x) = \varepsilon \int_\Gamma G(x-\gamma)(tr_\Gamma u)_+(\gamma)\, \mu(d\gamma) + \varepsilon \int_{\mathbb{R}^n} K(x-y)Qu(y)\, dy + H(x) \tag{6.2}$$

and

$$(Bu)(x) = \varepsilon \int_\Gamma G(x-\gamma)(tr_\Gamma u)_+(\gamma)\,\mu(d\gamma) + \varepsilon \int_{\mathbb{R}^n} K(x-y)u_+(y)\,dy + H(x). \quad (6.3)$$

Let us consider the first term of the operator B in $\mathbb{B}_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$. By (6.1) we have $s + \frac{n-d}{p} < 2 - \frac{n-d}{p'}$. By the embedding properties of $B_{pq}^s(\mathbb{R}^n)$ (see [5]) we have (all unimportant constants in the following estimates will be denoted by the same letter c)

$$\|(-\Delta+\mathrm{id})^{-1}\,\mathrm{id}_\Gamma(tr_\Gamma u)_+ \mid B_{pq}^{s+\frac{n-d}{p}}\| \le c\|(-\Delta+\mathrm{id})^{-1}\,\mathrm{id}_\Gamma(tr_\Gamma u)_+ \mid B_{p,\infty}^{2-\frac{n-d}{p'}}\| \quad (6.4)$$

(we omit $\mathbb{R}^n$ for brevity). By the lifting property of the operator $(-\Delta+\mathrm{id})^{-1}$ the last expression is equivalent to

$$\|\,\mathrm{id}_\Gamma(tr_\Gamma u)_+ \mid B_{p,\infty}^{-\frac{n-d}{p'}}(\mathbb{R}^n)\|. \quad (6.5)$$

By (3.9) we can estimate the last expression from above by

$$c\|(tr_\Gamma u)_+ \mid L_p(\Gamma)\|. \quad (6.6)$$

Using the boundedness of the operator T_Γ^+ on $L_p(\Gamma)$, (3.5) and known embedding properties of $B_{pq}^s(\mathbb{R}^n)$ we can estimate this from above by

$$c\|tr_\Gamma u \mid L_p(\Gamma)\| \le c\|u \mid B_{p1}^{\frac{n-d}{p}}(\mathbb{R}^n)\| \le c\|u \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\|. \quad (6.7)$$

Finally it follows that

$$\|(-\Delta+\mathrm{id})^{-1}\,\mathrm{id}_\Gamma(tr_\Gamma u)_+ \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\| \le c\|u \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\|. \quad (6.8)$$

For the volume term in (6.3) and (6.2) we have the same estimate (see [8])

$$\left\| \int_{\mathbb{R}^n} K(y)T^+u(x-y)\,dy \middle| B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\right\| \le c\|u \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\|, \quad (6.9)$$

and the same with Q-operator

$$\left\| \int_{\mathbb{R}^n} K(y)Qu(x-y)\,dy \middle| B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\right\| \le c\|u \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\|. \quad (6.10)$$

Hence it follows

$$\|Bu \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\| \le c\varepsilon\|u \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\| + \|H \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\|, \quad (6.11)$$

$$\|B^Q u \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\| \le c\varepsilon\|u \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\| + \|H \mid B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)\|. \quad (6.12)$$

Recall that in contrast to the operator Q the operator T^+ is not Lipschitz continuous in $B^s_{pq}(\mathbb{R}^n)$, with $n\left(\frac{1}{p}-1\right)_+ < s < 1+\frac{1}{p}$, $0 \le \frac{1}{p} < \infty$, see [7] and [9], Section 25. By the Lipschitz continuity of Q in $B^s_{pq}(\mathbb{R}^n)$ and the Lipschitz continuity of T^+ in $L_p(\Gamma)$ we can also prove the Lipschitz continuity of B^Q. As in (6.4)-(6.6) we have ($\mathbb{R}^n$ is omitted for brevity)

$$\begin{aligned}&\|(-\Delta+\mathrm{id})^{-1}\,\mathrm{id}_\Gamma(tr_\Gamma u_1)_+ - (-\Delta+\mathrm{id})^{-1}\,\mathrm{id}_\Gamma(tr_\Gamma u_2)_+ \mid B^{s+\frac{n-d}{p}}_{pq}\|\\ &\le c\|(tr_\Gamma u_1)_+ - (tr_\Gamma u_2)_+ \mid L_p(\Gamma)\|. \end{aligned}\tag{6.13}$$

Here we have used the linearity of $(-\Delta+\mathrm{id})^{-1}$ and id_Γ. Now by the Lipschitz continuity of T^+_Γ on $L_p(\Gamma)$ and the boundedness and linearity of tr_Γ the last expression can be estimated from above by

$$\begin{aligned}\|tr_\Gamma u_1 - tr_\Gamma u_2 \mid L_p(\Gamma)\| &\le \|tr_\Gamma(u_1-u_2) \mid L_p(\Gamma)\|\\ &\le \|u_1-u_2 \mid B^{s+\frac{n-d}{p}}_{pq}\|.\end{aligned}\tag{6.14}$$

The volume term in (6.2) is Lipschitz continuous by [8]. Finally we have

$$\|B^Q(u-v) \mid B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\| \le c\varepsilon\|u-v \mid B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\|. \tag{6.15}$$

We remark that there is no counterpart of (6.15) for the operator B. Hence for ε small enough the operator B^Q yields a contraction in $B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ and by the Banach contraction theorem it has a fixed point $u^0(x) \in B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$. By $G(y) \ge 0, K(y) \ge 0$ and (4.7) it follows that

$$u^0(x) \ge \varepsilon\int_\Gamma G(x-\gamma)(tr_\Gamma u)_+(\gamma)\,\mu(d\gamma) + \varepsilon\int_{\mathbb{R}^n} K(x-y)T^+u(y)\,dy + H(x) \tag{6.16}$$

is a supersolution for (5.6). By iteration, for $j \in \mathbb{N}$, we define

$$u^j(x) = (Bu^{j-1})(x) \tag{6.17}$$

$$= \varepsilon\int_\Gamma G(x-\gamma)(tr_\Gamma u^{j-1})_+(\gamma)\,\mu(d\gamma) + \varepsilon\int_{\mathbb{R}^n} K(x-y)T^+u^{j-1}(y)\,dy + H(x). \tag{6.18}$$

Again by $G(y) \ge 0, K(y) \ge 0$ and (4.7) we have that $u^j(x)$ is a monotonically decreasing sequence bounded from below

$$u^0(x) \ge u^1(x) \ge \ldots \ge u^j(x) \ge \ldots \ge H(x). \tag{6.19}$$

By (6.17) for ε small enough we have

$$\|u^1 \mid B^{s+\frac{n-d}{p}}_{pq}\| \le 2\|h \mid B^{s+\frac{n-d}{p}}_{pq}\|, \tag{6.20}$$

and

$$\|u^j \mid B_{pq}^{s+\frac{n-d}{p}}\| \leq \frac{1}{2}\|u^{j-1} \mid B_{pq}^{s+\frac{n-d}{p}}\| + \|h \mid B_{pq}^{s+\frac{n-d}{p}}\|, \quad j \in \mathbb{N} \tag{6.21}$$

and, by iteration,

$$\|u^j \mid B_{pq}^{s+\frac{n-d}{p}}\| \leq 2\|h \mid B_{pq}^{s+\frac{n-d}{p}}\|, \quad j \in \mathbb{N}, \tag{6.22}$$

where we omit $\mathbb{R}^n$ for brevity. By (6.19) and Lebesgue's bounded convergence theorem we have

$$u^j(x) \to u(x) \quad \text{in} \quad L_p. \tag{6.23}$$

We have that this sequence is bounded in $B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$ and converges to $u(x)$ in $S'(\mathbb{R}^n)$. Then by the Fatou property ([4], p. 15) it follows $u(x) \in B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$. By (6.19) and the monotone pointwise convergence in (6.17) we have that $u(x)$ is the solution:

$$u(x) = \varepsilon \int_\Gamma G(x-\gamma)(tr_\Gamma u)_+(\gamma)\,\mu(d\gamma) + \varepsilon \int_{\mathbb{R}^n} K(x-y)T^+u(y)\,dy + H(x), \tag{6.24}$$

what proves the theorem. ■

Remark 6.1. We note that it makes no sense to consider equation (5.6) in $L_p(\mathbb{R}^n)$ because it is impossible to define traces of functions in $L_p(\mathbb{R}^n)$ on d-sets. By this reason we cannot use the same method as in Theorem 2.1 and we cannot prove the uniqueness of the solution as it was done in the proof of Theorem 2.1.

Remark 6.2. If the fractional term in (5.6) is absent, i.e., if we put $G(x) = 0$ on Γ, then we have the same equation as in [7], where it was proved that the solution is unique. We can also state the following:

If $K = 0$ in the previous theorem, then (5.6) has a unique solution.

Proof. Let u and v be two functions from $\mathbb{B}_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$. By the same arguments as at the beginning of the proof of the previous theorem we get for the operator B the following (as a counterpart of (6.4)-(6.6))

$$\|Bu - Bv \mid B_{pq}^{s+\frac{n-d}{p}}\| = \varepsilon\|(-\Delta + \mathrm{id})^{-1}\,\mathrm{id}_\Gamma((tr_\Gamma u)_+ - (tr_\Gamma v)_+) \mid B_{pq}^{s+\frac{n-d}{p}}\| \tag{6.25}$$

$$\leq c'\varepsilon\|(tr_\Gamma u)_+ - (tr_\Gamma v)_+ \mid L_p(\Gamma)\|, \tag{6.26}$$

here we have used the linearity of $(-\Delta + \mathrm{id})^{-1}$ and id_Γ. Then we can use the Lipschitz continuity of the operator T_Γ^+ on $L_p(\Gamma)$ and estimate the last expression from above by

$$c\varepsilon\|tr_\Gamma u - tr_\Gamma v \mid L_p(\Gamma)\| = c\varepsilon\|tr_\Gamma(u-v) \mid L_p(\Gamma)\|, \tag{6.27}$$

here we used the linearity of tr_Γ. By (3.5) and the embedding properties of the $B^s_{pq}(\mathbb{R}^n)$ scale we get the following estimate from above

$$c\varepsilon \| tr_\Gamma(u-v) \mid L_p(\Gamma)\| \le c\varepsilon \|u-v \mid B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\| \le c\varepsilon\|u-v \mid B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\|. \quad (6.28)$$

Then it follows that for ε small enough operator B yields a contraction in $B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ and hence, by the Banach contraction theorem, it has a unique fixed point u, which is the solution that we are looking for. ■

References

[1] D. Adams and L. Hedberg, *Function spaces and potential theory*, Springer, Berlin 1996.

[2] M. Bricchi, *On the relationship between Besov spases $B^{s,\Psi}_{pq}(\mathbb{R}^n)$ and L_p-spaces defined on a (d,Ψ)-set*, Jenaer Schriften zur Mathematik und Informatik **13**, 2000.

[3] A. Jonsson and H. Wallin, *Function spaces on subsets of $\mathbb{R}^n$*, Math. reports 2, volume 1, London, Harwood acad. publ. 1984.

[4] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential operators*, Walter de Gruyter, Berlin 1996.

[5] W. Sickel and H. Triebel, *Hölder Inequalities and Sharp Embeddings in Function Spaces of B^s_{pq} and F^s_{pq} Type*, Journal for Analysis and its Applications, **14** (1) (1995), 105–140.

[6] H. Triebel, *Fractals and Spectra*, Birkhäuser, Basel 1997.

[7] H. Triebel, *Truncations of functions*, Forum Math. **12** (2000), 731-756.

[8] H. Triebel, *Regularity theory for some semi-linear equations: the Q-method*, Forum Math. **13** (2001), 1–19.

[9] H. Triebel, *The structure of functions*, Birkhäuser, Basel 2001.

[10] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel 1983.

[11] H. Triebel, *Refined solutions of some integral equations*, Functiones et Approximatio, to appear, 2002.

[12] S. Dachkovski, *Application of the Q-method to some semilinear differential and integral equations*, Jenaer Schriften zur Mathematik und Informatik **15**, 2000.

[13] S. Dachkovski, *Anisotropic function spaces and related hypoelliptic equations*, Math. Nachr., to appear 2002.

Address: Mathematisches Institut, Fridrich-Schiller-Universität Jena,
Ernst-Abbe-Platz 1-4, D-07743, Jena, Germany
E-mail: dsn@minet.uni-jena.de

Received: 13 January 2002