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COMPACT EMBEDDINGS BETWEEN BESOV SPACES DEFINED ON h-SETS

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Abstract: We study the compactness of the embedding between Besov spaces defined on some type of isotropic fractal sets in the Euclidean space. The "degree of compactness" of such an embedding is expressed in terms of its entropy numbers.
Keywords: generalised Besov spaces, entropy numbers, fractal sets.

1. Introduction

This paper is a natural continuation of our previous work [7]: there we have discussed in detail the definition of generalised Besov-type spaces $B^{\sigma}_{p,q}(\Gamma)$ on a class of isotropic fractal sets Γ in $\mathbb{R}^n$; more precisely, we have taken into consideration the class of h-sets, which we have introduced and studied in [8], [6], [5].

The term "generalised" used in this context of function spaces means that we are considering a real sequence $\sigma = \{\sigma_j\}_{j\in\mathbb{N}_0}$ as a regularity index and not only a number s. This "scalar" case is however subsumed in the general setting, letting $\sigma_j = 2^{js}$, $j \in \mathbb{N}_0$. We shall be more precise in the sequel, here we remark that as a further specialization of the scale $B^s_{p,q}(\Gamma)$, one has also a suitable definition of $H^s(\Gamma)$ (that is, of course, $W^s_2(\Gamma)$), for $s \geqslant 0$.

Here we wish to present an application of the theory developed so far: we examine namely the compactness of the embedding between two spaces of this kind, say, $B^{\sigma}_{p,q}(\Gamma)$ and $B^{\tau}_{u,v}(\Gamma)$.

The main assertion of this note (Theorem 7.12) on the one hand shows that our definition of the scale $B^{\sigma}_{p,q}(\Gamma)$ is reasonable (and in some sense optimal) and, on the other hand, provides an useful tool for further applications (we have in mind spectral properties of pseudo-differential fractal operators and PDE's on bounded regions with fractal boundary) which will be the subject of our study in forthcoming papers.

This article is written in a self-contained way, that is, we summarize also the main results proved in [7] and [6], although we shall omit related details.

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The proof of the main theorem 7.12 exploits essentially the possibility to represent the elements in $B^{\sigma}_{p,q}(\mathbb{R}^n)$ as a sum of localized smooth building blocks (quarks). That is why we shall briefly present also the quarkonial representation theorems for these spaces (subsections (5.1) and (5.2)), which, apart from their own interest, turn out to be powerful tools in applications.

2. General notation

In this note we shall adopt the following general notation: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^n$, $n \in \mathbb{N}$, denotes the Euclidean n-dimensional space, $\mathbb{R} = \mathbb{R}^1$, $\mathbb{N}_0^n$ is the set of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_i \in \mathbb{N}_0$ for $i = 1, \dots, n$. In this case we also define $|\alpha| = \alpha_1 + \dots + \alpha_n$. Finally, $\mathbb{Z}^n$ denotes the set of all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with integer coordinates. Throughout this work we use the equivalence "$\sim$" in

$$a_k \sim b_k, \qquad \varphi(r) \sim \psi(r) \qquad \text{or} \qquad \mu(A) \sim \nu(A)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1 a_k \leqslant b_k \leqslant c_2 a_k, \quad c_1\varphi(r) \leqslant \psi(r) \leqslant c_2\varphi(r) \quad \text{or} \quad c_1\mu(A) \leqslant \nu(A) \leqslant c_2\mu(A),$$

for all admitted values of the discrete variable k or the continuous variable r or for all admitted Borel sets $A \subset \mathbb{R}^n$, respectively. Here a_k, b_k are positive numbers, φ, ψ are positive functions and μ, ν are (positive) finite Borel outer measures. From now on we shall speak simply of *measures* instead of *outer measures*. The word "positive" is always used to mean "strictly positive", both for functions and for real numbers.

If not otherwise indicated, log is always taken with respect to base 2. All unimportant constants are denoted by c, occasionally primed or with subscripts.

3. Sequences and related indices

Definition 3.1. Let $\sigma = \{\sigma_j\}_{j\in\mathbb{N}_0}$ be a sequence of positive numbers. We say that σ is an *admissible sequence* if there are two positive constants d_0 and d_1 such that

$$d_0\sigma_j \leqslant \sigma_{j+1} \leqslant d_1\sigma_j, \qquad j \in \mathbb{N}_0. \tag{3.1}$$

Remark 3.2. If σ and τ are two admissible sequences, then, for $\alpha \in \mathbb{R}$, also $\sigma^\alpha = \{\sigma_j^\alpha\}_{j\in\mathbb{N}_0}$ and $\sigma\tau = \{\sigma_j\tau_j\}_{j\in\mathbb{N}_0}$ are admissible sequences.
Examples of admissible sequences are $\sigma = \{2^{sj}j^a\}_{j\in\mathbb{N}_0}$, for any $s, a \in \mathbb{R}$ and $\sigma = \{h(2^{-j})\}_{j\in\mathbb{N}_0}$, if $h\colon (0,1] \rightarrow \mathbb{R}$ is a measure function (see Definition 4.2 below).

Definition 3.3. Let σ be an admissible sequence. Then we let

$$\underline{\sigma}_j = \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k}, \qquad \overline{\sigma}_j = \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \tag{3.2}$$

and we define

$$\beta_\sigma = \lim_{j\to\infty} \frac{\log \underline{\sigma}_j}{j}, \qquad \alpha_\sigma = \lim_{j\to\infty} \frac{\log \overline{\sigma}_j}{j}. \tag{3.3}$$

We refer to β_σ and α_σ as to the *lower* and the *upper Boyd index* of σ, respectively.

Remark 3.4. The sequence $\log \overline{\sigma}_j$ is sub-additive, thus the definition of α_σ is well posed. Since $\beta_\sigma = -\alpha_{\sigma^{-1}}$, also β_σ is well defined. Of course, these indices are finite for any admissible sequence σ.

The use of the name *Boyd indices* in this context is justified: we do not want to go into details and we refer to [2] where we have treated with some care this aspect. We remark only that the following identities hold true for any admissible sequence σ and τ and any $\varepsilon > 0$ (see also [23]):

$$\beta_\sigma = -\alpha_{\sigma^{-1}}, \tag{3.4}$$

$$\alpha_{\sigma\tau} = \alpha_\sigma + \alpha_\tau \quad \text{and} \quad \beta_{\sigma\tau} = \beta_\sigma + \beta_\tau, \tag{3.5}$$

$$c_1 2^{(\beta_\sigma-\varepsilon)j} \leqslant \frac{\sigma_{j+k}}{\sigma_k} \leqslant c_2 2^{(\alpha_\sigma+\varepsilon)j}, \qquad j,k \in \mathbb{N}_0, \tag{3.6}$$

for some positive constants $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(\varepsilon)$.

We shall also adopt the following convention: if $h\colon (0,1] \to \mathbb{R}$ is a positive function such that $h(2^{-j}) \sim h(2^{-j-1})$, i.e., such that the sequence $\{h(2^{-j})\}_{j\in\mathbb{N}_0}$ is admissible, then we let

$$\alpha_h = \alpha_{\{h(2^{-j})\}} \qquad \text{and} \qquad \beta_h = \beta_{\{h(2^{-j})\}}. \tag{3.7}$$

4. h-sets

In this section we specify the class of compact fractal sets we shall take into consideration. In what follows we refer closely to [6], [5] and [8].

Definition 4.1. We denote by $\mathbb{H}$ the class of all positive non-decreasing continuous functions defined on $(0,1]$. We refer to $\mathbb{H}$ as to the class of all *gauge functions*.

Definition 4.2. Let h be a gauge function. Then a non-empty compact set $\Gamma \subset \mathbb{R}^n$ is called *h-set* if there exists a finite Radon measure μ with

1. $\operatorname{supp}\mu = \Gamma$,
2. $\mu(B(\gamma,r)) \sim h(r)$, for $\gamma \in \Gamma$, $r \in (0,1]$,

where supp denotes the support of the measure μ and $B(\gamma,r)$ is the closed ball centered at γ with radius r.

Any finite Radon measure fulfilling the above two conditions will be called *h-measure (related to* Γ*)*. Any gauge function h for which there exists an h-set (in $\mathbb{R}^n$) is called *measure function (in* $\mathbb{R}^n$*)*.

Remark 4.3. This definition generalises the notion of d-sets, introduced by A. Jonnson and H. Wallin in [17], and the definition of (d, Ψ)-sets, considered by D. Edmunds and H. Triebel in [13]. The former are subsumed in the above definition choosing $h(r) = r^d$, the latter are obtained letting $h(r) \sim r^d \Psi(r)$ (we shift to the next subsection the necessary explanations).

The structure of h-sets has been extensively studied in [6] and [8], whereas unbounded versions of h-sets have been taken into consideration in [5].

Here we summarize the main results concerning these sets.

Theorem 4.4 (Characterization). *Let h be a gauge function. Then h is a measure function in $\mathbb{R}^n$ if, and only if,*

$$\frac{h(2^{-j-k})}{h(2^{-j})} \gtrsim 2^{-kn}, \qquad j, k \in \mathbb{N}_0. \tag{4.1}$$

Remark 4.5. Here we have used a compact notation. The symbol $\gtrsim$ in (4.1) is to be understood in the following way: there exists a gauge function $h' \sim h$ which satisfies the above estimation with the symbol $\geqslant$ in place of $\gtrsim$.

Notice that if (4.1) holds only definitively in k, then again one can conclude that h is a measure function: this turns out to be particularly useful when dealing with concrete examples.

Remark 4.6. Observe that if h is a measure function, then the sequence $\{h(2^{-j})\}_{j \in \mathbb{N}_0}$ is admissible and one has

$$-n \leqslant \beta_h \leqslant \alpha_h \leqslant 0, \tag{4.2}$$

according to the notation introduced in (3.7).

Proposition 4.7. *Let Γ be an h-set. Then the following assertions hold true.*

(i) All h-measures related to Γ are equivalent to $\mathcal{H}^h|\Gamma$;

(ii) For any $t \in (0, 1]$ and $\gamma \in \Gamma$ one has

$$\dim_{\mathcal{H}}(\Gamma \cap B(\gamma, t)) = \liminf_{r \to 0} \frac{\log h(r)}{\log r} \tag{4.3}$$

and

$$\dim_{\mathcal{P}}(\Gamma \cap B(\gamma, t)) = \limsup_{r \to 0} \frac{\log h(r)}{\log r}, \tag{4.4}$$

where $\dim_{\mathcal{H}} A$ and $\dim_{\mathcal{P}} A$ denote the Hausdorff dimension and packing dimension of the set A, respectively.

Remark 4.8. The measure $\mathcal{H}^h|\Gamma$ appearing in (i) above is the restriction to Γ of the *generalised Hausdorff measure* $\mathcal{H}^h$ *related to the gauge function* h. If $h(r) = r^d$, this measure coincides with the usual d-dimensional Hausdorff measure $\mathcal{H}^d$. We refer to [26] for definitions and properties of Hausdorff-type measures and to [22] for results concerning Hausdorff and packing dimensions.

We remark here that one always has

$$0 \leqslant \dim_{\mathcal{H}} A \leqslant \dim_{\mathcal{P}} A \leqslant n, \tag{4.5}$$

for any $A \subset \mathbb{R}^n$. Therefore (4.3) and (4.4) can be regarded as further necessary conditions on the decay of h near 0.

Later on we shall restrict our attention to those h-sets which fulfill a certain "porosity" condition. We quote the necessary definition.

Definition 4.9. A non-empty Borel set Γ satisfies the *ball condition* if there exists a number $0 < \eta < 1$ with the following property:

for any ball $B(\gamma, r)$ centered at $\gamma \in \Gamma$ and of radius $0 < r \leqslant 1$ there is a ball $B(x, \eta r)$ centered at some $x \in \mathbb{R}^n$, such that

$$B(\gamma, r) \supset B(x, \eta r) \qquad \text{and} \qquad B(x, \eta r) \cap \overline{\Gamma} = \emptyset. \tag{4.6}$$

This definition coincides essentially with [28, Definition 18.10, p. 142].

In [29, Proposition 9.18, p. 139] one finds a necessary and sufficient condition on a measure function h under which an h-set Γ satisfies the ball condition. As a straightforward consequence of this statement we have the following useful proposition.

Proposition 4.10. *Let Γ be an h-set in $\mathbb{R}^n$. Then Γ fulfills the ball condition if, and only if, $\beta_h > -n$.*

Moreover, any h-set with the ball-condition has Lebesgue measure zero.

4.1. Slowly varying functions

The aim of this subsection is to provide a class of examples of measure functions. The material presented here will be used effectively in Subsection 7.3. We shall rely on the beautiful theory of regular variations, pioneered by J. Karamata in the early 30's, following closely the monograph [1].

Definition 4.11. Let H be a positive and measurable function defined on $(0, 1]$ satisfying

$$\lim_{r \to 0} \frac{H(sr)}{H(r)} = 1, \qquad \text{for every } s \in (0, 1]. \tag{4.7}$$

Then H is said to be a *slowly varying function* (in *Karamata's sense*).

Proposition 4.12. *Let H be a slowly varying function. Then the following properties hold true:*

(i) There exists a C^∞ slowly varying function $\widetilde{H} \sim H$. Moreover, if H is eventually monotone, so is $\widetilde{H}$ (see [1, Proposition 1.3.4, p. 15]).

(ii) For any $d > 0$ there exists a C^∞ non-decreasing function $h(r) \sim r^d H(r)$, $r \in (0, 1]$ (see [1, Proposition 1.8.2, p. 45]).

(iii) There exists a uniquely determined (up to equivalence) slowly varying function $H^\sharp$ such that $H(r)H^\sharp(rH(r)) \sim H^\sharp(r)H(rH^\sharp(r)) \sim 1$ (see [1, Proposition 1.5.13, p. 29]). The function $H^\sharp$ is called Bruijn conjugate of H.

The following theorem characterizes all slowly varying functions (see [1, Theorem 1.3.1, p. 12]).

Theorem 4.13. *A positive function H defined on $(0,1]$ is slowly varying if, and only if,*

$$H(r) = c(r)\exp\Big\{-\int_r^1 \varepsilon(u)\,\frac{du}{u}\Big\}, \qquad r \in (0,1], \tag{4.8}$$

where c and ε are measurable functions with $c(r) \to b \in (0,\infty)$ and $\varepsilon(u) \to 0$, as $r \to 0$.

Remark 4.14. As an immediate consequence of this theorem we have that given a function $l(r)$ with $rl'(r)/l(r)$ continuous in $(0,1]$ and $o(1)$, as $r \to 0$, then l is slowly varying. To see this consider simply $c(r) \equiv 1/l(1)$ and $\varepsilon(u) = ul'(u)/l(u)$ in the above representation theorem.

Example 4.15. Using this representation theorem, specific examples of slowly varying functions can be constructed at will (in the following examples r is to be taken small enough to avoid awkward formulations). Trivially, positive measurable functions with positive limit at 0 (in particular positive constants) are slowly varying. The first non-trivial example is $L_1(r) = |\log r|$. The iterated logarithms $L_2(r) = \log|\log r|$,

$$L_k(r) = \underbrace{\log\cdots\log}_{k-1 \text{ times}} |\log r| \tag{4.9}$$

are also slowly varying, as the powers of L_k, rational functions with positive coefficients formed with L_k, and so forth. Non-logarithmic examples are given by

$$H(r) = e^{L_1(r)^{\varkappa_1}\cdots L_k(r)^{\varkappa_k}}, \qquad 0 < \varkappa_k < 1, \tag{4.10}$$

and

$$G(r) = e^{\frac{L_1(r)}{L_2(r)}}. \tag{4.11}$$

Note that a slowly varying function may oscillate, an example being

$$H(r) = e^{|\log r|^{\frac{1}{3}}\cos(|\log r|^{\frac{1}{3}})}. \tag{4.12}$$

We make now a short digression.

In [13], [14] D. Edmunds and H. Triebel have introduced the class of admissible functions as the class of all positive monotone functions $\Psi\colon (0,1] \to \mathbb{R}$ satisfying

$$\Psi(2^{-2j}) \sim \Psi(2^{-j}), \qquad j \in \mathbb{N}_0. \tag{4.13}$$

In [13], [14], [24], [25] this class of weights is extensively studied in connection to function spaces of Besov and Triebel-Lizorkin type. We also contributed in [3] and [4] to this subject. Presently, we want to show that any admissible function is essentially a slowly varying function (see [8, p. 37–38] for the proof).

Proposition 4.16. *Let Ψ be an admissible function. Then there exists an admissible function $\widetilde{\Psi}$ equivalent to Ψ which is slowly varying.*

Finally, we remark some elementary, but significant, properties of slowly varying functions (see [1, Proposition 1.3.6, p. 16]).

Proposition 4.17. *The following properties hold true:*

1. If H varies slowly, then $\log H(r) = o(\log r)$, as $r \to 0$.

2. If H varies slowly, so does $H^a(r)$, for any $a \in \mathbb{R}$ and $H(r^a)$, for any $a > 0$.

3. If H_1 and H_2 vary slowly, so do $H_1 H_2$, $H_1 + H_2$ and $H_1 \circ H_2$ (provided $H_2(r) \to 0$ in this last case).

We quote now the main theorem concerning slowly varying functions and h-sets (see [8, Theorem 1.9.15])

Theorem 4.18. *Let H be a slowly varying function. Then*
(i) $h(r) \sim r^d H(r)$ is a measure function in $\mathbb{R}^n$, for all $0 < d < n$;
(ii) $h(r) \sim r^n H(r)$ is a measure function in $\mathbb{R}^n$, provided H is non-increasing;
(iii) $h(r) \sim H(r)$ is a measure function in $\mathbb{R}^n$, provided H is non-decreasing.

Remark 4.19. The $\sim$ in assertions (i), (ii) and (iii) means 'after appropriate replacement with an equivalent gauge function', which is possible by the Proposition 4.12.

Remark 4.20. There are measure functions h in $\mathbb{R}^n$ which are not of the form $r^d H(r)$, with $0 \leqslant d \leqslant n$ and H slowly varying. We refer to [8], [5] for details.

5. Besov spaces in $\mathbb{R}^n$

Besov and Triebel-Lizorkin spaces defined in terms of a perturbed smoothness have been already considered in some generality: see for instance [23], [10], [19], [18] and [16]. We shall follow closely the recent work [15] of W. Farkas and H.-G. Leopold which represents a general and unified approach.

Remark 5.1. We mention here that in complete analogy to the theory of Besov spaces with generalised smoothness briefly presented in this section, one can consider the generalised variant of the other famous scale of function spaces: the F-scale of Triebel-Lizorkin spaces (this is done in the papers mentioned above). However, although what follows in this section has indeed a perfect counterpart for the F-scale, we prefer, in view of our application to fractal sets, to restrict our attention to the B-scale only.

Before the main definition, we collect usual notation and basic concepts. $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of all C^∞ rapidly decreasing functions together with all their derivatives and $\mathcal{S}'(\mathbb{R}^n)$ is the space of all tempered distributions. The Fourier transform and the inverse Fourier transform of a tempered distribution f are denoted by $\mathcal{F}f$ and by $\mathcal{F}^{-1}f$, respectively.

Definition 5.2. By a *resolution of unity* $\Phi = \{\varphi_j\}_{j\in\mathbb{N}_0}$ (in $\mathbb{R}^n$) we mean a sequence of compactly supported smooth functions, such that

$$\mathrm{supp}\varphi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}; \tag{5.1}$$

$$\mathrm{supp}\varphi_j \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}, \qquad j = 1, 2, \ldots; \tag{5.2}$$

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1, \qquad \xi \in \mathbb{R}^n; \tag{5.3}$$

$$\sup_{\xi\in\mathbb{R}^n} |D^\alpha \varphi_j(\xi)| \leqslant c_\alpha 2^{-j|\alpha|}, \qquad \alpha \in \mathbb{N}_0^n. \tag{5.4}$$

Definition 5.3. If $\Phi = \{\varphi_j\}_{j\in\mathbb{N}_0}$ is a resolution of unity and $f \in \mathcal{S}'(\mathbb{R}^n)$, then we set, for $j = 0, 1, \ldots,$

$$\varphi_j(D)f = \mathcal{F}^{-1}(\varphi_j \mathcal{F} f). \tag{5.5}$$

Definition 5.4. If $\{a_j(x)\}_{j\in\mathbb{N}_0}$ is a sequence of functions defined in $\mathbb{R}^n$, then we put, for $0 < p, q \leqslant \infty$,

$$\|a_j \mid \ell_q(L_p)\| = \left\|\left\{\|a_j \mid L_p(\mathbb{R}^n)\|\right\}_j \mid \ell_q\right\|. \tag{5.6}$$

Now we are ready for the main definition.

Definition 5.5. Let $\Phi = \{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity. Consider an admissible sequence $\sigma = \{\sigma_j\}_{j\in\mathbb{N}_0}$ and fix $0 < p, q \leqslant \infty$. Then

$$B^\sigma_{p,q}(\mathbb{R}^n) = \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f \mid B^\sigma_{p,q}(\mathbb{R}^n)\| = \|\sigma_j\varphi_j(D)f \mid \ell_q(L_p)\| < \infty\right\}. \tag{5.7}$$

Remark 5.6. (i) As for the classical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ (subsumed in the above definition for $\sigma = \{2^{sj}\}_{j\in\mathbb{N}_0}$), also the spaces $B^\sigma_{p,q}(\mathbb{R}^n)$ are independent, up to equivalent quasi-norms, from the chosen resolution of unity appearing in their definition.

(ii) If Ψ is an admissible sequence in the sense of [13], then $\sigma = \{2^{sj}\Psi(2^{-j})\}_{j\in\mathbb{N}_0}$ is an admissible sequence and the resulting Besov spaces coincide with the perturbed spaces called $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$ in [13], [14].

In both cases (classical and perturbed spaces) we stick at the original notation.

Some known facts valid for the classical spaces remain valid, after adequate modifications, also in this generalised context. We quote only those results we shall use in the sequel and we refer to the list of works quoted above for specific results and comments.

If $0 < q, r \leqslant \infty$ we let $1 \leqslant (q_r)' \leqslant \infty$ defined by

$$\begin{cases} \dfrac{1}{(q_r)'} + \dfrac{1}{q} = \dfrac{1}{r}, & \text{if} \quad q > r, \\ \infty, & \text{otherwise}. \end{cases} \tag{5.8}$$

Then one can easily prove the following assertion.

Proposition 5.7. *Let $0 < p, q, r \leqslant \infty$ be fixed indices and let σ and τ be two admissible sequences. Then*

$$B^{\sigma}_{p,q}(\mathbb{R}^n) \subset B^{\tau}_{p,r}(\mathbb{R}^n), \qquad \text{if} \quad \sigma^{-1}\tau \in \ell_{(q_r)'}. \tag{5.9}$$

The result concerning the embedding in $L_p(\mathbb{R}^n)$ reads as follows. Remember that for $a \in \mathbb{R}$, a_+ means $\max(a, 0)$.

Proposition 5.8. *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and let σ be an admissible sequence with $\beta_\sigma > n(1/\min(1,p) - 1)$. Then*

$$B^{\sigma}_{p,q}(\mathbb{R}^n) \subset L_p(\mathbb{R}^n), \qquad \text{if} \quad p \geqslant 1, \tag{5.10}$$

and

$$B^{\sigma}_{p,q}(\mathbb{R}^n) \subset L_1(\mathbb{R}^n) \cap L_p(\mathbb{R}^n), \qquad \text{if} \quad p < 1. \tag{5.11}$$

This assertion can be easily proved using the atomic representation theorem formulated in full generality and proved in [15].

5.1. Quarks and quarkonial decompositions: the regular case

Quite recently H. Triebel has introduced in [28] the idea of quarks as refined building blocks in the function spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$. In what follows we present on the one hand the quarkonial representation theorem for $B^{\sigma}_{p,q}(\mathbb{R}^n)$ in the case where no moment conditions are needed. On the other hand, in the next subsection we shall consider the general case. The main theorems 5.13 and 5.17 are stated and proved in [8] (where we also consider the F-case), since they do not appear in the paper [15]. However, our proof is mostly indebted to the results contained in that paper on the one hand and on the techniques used by H. Triebel in his new book [29] on the other (see later on for specific references).

This decomposition theorem will turn out to be extremely useful in order to deal with the estimation of entropy numbers of compact embeddings between Besov spaces defined on h-sets (Section 7).

We begin with some preparations.

Definition 5.9. A *mother function* Θ is a non-negative smooth function with the following properties:

1. $\operatorname{supp}\Theta \subset \{x \in \mathbb{R}^n : |x| < 2^{\varrho}\}$, for some $\varrho > 0$;
2. $\sum_{m \in \mathbb{Z}^n} \Theta(x - m) = 1$, for all $x \in \mathbb{R}^n$.

If Θ is a mother function and $\beta \in \mathbb{N}^n_0$, we let

$$\Theta^{\beta}(x) = x^{\beta}\Theta(x), \qquad x \in \mathbb{R}^n, \tag{5.12}$$

where $x^{\beta} = x_1^{\beta_1} \cdots x_n^{\beta_n}$ if $x = (x_1, \ldots, x_n)$.

Definition 5.10. Let σ be an admissible sequence, $0 < p \leqslant \infty$ and consider a mother function Θ as in 5.9. Then the expression

$$\beta\text{-qu}_{j,m}(x) = \sigma_j^{-1} 2^{\frac{n}{p}j} \Theta^{\beta}(2^j x - m), \qquad \beta \in \mathbb{N}^n_0,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{5.13}$$

is called (σ, p)*–β-quark.*

Occasionally, we prefer the less rigorous notation "(σ_j, p)–β-quark" to "(σ, p)–β-quark", mostly if $\sigma = \{\sigma_j\}_{j \in \mathbb{N}_0}$ is explicitly given.

Remark 5.11. The quarks are well localized: we have, for some $d > 1$,

$$\operatorname{supp}\beta\text{-qu}_{j,m} \subset B(2^{-j}m, d2^{-j}), \tag{5.14}$$

where $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Moreover, one has, for every $x \in \mathbb{R}^n$,

$$|\Theta^\beta(x)| \leqslant 2^{\varrho|\beta|}, \qquad \beta \in \mathbb{N}_0^n. \tag{5.15}$$

The factor $\sigma_j^{-1} 2^{\frac{n}{p}j}$ is a normalising term by which quarks become building blocks:

$$c_1^\beta \leqslant \|\beta\text{-qu}_{j,m} \mid B^\sigma_{p,p}(\mathbb{R}^n)\| \leqslant c_2^\beta, \tag{5.16}$$

for some positive constants c_1^β and c_2^β which are independent of $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

Quarkonial decompositions rely on an interplay between sufficiently smooth building blocks (the quarks) and some sequence spaces. In the following definition we precise the type of sequence spaces we need.

Definition 5.12. Let $0 < p, q \leqslant \infty$ and

$$\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}. \tag{5.17}$$

Then we define

$$b_{p,q} = \left\{\lambda : \|\lambda \mid b_{p,q}\| = \left(\sum_{j=0}^{\infty}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p\Big)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty\right\} \tag{5.18}$$

(modification if p and/or q is infinite).

We can now state the quarkonial representation theorem for the spaces $B^\sigma_{p,q}(\mathbb{R}^n)$.

Theorem 5.13. *Fix a mother function Θ as in 5.9, in particular with ϱ given by 5.9-(i). Let $0 < p, q \leqslant \infty$ and consider an admissible sequence σ with*

$$\beta_\sigma > n\Big(\frac{1}{p} - 1\Big)_+. \tag{5.19}$$

Fix a number $\kappa > \varrho$. Then the following assertion holds true.

Any f in $S'(\mathbb{R}^n)$ belongs to $B^\sigma_{p,q}(\mathbb{R}^n)$ if, and only if, there exists a sequence $\lambda = \{\lambda^\beta\}_{\beta\in\mathbb{N}_0^n}$ of elements of $b_{p,q}$ such that

$$f = \sum_{\beta\in\mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{j,m}\beta\text{-qu}_{j,m}(x), \qquad \textit{in } S'(\mathbb{R}^n) \tag{5.20}$$

and

$$\|\lambda \mid b_{p,q}\|_\kappa = \sup_{\beta\in\mathbb{N}_0^n} 2^{\kappa|\beta|}\|\lambda^\beta \mid b_{p,q}\| < \infty. \tag{5.21}$$

Moreover,

$$\|\lambda \mid b_{pq}\|_\kappa \sim \|f \mid B^\sigma_{p,q}(\mathbb{R}^n)\|. \tag{5.22}$$

Further, the coefficients $\lambda = \{\lambda^\beta_{j,m}\}$ *can be chosen linearly dependent on* f.

Remark 5.14. We do not want to go into details and we refer to Chapter 1, especially Sections 1, 2 and 3 of [29] for heuristics, motivations, results and comments on quarkonial expansions. We only remark the convergence in $S'(\mathbb{R})$ of the sum (5.20) is not an additional requirement of the theorem, but it is always implied by (5.21). Moreover the convergence of this sum is unconditional and hence we shall briefly write form now on (and also in the general case, where an analogous assertion holds true)

$$\sum_{\beta,j,m} \quad \text{in place of} \quad \sum_{\beta\in\mathbb{N}_0^n}\sum_{j=0}^\infty\sum_{m\in\mathbb{Z}^n}. \tag{5.23}$$

5.2. Quarkonial decompositions: the general case

The quarkonial representation theorem 5.13 cannot be applied to $B^\sigma_{p,q}(\mathbb{R}^n)$ if $\beta_\sigma \leqslant n(1/p-1)_+$. This is due to the lack of quarks with moment conditions, which cannot be avoided for general sequences σ. Here we point out the necessary changes in order to deal with the general case. Again, we follow [29, Section 3].

Remember that, for $k \in \mathbb{N}_0$,

$$(-\Delta)^k = \Big(-\sum_{l=1}^n \frac{\partial^2}{\partial x_l^2}\Big)^k = (-1)^k \sum_{k_1+\cdots+k_n=k} \frac{\partial^{2k_1}}{\partial x_1^{2k_1}}\cdots\frac{\partial^{2k_n}}{\partial x_n^{2k_n}}. \tag{5.24}$$

Definition 5.15. Let Θ be a mother function, σ an admissible sequence and $0<p\leqslant\infty$. Let $(L+1)/2 \in \mathbb{N}_0$ and consider $\beta\in\mathbb{N}_0^n$. Then

$$\beta\text{-qu}^L_{j,m}(x) = \sigma_j^{-1} 2^{\frac{n}{p}j}\big((-\Delta)^{\frac{L+1}{2}}\Theta^\beta\big)(2^jx-m) \tag{5.25}$$

is called $(\sigma,p)^L$*–*β*-quark*. Here $j\in\mathbb{N}_0$ and $m\in\mathbb{Z}^n$.

Remark 5.16. If $L=-1$, then the above definition coincides with Definition 5.10. In particular, $\beta\text{-qu}^{-1}_{j,m} = \beta\text{-qu}_{j,m}$ are called as before (σ,p)–β-quarks. If $L>-1$, then moment conditions up to order L

$$\int x^\alpha \beta\text{-qu}^L_{j,m}(x)\,dx = 0, \qquad |\alpha|\leqslant L, \tag{5.26}$$

are ensured.

The general decomposition theorem reads as follows.

Theorem 5.17. *Fix a mother function Θ as in 5.9, in particular with ϱ given by 5.9-(i). Let $0 < p, q \leqslant \infty$ and consider an admissible sequence σ. Choose a number L and an auxiliary admissible sequence τ such that $(L+1)/2 \in \mathbb{N}_0$ and*

$$L > -1 + n\Big(\frac{1}{p} - 1\Big)_+ - \beta_\sigma, \quad \beta_\tau > n\Big(\frac{1}{p} - 1\Big)_+ \quad \text{and} \quad \beta_{\sigma^{-1}\tau} > 0. \tag{5.27}$$

and let β-$\mathrm{qu}_{j,m}$ and β-$\mathrm{qu}^L_{j,m}$ be $(\tau,p)-\beta$-quarks and $(\sigma,p)^L-\beta$-quarks generated by Θ, respectively. Fix a number $\kappa > \varrho$. Then the following assertion holds true.

Any f in $S'(\mathbb{R}^n)$ belongs to $B^\sigma_{p,q}(\mathbb{R}^n)$ if, and only if, there exist two sequences $\lambda = \{\lambda^\beta\}_{\beta\in\mathbb{N}_0^n}$ and $\eta = \{\eta^\beta\}_{\beta\in\mathbb{N}_0^n}$ of elements of $b_{p,q}$ such that

$$f = \sum_{\beta,j,m} \big(\eta^\beta_{j,m}\beta\text{-}\mathrm{qu}_{j,m}(x) + \lambda^\beta_{j,m}\beta\text{-}\mathrm{qu}^L_{j,m}(x)\big), \qquad \text{in } S'(\mathbb{R}^n) \tag{5.28}$$

and

$$\|\eta \mid b_{p,q}\|_\kappa + \|\lambda \mid b_{p,q}\|_\kappa < \infty. \tag{5.29}$$

Moreover, we have that

$$\|\eta \mid b_{p,q}\|_\kappa + \|\lambda \mid b_{p,q}\|_\kappa \sim \|f \mid B^\sigma_{p,q}(\mathbb{R}^n)\|. \tag{5.30}$$

Further, The coefficients λ and η can be chosen linearly dependent on f.

6. Function spaces on h-sets

In this section we give the definition of $B^\sigma_{p,q}(\Gamma)$, where Γ is an h-set. We follow closely [8], [7].

First, observe that we can canonically define the spaces $L_p(\Gamma)$, for $0 < p \leqslant \infty$. The related measure is any h-measure associated to Γ by its definition: since any two such measures are mutually equivalent (by virtue of Proposition 4.7-(i)) and sufficiently regular, they originate the same space (up to equivalent quasi-norms). We shall always think of $L_p(\Gamma)$ endowed with one of these measures, say $\mathcal{H}^h|\Gamma$.

Suppose now that there exists a positive constant c such that

$$\|\varphi|_\Gamma \mid L_p(\Gamma)\| \leqslant \mathrm{c}\|\varphi \mid B^\sigma_{p,q}(\mathbb{R}^n)\|, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{6.1}$$

Then, for $\max(p,q) < \infty$, exploiting the density of $\mathcal{S}(\mathbb{R}^n)$ in $B^\sigma_{p,q}(\mathbb{R}^n)$, we can define by completion the *trace* of any function $f \in B^\sigma_{p,q}(\mathbb{R}^n)$ on Γ and denote it by $\mathrm{tr}_\Gamma f$ (*trace of f on Γ*). We have that the following optimal assertion holds true ([7, Theorem 5.9]).

Theorem 6.1. *Let $0 < p < \infty$, $0 < q \leqslant \min(1,p)$ and let $\Gamma \subset \mathbb{R}^n$ be an h-set satisfying the ball condition, according to Definition 4.9. Then*

$$L_p(\Gamma) = \mathrm{tr}_\Gamma B^{h_p}_{p,q}(\mathbb{R}^n). \tag{6.2}$$

Here we have used the following compact notation for the admissible sequence h_p, for $0 < p \leqslant \infty$ which will be also exploited throughout this note:

$$h_p \qquad \text{denotes the sequence} \qquad \{h(2^{-j})^{1/p}2^{\frac{n}{p}j}\}_{j\in\mathbb{N}_0}. \tag{6.3}$$

Notice that h_∞ is the constant sequence $(1,1,\ldots)$.

Theorem 6.1 gives the motivation for the following definition, which is at the heart of this section.

Definition 6.2. Consider an h-set $\Gamma \subset \mathbb{R}^n$ satisfying the ball condition. Let σ be an admissible sequence with $\beta_\sigma > 0$ and let $0 < p, q \leqslant \infty$. Then we define

$$B^\sigma_{p,q}(\Gamma) = \mathrm{tr}_\Gamma B^{\sigma h_p}_{p,q}(\mathbb{R}^n), \tag{6.4}$$

endowed with the quasi-norm

$$\|f \mid B^\sigma_{p,q}(\Gamma)\| = \inf\left\|g \mid B^{\sigma h_p}_{p,q}(\mathbb{R}^n)\right\|, \tag{6.5}$$

where the infimum is taken over all $g \in B^{\sigma h_p}_{p,q}(\mathbb{R}^n)$ with $\mathrm{tr}_\Gamma\, g = f$.

By Theorem 6.1 we can complement this definition letting

$$B^\sigma_{p,q}(\Gamma) = L_p(\Gamma), \tag{6.6}$$

for all $0 < p, q \leqslant \infty$ and all admissible sequences σ with $\beta_\sigma = 0$.

Remark 6.3. If $\beta_\sigma > 0$, then by Proposition 5.7

$$B^{\sigma h_p}_{p,q}(\mathbb{R}^n) \subset B^{h_p}_{p,\min(p,1)}(\mathbb{R}^n). \tag{6.7}$$

Therefore the above definition makes sense also for $q = \infty$, at least if $p < \infty$. But if $p = \infty$ we have that

$$B^\sigma_{\infty,q}(\mathbb{R}^n) \subset B^0_{\infty,1}(\mathbb{R}^n) \subset C(\mathbb{R}^n), \tag{6.8}$$

where $C(\mathbb{R}^n)$ is the space of all bounded and uniformly continuous functions in $\mathbb{R}^n$, normed in the usual way. This follows from 5.7 and the known (sharp) embedding of $B^0_{\infty,1}(\mathbb{R}^n)$ in $C(\mathbb{R}^n)$ (see, for instance [27, Thoerem 1, p. 32]). Therefore the trace operator, in this case, is simply the pointwise restriction.

Remark 6.4. Definition 6.2 generalises the definition of the Besov-type spaces $B^{(s,\Psi^a)}_{p,q}(\Gamma)$, which are defined in [14] for a (d,Ψ)-set Γ as follows:

$$B^{(s,\Psi^a)}_{p,q}(\Gamma) = \mathrm{tr}_\Gamma B^{(s+\frac{n-d}{p},\Psi^{\frac{1}{p}+a})}_{p,q}(\mathbb{R}^n) \tag{6.9}$$

(quasi-normed analogously to (6.5)), where $0 < p, q \leqslant \infty$, $s > 0$, $a \in \mathbb{R}$ and $0 < d < n$. In fact, these spaces are clearly obtained with $\sigma = \{2^{js}\Psi^a(2^{-j})\}_{j \in \mathbb{N}_0}$ in 6.2 (remember that in this case $h(2^{-j}) = 2^{-dj}\Psi(2^{-j})$, $j \in \mathbb{N}_0$).

There is the possibility to provide more direct characterizations of the spaces $B^\sigma_{p,q}(\Gamma)$, relying on atoms, quarks, local polynomial approximations and differences. Some of these procedures are described in [8].

7. Compact embeddings

Here we present an application of the theory developed so far, which is the heart of this note: we shall examine under which condition the embedding

$$B^\sigma_{p_1,q_1}(\Gamma) \xrightarrow{id} B^\tau_{p_2,q_2}(\Gamma) \tag{7.1}$$

turns out to be compact. Moreover, in this case we shall study the entropy numbers $e_j(\mathrm{id})$, $j \in \mathbb{N}_0$ related to this embedding.

In order to get the main result (Theorem 7.12) we present basic definitions and results on compact embeddings between weighted sequence spaces, which play a decisive role in what follows. As it will be clear the quarkonial representation theorem 5.17 also plays an outstanding role, reducing the problem of the upper estimation of the entropy numbers of (7.1) to the known estimation of the entropy numbers of the embeddings between the mentioned weighted sequence spaces.

References on this subject are given below.

7.1. Sequence spaces

In this subsection we collect definitions and results for weighted sequence spaces. Their importance, apart from their own interest, will appear in subsection 7.2.

The material presented in this subsection closely refers to the articles of H.-G. Leopold [20], [21].

We give the general definition of the spaces we have in mind, later on we shall restrict ourselves to a particular class of sequence spaces.

Throughout this chapter, by a *general weight sequence* $\{\mathfrak{p}_j\}_{j \in \mathbb{N}_0}$ we mean a sequence of real non-negative numbers $\mathfrak{p}_j$.

Definition 7.1. Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, let $\{\mathfrak{p}_j\}_{j \in \mathbb{N}_0}$ be a general weight sequence and $\{M_j\}_{j \in \mathbb{N}}$ be a sequence of natural numbers. Then

$$\ell_q(\mathfrak{p}_j \ell_p^{M_j}) = \Bigg\{ \zeta : \zeta = \{\zeta_{j,m}\}_{j \in \mathbb{N}_0,\, m=1,\ldots,M_j} \text{ with}$$

$$\|\zeta \mid \ell_q(\mathfrak{p}_j \ell_p^{M_j})\| = \left(\sum_{j=0}^{\infty} \mathfrak{p}_j^q \Big(\sum_{m=1}^{M_j} |\zeta_{j,m}|^p \Big)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \Bigg\} \tag{7.2}$$

(with obvious modifications if p and/or q are infinity).

It is clear that these sequence spaces are quasi-Banach spaces. This definition coincides with [28, (8.2), p. 38], with $\mathfrak{p}_j = 2^{\delta j}$ and $M_j \sim 2^{jd}$.

We are interested in the embedding

$$\mathrm{id}\colon \ell_{q_1}(\mathfrak{p}_j \ell_{p_1}^{M_j}) \to \ell_{q_2}(\ell_{p_2}^{M_j}). \tag{7.3}$$

The following theorem clarifies when such an embedding makes sense. We recall again that $a_+ = \max(a, 0)$, for $a \in \mathbb{R}$.

Theorem 7.2. *Let $0 < p_1, p_2 \leqslant \infty$ and $0 < q_1, q_2 \leqslant \infty$. Let $\{\mathfrak{p}_j\}_{j\in\mathbb{N}_0}$ be a general weight sequence and $\{M_j\}_{j\in\mathbb{N}}$ be an arbitrary sequence of natural numbers. Then the embedding (7.3) is bounded if, and only if,*

$$\left\{\mathfrak{p}_j^{-1} M_j^{(\frac{1}{p_2}-\frac{1}{p_1})_+}\right\}_{j\in\mathbb{N}_0} \in \ell_{q^*}, \tag{7.4}$$

where

$$\frac{1}{q^*} = \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+, \quad \text{i.e.,} \quad q^* = \begin{cases} \infty, & \text{if } 0 < q_1 \leqslant q_2 \leqslant \infty, \\ \dfrac{1}{q_2} - \dfrac{1}{q_1}, & \text{if } 0 < q_2 < q_1 < \infty, \\ q_2, & \text{if } q_1 = \infty. \end{cases} \tag{7.5}$$

Moreover, if this embedding is bounded, the norm in ℓ_{q^*} of the sequence appearing in (7.4) is an upper bound for $\|\mathrm{id} \mid \ell_{q_1}(\mathfrak{p}_j \ell_{p_1}^{M_j}) \to \ell_{q_2}(\ell_{p_2}^{M_j})\|$.

When the identity operator id in (7.3) exists, one may ask for the compactness of this operator. The following theorem provides the answer.

Theorem 7.3. *Let $0 < p_1, p_2 \leqslant \infty$ and $0 < q_1, q_2 \leqslant \infty$. Let $\{\mathfrak{p}_j\}_{j\in\mathbb{N}_0}$ be a general weight sequence and $\{M_j\}_{j\in\mathbb{N}_0}$ be an arbitrary sequence of natural numbers. Then the embedding (7.3) is compact if, and only if,*

$$\left\{\mathfrak{p}_j^{-1} M_j^{(\frac{1}{p_2}-\frac{1}{p_1})_+}\right\}_{j\in\mathbb{N}_0} \in \ell_{q^*}, \qquad \text{if } q^* < \infty \tag{7.6}$$

or

$$\lim_{j\to\infty} \mathfrak{p}_j^{-1} M_j^{(\frac{1}{p_2}-\frac{1}{p_1})_+} = 0, \qquad \text{if } q^* = \infty. \tag{7.7}$$

The next step is the calculation of the entropy numbers of the above embedding. We briefly recall the definition and some properties of entropy numbers. As far as this topic is concerned, the reader may refer, for instance, to [11] or [12].

Recall that if A and B are two quasi-Banach (complex) spaces, then $\mathcal{L}(A, B)$ denotes the space of all linear and bounded operators normed in the natural way. For short, we let $\mathcal{L}(A, A) = \mathcal{L}(A)$.

Definition 7.4. Let A and B be two quasi-Banach complex spaces and let $T: A \to B$ be a linear and bounded operator. Then for all $j \in \mathbb{N}$, the j-th (dyadic) entropy number of T is defined by

$$e_j(T) = \inf\Big\{\varepsilon > 0 : T(\mathcal{B}_A) \subset \bigcup_{l=1}^{2^{j-1}} (b_l + \varepsilon \mathcal{B}_B), \text{ for some } b_1, \dots, b_{2^{j-1}} \in B\Big\}, \quad (7.8)$$

where $\mathcal{B}_A$ and $\mathcal{B}_B$ denote the closed unitary balls in A and in B, respectively.

The main properties of entropy numbers are summarized in the following proposition.

Proposition 7.5. *Let A, B and C be three quasi-Banach complex spaces. Let $S, T \in \mathcal{L}(A, B)$ and $R \in \mathcal{L}(B, C)$. Then*

1. $\|T\| \geqslant e_1(T) \geqslant e_2(T) \geqslant \cdots$ *($e_1(T) = \|T\|$ if B is a Banach space);*
2. *for all $j, l \in \mathbb{N}$*

$$e_{j+l-1}(R \circ S) \leqslant e_j(R) e_l(S); \quad (7.9)$$

3. *if $A = B$,*

$$T \text{ is a compact operator if, and only if, } e_j(T) \to 0, \quad \text{as } j \to \infty. \quad (7.10)$$

Let $T \in \mathcal{L}(A)$ be a compact operator. Then the spectrum of T, apart from the point 0, consists solely of eigenvalues of finite algebraic multiplicity: let $\{\lambda_j(T)\}_{j \in \mathbb{N}}$ be the sequence of all non-zero eigenvalues of T, repeated according to their algebraic multiplicity and ordered so that

$$|\lambda_1(T)| \geqslant |\lambda_2(T)| \geqslant \cdots \to 0. \quad (7.11)$$

If T has only m distinct eigenvalues and M is the sum of their algebraic multiplicity we put $\lambda_n(T) = 0$, for all $n > M$.

Then the following theorem (proved by B. Carl in [9] in the context of Banach spaces and in [12] in the context of quasi-Banach spaces) sheds some light on the connection between entropy numbers (which express a geometrical property of the compact operator T) with the eigenvalues of T (which are related to the spectral properties of the operator).

Theorem 7.6. *Let T and $\{\lambda_m(T)\}_{m \in \mathbb{N}}$ as above. Then*

$$\Big(\prod_{m=1}^{j} |\lambda_m(T)|\Big)^{\frac{1}{j}} \leqslant \inf_{n \in \mathbb{N}} 2^{\frac{n}{2j}} e_n(T), \qquad j \in \mathbb{N}. \quad (7.12)$$

As a consequence of (7.11) and 7.6 we get the famous Carl's inequality

$$|\lambda_j(T)| \leqslant \sqrt{2}\, e_j(T), \qquad j \in \mathbb{N}, \quad (7.13)$$

which is a clear example of the above mentioned connection between entropy numbers and eigenvalues of a compact operator.

Let us come back to the estimations of the entropy numbers related to the operator

$$\mathrm{id}\colon \ell_{q_1}(\mathfrak{p}_j \ell_{p_1}^{M_j}) \to \ell_{q_2}(\ell_{p_2}^{M_j}). \tag{7.14}$$

It turns out that under some restrictions on the weight sequence $\{\mathfrak{p}_j\}_{j\in\mathbb{N}_0}$ and the sequence $\{M_j\}_{j\in\mathbb{N}}$, the entropy numbers of the above embedding can be completely estimated (up to equivalence). For the following definition and the next relevant Theorem 7.9 we refer to [21, Theorem 3].

Definition 7.7. A sequence $\{a_j\}_{j\in\mathbb{N}_0}$ of positive real numbers is called *almost strongly increasing* if there is a constant $\kappa_0 \in \mathbb{N}$ such that

$$2a_j \leqslant a_k, \qquad \text{for every } j \text{ and } k \text{ with } k \geqslant j+\kappa_0. \tag{7.15}$$

Geometric sequences, i.e., sequences of the form $a_j = 2^{bj}$, $b>0$, are clearly almost strongly increasing.

More generally, if we consider a measure function h which does not decay too slowly near zero, then it turns out that $\{h_j^{-1}\}_{j\in\mathbb{N}_0}$ is an almost strongly increasing sequence, where $h_j = h(2^{-j})$, for $j \in \mathbb{N}_0$, as we point out in the following proposition. The reason for considering this type of sequences will be clear in the next subsection.

Proposition 7.8. *Let h be a measure function with $\alpha_h < 0$. Then $\{h(2^{-j})^{-1}\}_{j\in\mathbb{N}_0}$ is an almost strongly increasing sequence.*

Proof. Let $h_j = h(2^{-j})$, for $j \in \mathbb{N}_0$. In order to prove the assertion we have to find $\kappa_0 \in \mathbb{N}$ such that

$$\frac{h_{j+l+\kappa_0}}{h_j} \leqslant \frac{1}{2}, \qquad j,l \in \mathbb{N}_0. \tag{7.16}$$

Let $\varepsilon > 0$ be chosen such that $\alpha_h + \varepsilon < 0$. By assumption and the definition of α_h there exists a constant $c = c(\varepsilon) > 0$ such that

$$\frac{h_{j+l+\kappa_0}}{h_j} \leqslant \overline{h}_{l+\kappa_0} \leqslant c 2^{(\alpha_h+\varepsilon)(l+\kappa_0)}, \qquad j,l \in \mathbb{N}_0, \tag{7.17}$$

holds for any $\kappa_0 \in \mathbb{N}_0$. Choosing now κ_0 big enough we conclude the proof. ∎

The role played by almost strongly increasing sequences is explained by the following relevant theorem.

Theorem 7.9. *Let $0 < p_1 \leqslant p_2 \leqslant \infty$ and $0 < q_1, q_2 \leqslant \infty$. Let $\{\mathfrak{p}_j\}_{j\in\mathbb{N}_0}$ be an almost strongly increasing weight sequence and let $\{M_j\}_{j\in\mathbb{N}_0}$ be an almost strongly increasing sequence of natural numbers. Then*

$$e_{2M_L}\big(\mathrm{id}\colon \ell_{q_1}(\mathfrak{p}_j \ell_{p_1}^{M_j}) \to \ell_{q_2}(\ell_{p_2}^{M_j})\big) \sim \mathfrak{p}_L^{-1} M_L^{-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}, \qquad L \in \mathbb{N}_0. \tag{7.18}$$

Unfortunately, this theorem is not completely sufficient for our later purposes. We need something like an ℓ_u-version of the above assertion. But fortunately, it comes out that these generalisations are nothing more than a technical appendix to the just mentioned result.

Let again $\ell_q(\mathfrak{p}_j \ell_p^{M_j})$ be the sequence space introduced in Definition 7.1. Let, in addition, $\varrho > 0$ and $0 < u \leqslant \infty$. Then

$$\ell_u\big[2^{\varrho l}\ell_q(\mathfrak{p}_j \ell_p^{M_j})\big] \tag{7.19}$$

denotes the linear space of all $\ell_q(\mathfrak{p}_j \ell_p^{M_j})$-valued sequences $x = \{x^l\}_{l\in\mathbb{N}_0}$ endowed with the quasi-norm

$$\big\|x \mid \ell_u\big[2^{\varrho l}\ell_q(\mathfrak{p}_j \ell_p^{M_j})\big]\big\| = \Big(\sum_{l=0}^{\infty} 2^{\varrho l u}\|x^l \mid \ell_q(\mathfrak{p}_j \ell_p^{M_j})\|^u\Big)^{\frac{1}{u}}, \tag{7.20}$$

with obvious modification if $u = \infty$. For these spaces one can prove the following theorem.

Theorem 7.10. *Let $0 < p_1 \leqslant p_2 \leqslant \infty$, $0 < u_1, u_2 \leqslant \infty$ and $0 < q_1, q_2 \leqslant \infty$. Let $\{\mathfrak{p}_j\}_{j\in\mathbb{N}_0}$ be an almost strongly increasing weight sequence and let $\{M_j\}_{j\in\mathbb{N}_0}$ be an almost strongly increasing sequence of natural numbers. Then, for $\varrho > 0$, the embedding*

$$\ell_{u_1}\big[2^{\varrho l}\ell_{q_1}(\mathfrak{p}_j \ell_{p_1}^{M_j})\big] \subset \ell_{u_2}\big[\ell_{q_2}(\ell_{p_2}^{M_j})\big] \tag{7.21}$$

is compact and for the related entropy numbers we have

$$e_{2M_L}\big(\mathrm{id}: \ell_{u_1}\big[2^{\varrho l}\ell_{q_1}(\mathfrak{p}_j \ell_{p_1}^{M_j})\big] \to \ell_{u_2}\big[\ell_{q_2}(\ell_{p_2}^{M_j})\big]\big) \sim \mathfrak{p}_L^{-1} M_L^{-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}. \tag{7.22}$$

The proof can be given following the analogous proof of [28, Theorem 9.2, p. 47] and the estimates of entropy numbers given in Theorem 7.9.

7.2. Compact embeddings between Besov spaces

In this subsection we consider the embedding id between the Besov spaces

$$B^{\sigma}_{p_1,q_1}(\Gamma) \xrightarrow{id} B^{\tau}_{p_2,q_2}(\Gamma), \tag{7.23}$$

where σ and τ are admissible sequences with β_σ and $\beta_\tau \geqslant 0$. If Γ is a d-set, then the above embedding is studied in [28] and the outcome is the following important theorem ([28, Theorem 20.6, p. 166] is even more general).

Theorem 7.11. *Let Γ be a compact d-set in $\mathbb{R}^n$ with $0 < d < n$. Let*

$$0 \leqslant s_1 < s_2 < \infty \tag{7.24}$$

$$1 < p_1, p_2 < \infty, \quad 0 < q_1, q_2 \leqslant \infty, \tag{7.25}$$

and

$$s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0. \tag{7.26}$$

Then the embedding of $B^{s_1}_{p_1,q_1}(\Gamma)$ *into* $B^{s_2}_{p_2,q_2}(\Gamma)$ *is compact and for the related entropy numbers one has:*

$$e_j\big(\mathrm{id}\colon B^{s_1}_{p_1,q_1}(\Gamma) \to B^{s_2}_{p_2,q_2}(\Gamma)\big) \sim j^{-\frac{s_1-s_2}{d}}, \qquad j \in \mathbb{N}. \tag{7.27}$$

Recently, S. de Moura has proved an analogous theorem for a (d,Ψ)-set Γ (see [25] and also the discussion on this subject after Proposition 7.16). Accordingly, the Besov spaces on Γ taken now into consideration are defined via trace procedures from the tailored spaces $B^{(s,\Psi^{1/p})}_{p,q}(\mathbb{R}^n)$, as we have remarked in 6.4.

In both cases, the underlying main tool exploited in the estimations for the entropy numbers of these embeddings is the quarkonial representation of the elements in the considered Besov spaces and the results concerning compact embeddings for the sequence spaces studied in Subsection 7.1.

Thanks to 5.13 and 5.17, we have also a characterization of $B^{\sigma}_{p,q}(\mathbb{R}^n)$ with quarks and this allows us to state and prove the following theorem.

Remember that to avoid awkward formulations we agree on

$$B^{\sigma}_{p,q}(\Gamma) = L_p(\Gamma), \tag{7.28}$$

for all $0 < p, q \leqslant \infty$, if $\beta_\sigma = 0$, simply as a notation. Moreover, if $\{\alpha_j\}$ is an unbounded increasing sequence of positive numbers we write for entropy numbers e_{α_j} instead of $e_{[\alpha_j]}$, where $[\cdot]$ denotes the integer-part function.

Theorem 7.12. *Let* $\Gamma \subset \mathbb{R}^n$ *be an* h*-set, with* $-n < \beta_h \leqslant \alpha_h < 0$ *and let* $d = \dim_{\mathcal{P}} \Gamma$. *Let* σ *and* τ *be two admissible sequences with* $\beta_\tau \geqslant 0$. *Consider* $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leqslant \infty$ *and suppose*

$$\beta_\sigma - \alpha_\tau > d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+. \tag{7.29}$$

Then the embedding of $B^{\sigma}_{p_1,q_1}(\Gamma)$ *into* $B^{\tau}_{p_2,q_2}(\Gamma)$ *is compact and for the related entropy numbers one has:*

$$e_{h_j^{-1}}\big(\mathrm{id}\colon B^{\sigma}_{p_1,q_1}(\Gamma) \to B^{\tau}_{p_2,q_2}(\Gamma)\big) \sim \sigma_j^{-1}\tau_j, \qquad j \in \mathbb{N}, \tag{7.30}$$

where h_j *denotes the sequence* $\{h(2^{-j})\}_{j\in\mathbb{N}_0}$.

Remark 7.13. Before giving the proof of the above theorem we point out some comments: the assumption $-n < \beta_h \leqslant \alpha_h < 0$ is in some sense necessary: it guarantees that the fractal Γ preserves the ball condition (Proposition 4.10) and that the sequence $\{h_j^{-1}\}$ is almost strongly increasing (Proposition 7.8). The relevant assumption is of course (7.29). One could prove that this condition (which is a generalization of (7.26)) is necessary. If it fails, then either one has no embedding, or the embedding turns out to be non-compact. We refer to [28, 20.7, p. 169] where this last observation is discussed in detail for d-sets, but with immediate modifications one could treat the general case. The asymptotic behaviour (7.30) is also a generalization of (7.27): we shift to Remark 7.14 more detailed comments about its structure.

Proof of Theorem 7.12. Step 1. Let $p_2 \geqslant p_1$. Let us consider $f \in B^{\sigma}_{p_1,q_1}(\Gamma)$. Then there exists a (non-linear) bounded extension operator $g = \mathrm{ext}\, f$ such that

$$\mathrm{tr}_\Gamma\, g = f \qquad \text{and} \qquad \left\| g \mid B^{\sigma h_{p_1}}_{p_1,q_1}(\mathbb{R}^n) \right\| \leqslant 2 \| f \mid B^{\sigma}_{p_1,q_1}(\Gamma) \|. \qquad ()$$

We recall that for $p \in (0,\infty]$ the sequence σh_p stands for $\{\sigma_j h(2^{-j})^{\frac{1}{p}} 2^{\frac{n}{p}j}\}_{j\in\mathbb{N}_0}$, according to Remark 3.2 and formula (6.3).

To fix the imagination we may assume that g is zero outside of a fixed neighborhood of Γ. We expand g according to 5.17 in terms of (φ_j, p_1)–β-quarks and $(\sigma_j h_j^{1/p_1} 2^{\frac{n}{p_1}j}, p_1)^L$–$\beta$-quarks, where φ is an admissible sequence with appropriately big index β_φ, according to 5.17. The idea of the proof is to reduce everything to the building blocks introduced in 5.15 and to the knowledge that the involved sequences of complex numbers in (5.28) belong to the space b_{p_1,q_1}. Hence, it does not matter very much to look at one of the two terms of (5.28) and we assume that we can apply the somewhat simpler situation of 5.13 to $B^{\sigma}_{p_1,q_1}(\Gamma)$ and $B^{\sigma}_{p_2,q_2}(\Gamma)$. The necessary technical modifications are clear in all cases.

Hence, without restriction of generality, we assume

$$g = \sum_{\beta,j,m} \lambda^{\beta}_{j,m} \sigma_j^{-1} h_j^{-\frac{1}{p_1}} \Theta^{\beta}(2^j x - m), \qquad (7.32)$$

with

$$\|\lambda \mid b_{p_1,q_1}\|_{\kappa_1} \sim \|g \mid B^{\sigma}_{p_1,q_1}(\mathbb{R}^n)\|, \qquad (7.33)$$

where Θ is a fixed mother function, according to 5.9 and $\kappa_1 > \varrho$, where ϱ is the number related to Θ according to 5.9-(i).

We decompose the embedding id as $\mathrm{id} = \mathrm{tr}_\Gamma \circ B \circ \mathrm{id}_{\mathrm{seq}} \circ A \circ \mathrm{ext}$, according to the following commutative diagram.

$$\begin{array}{ccccc} B^{\sigma}_{p_1,q_1}(\Gamma) & \xrightarrow{ext} & B^{\sigma h_{p_1}}_{p_1,q_1}(\mathbb{R}^n) & \xrightarrow{A} & \ell_\infty[2^{\kappa_1|\beta|}\ell_{q_1}(\mathfrak{p}_j \ell^{M_j}_{p_1})] \\ {\scriptstyle id}\big\downarrow & & & & \big\downarrow{\scriptstyle id_{seq}} \\ B^{\tau}_{p_2,q_2}(\Gamma) & \xleftarrow[tr_\Gamma]{} & B^{\tau h_{p_2}}_{p_2,q_2}(\mathbb{R}^n) & \xleftarrow[B]{} & \ell_\infty[2^{\kappa_2|\beta|}\ell_{q_2}(\ell^{M_j}_{p_2})] \end{array} \qquad (7.34)$$

where $\mathfrak{p}_j = \sigma_j \tau_j^{-1} h_j^{\frac{1}{p_1}-\frac{1}{p_2}}$ and $M_j \sim h_j^{-1}$. Now we clarify the definition of the operators involved in the above path.

• The operator ext has been already taken into consideration. Notice that, although not linear, ext is bounded and maps the unit ball $B(0,1)$ in $B^{\sigma}_{p_1,q_1}(\Gamma)$ into the ball $B(0,2)$ of $B^{\sigma h_{p_1}}_{p_1,q_1}(\mathbb{R}^n)$.

- As far as A is concerned, we let

$$f = \sum_{\beta,j,m} \lambda^{\beta}_{j,m} \sigma_j^{-1} h_j^{-\frac{1}{p_1}} \Theta^{\beta}(2^j x - m) \xrightarrow{A} \{\eta^{\beta,\Gamma}\}_{\beta \in \mathbb{N}_0^n}, \tag{7.35}$$

where,

$$\eta^{\beta,\Gamma} = \{\sigma_j^{-1} \tau_j h_j^{\frac{1}{p_2} - \frac{1}{p_1}} \lambda^{\beta}_{j,m} : CQ_{j,m} \cap \Gamma \neq 0\}. \tag{7.36}$$

We may assume that $C > 1$ is fixed and sufficiently large such that what follows is justified. By usual arguments we have that, for each fixed $j \in \mathbb{N}_0$, the number of indices m such that $CQ_{j,m} \cap \Gamma \neq 0$ is equivalent to h_j^{-1}.

The spaces $\ell_\infty\big[2^{\varrho|\beta|}(\ell_q \mathfrak{p}_j \ell_p^{M_j})\big]$ have been essentially introduced in (7.19) with (7.20), based on 7.1: of course we adapt these definitions to our present situation.

We have (modifications if p and/or q is infinity):

$$\begin{aligned}\big\| Af \mid \ell_\infty\big[2^{\kappa_1|\beta|}\ell_{q_1}(\mathfrak{p}_j \ell_{p_1}^{M_j})\big]\big\| &= \sup_{\beta \in \mathbb{N}_0^n} 2^{\kappa_1|\beta|} \|\eta^{\beta,\Gamma} \mid \ell_{q_1}(\mathfrak{p}_j \ell_{p_1}^{M_j})\| \\ &= \sup_{\beta \in \mathbb{N}_0^n} 2^{\kappa_1|\beta|} \Big(\sum_{j=0}^{\infty}\Big(\sum_{m=1}^{M_j} |\lambda_{j,m}|^{p_1}\Big)^{\frac{q_1}{p_1}}\Big)^{\frac{1}{q_1}} \\ &\leqslant \|\lambda \mid b_{p_1,q_1}\|_{\kappa_1} \leqslant c\big\| f \mid B^{\sigma h_{p_1}}_{p_1,q_1}(\mathbb{R}^n)\big\|. \end{aligned} \tag{7.37}$$

Consequently A is a bounded operator which maps the unit ball $B(0,2)$ in $B^{\sigma h_{p_1}}_{p_1,q_1}(\mathbb{R}^n)$ into the ball $B(0,c)$ of $\ell_\infty\big[2^{\kappa_1|\beta|}\ell_{q_1}(\mathfrak{p}_j \ell_{p_1}^{M_j})\big]$.

- The linear and bounded operator $\mathrm{id}_{\mathrm{seq}}$ is the embedding between the sequence spaces which are shown in the Diagram (7.34). Of course, we assume $\kappa_1 > \kappa_2 > 0$. In this case the embedding makes sense: by 7.3 we claim that the embedding $\mathrm{id}_{\mathrm{seq}}$ is compact. First of all notice that by 7.8 $\{h_j^{-1}\}_{j \in \mathbb{N}_0}$ is an almost strongly increasing sequence, and the same holds for the weight sequence $\{\mathfrak{p}_j\}_{j \in \mathbb{N}_0}$. Then the non immediate case to check in (7.4) of 7.3 occurs if $p_1 < p_2$: we verify this one. In this case (7.6) reduces to

$$\sigma_j^{-1} \tau_j h_j^{-\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}. \tag{7.38}$$

By assumption $\alpha_{\sigma^{-1}\tau} < -d(1/p_1 - 1/p_2)$, where

$$d = \dim_{\mathcal{P}} \Gamma = \limsup_{r \to 0} \log h(r)/\log r. \tag{7.39}$$

Hence, there is some $\delta > 0$ such that

$$\sigma_j^{-1} \tau_j \leqslant c 2^{-\delta j - d(1/p_1 - 1/p_2)j}, \qquad j \in \mathbb{N}_0. \tag{7.40}$$

Therefore,

$$\sigma_j^{-1}\tau_j h_j^{-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} = c2^{-(\delta-(1/p_1-1/p_2)\varepsilon)j}2^{\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\left(\frac{|\log h_j|}{j}-(d+\varepsilon)\right)j}$$
$$\leqslant c' 2^{-(\delta-(1/p_1-1/p_2)\varepsilon)j}. \tag{7.41}$$

Choosing ε appropriately small, we have that (7.6) or (7.7) are satisfied. Therefore the embedding $\mathrm{id}_{\mathrm{seq}}$ is compact and, by (7.22), we have:

$$e_{h_j^{-1}}(\mathrm{idseq}) \sim e_{2M_j}(\mathrm{idseq}) \sim \sigma_j^{-1}\tau_j h_j^{-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} M_j^{-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \sim \sigma_j^{-1}\tau_j, \tag{7.42}$$

for $j \in \mathbb{N}_0$.

- The operator B is then defined by

$$\{\varkappa^\beta_{j,m}\}_{\beta,j,m} \xrightarrow{B} \sum_{\beta,j,m} \varkappa^\beta_{j,m} \underbrace{\tau_j^{-1} h_j^{-\frac{1}{p_2}} \Theta^\beta(2^j x - m)}_{(\tau h_{p_2})-\beta\text{-quarks}}, \tag{7.43}$$

where the sum over m is taken only for those indices m with $CQ_{j,m} \cap \Gamma \neq 0$. As we said, we may assume without loss of generality that Theorem 5.13 is applicable to $B^{\tau h_{p_2}}_{p_2,q_2}(\mathbb{R}^n)$. Then B is a linear and bounded map.

- The trace operator tr_Γ is linear and bounded. If $\beta_\tau = 0$, then by convention we have set $B^\tau_{p_2,q}(\Gamma) = L_{p_2}(\Gamma)$ for all $0 < q \leqslant \infty$. Therefore we can take $q_2 = \min(1, p_2)$ and then, by 6.1, tr_Γ is always well-defined.

We make the point of the situation: the linear and bounded operator id is factorized through

$$\mathrm{id} = \mathrm{tr}_\Gamma \circ B \circ \mathrm{id}_{\mathrm{seq}} \circ A \circ \mathrm{ext}, \tag{7.44}$$

where A is bounded, $\mathrm{id}_{\mathrm{seq}}$ is linear bounded and compact and, finally, B is a linear and bounded operator. Taking f in the starting space and following the indicated operations, we end up with f again when we finish. In particular the final outcome is independent of ambiguities in the nonlinear construction of ext and A. The unit ball in $B^\sigma_{p_1,q_1}(\Gamma)$ is mapped into a bounded set in $\ell_\infty\left[2^{\kappa_1|\beta|}\ell_{p_1}(\mathrm{p}_j \ell_{q_2}^{M_j})\right]$. By $\mathrm{id}_{\mathrm{seq}}$ this bounded set is mapped into a pre-compact set in $\ell_\infty\left[2^{\kappa_2|\beta|}\ell_{p_2}(\ell_{q_2}^{M_j})\right]$ which can be covered by $N_j \sim 2^{h_j^{-1}}$ balls with radius $ce_{h_j^{-1}}(\mathrm{id}_{\mathrm{seq}})$, with

$$e_{h_j^{-1}}(\mathrm{idseq}) \leqslant c\sigma_j^{-1}\tau_j, \qquad j \in \mathbb{N}_0. \tag{7.45}$$

Afterwards the two linear and bounded maps B and tr_Γ do not change this assertion (not to speak about constants). This is one half of the desired estimation, in the case $p_1 \leqslant p_2$.

Step 2. If $p_1 > p_2$, then we have

$$B^\sigma_{p_1,q_1}(\Gamma) \subset B^\sigma_{p_2,q_1}(\Gamma). \tag{7.46}$$

To see this, let us reason as follows: let $f \in B^{\sigma}_{p_1,q_1}(\Gamma)$ and we take an extension $\mathrm{ext}^{\Omega} f$ of f in $B^{\sigma h_{p_1}}_{p_1,q_1}(\mathbb{R}^n)$ with $\mathrm{supp}\,\mathrm{ext}^{\Omega} f \subset \Omega$, where Ω is an open set including Γ. By the characterization via local means of the spaces $B^{\sigma}_{p,q}(\mathbb{R}^n)$ ([15, Theorem 4.3.4]) and the monotonicity of the L_p spaces on bounded domains, the embedding (7.46) follows. Now the desired (half) assertion in this case follows from (7.46) and the previous step applied to $p_1 = p_2$.

Step 3. Now we have to show that there exists a positive constant c such that

$$e_{h_j^{-1}}\big(\mathrm{id}\colon B^{\sigma}_{p_1,q_1}(\Gamma) \to B^{\tau}_{p_2,q_2}(\Gamma)\big) \geqslant c\sigma_j^{-1}\tau_j, \qquad j \in \mathbb{N}_0. \tag{7.47}$$

Suppose that this is false. Then there exists a subsequence $\{h_{j_k}^{-1}\}_{k\in\mathbb{N}_0}$ of $\{h_j^{-1}\}_{j\in\mathbb{N}_0}$ such that

$$e_{h_{j_k}^{-1}}\big(\mathrm{id}\colon B^{\sigma}_{p_1,q_1}(\Gamma) \to B^{\tau}_{p_2,q_2}(\Gamma)\big)\sigma_{j_k}\tau_{j_k}^{-1} \to 0, \qquad k \to \infty. \tag{7.48}$$

By the multiplicative property 7.5-(*ii*) of the entropy numbers and thanks to the previous step we can assume $B^{\tau}_{p_2,q_2}(\Gamma) = L_{p_2}(\Gamma)$. In particular, we can also assume $1 < p_1 \leqslant \infty$. Therefore, we have

$$e_{h_j^{-1}}\big(\mathrm{id}\colon B^{\sigma}_{p_1,q_1}(\Gamma) \to L_{p_1}(\Gamma)\big) \leqslant \sigma_j^{-1}, \qquad j \in \mathbb{N}_0 \tag{7.49}$$

and

$$e_{h_{j_k}^{-1}}\big(\mathrm{id}\colon B^{\sigma}_{p_1,q_1}(\Gamma) \to L_{p_2}(\Gamma)\big)\sigma_{j_k} \to 0, \qquad k \to \infty. \tag{7.50}$$

Suppose $p_2 < 1$. Then for every $f \in L_{p_1}(\Gamma)$ we have

$$\|f \mid L_p(\Gamma)\| \leqslant \|f \mid L_{p_1}(\Gamma)\|^{1-\theta}\|f \mid L_{p_2}(\Gamma)\|^{\theta}, \tag{7.51}$$

where

$$0 < \theta < 1, \qquad \text{and} \qquad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}. \tag{7.52}$$

By the property of entropy numbers with respect to interpolation spaces (see [12, 1.3.2, p. 13]) we infer

$$\begin{aligned}
&e_{2h_{j_k}^{-1}}\big(\mathrm{id}\colon B^{\sigma}_{p_1,q_1}(\Gamma) \to L_p(\Gamma)\big)\sigma_{j_k} \\
&\leqslant c\Big(e_{h_{j_k}^{-1}}\big(\mathrm{id}\colon B^{\sigma}_{p_1,q_1}(\Gamma) \to L_{p_1}(\Gamma)\big)\Big)^{1-\theta}\Big(e_{h_{j_k}^{-1}}\big(\mathrm{id}\colon B^{\sigma}_{p_1,q_1}(\Gamma) \to L_{p_2}(\Gamma)\big)\Big)^{\theta}\sigma_{j_k} \\
&\leqslant c'\Big(e_{h_{j_k}^{-1}}\big(\mathrm{id}\colon B^{\sigma}_{p_1,q_1}(\Gamma) \to L_{p_2}(\Gamma)\big)\sigma_{j_k}\Big)^{\theta} \to 0, \qquad k \to \infty.
\end{aligned} \tag{7.53}$$

Since we can choose $p > 1$, we can without loss of generality suppose $p_2 > 1$ in what follows.

We make the point of the situation: in order to prove the desired inequality, we have to disprove that there exists a subsequence $\{j_k\}_{k\in\mathbb{N}_0}$ such that

$$e_{h_{j_k}^{-1}}\big(\mathrm{id}\colon B^{\sigma}_{p_1,q_1}(\Gamma) \to L_{p_2}(\Gamma)\big)\sigma_{j_k} \to 0, \qquad k \to \infty, \tag{7.54}$$

for $1 < p_1, p_2 \leqslant \infty$ and $\beta_\sigma > d(1/p_1 - 1/p_2)_+$. In other words, we have to show that there exists a constant $c > 0$ with

$$e_{h_j^{-1}}\left(\mathrm{id}\colon B^\sigma_{p_1,q_1}(\Gamma) \to L_{p_2}(\Gamma)\right) \geqslant c\sigma_j^{-1}, \qquad j \in \mathbb{N}_0. \tag{7.55}$$

Since Γ is compact we can surely assume $\mu(\Gamma) = 1$, where μ is the related h-measure. Let us choose a disjoint collection of balls $B_{j,m} = B(\gamma_{j,m}, 2^{-j})$. with $\gamma_{j,m} \in \Gamma$, for $m = 1, \ldots, M_j \sim h_j^{-1}$ (this is surely possible). Let us choose two smooth functions φ and ψ with support in the unit ball, such that

$$c_{j,m} h_j^{-1} \int_\Gamma \varphi(2^j(\gamma - \gamma_{j,m}))\psi(2^j(\gamma - \gamma_{j,m}))\, d\mu(\gamma) = 1, \tag{7.56}$$

for some positive constants $c_{j,m}$. We can assume that

$$c \leqslant c_{j,m} \leqslant c', \qquad j \in \mathbb{N}_0,\ m = 1, \ldots, M_j, \tag{7.57}$$

for some $c, c' > 0$ independent of j and m as above. Now we consider the following commutative diagram:

$$\begin{array}{ccc} \sigma_j h_j^{\frac{1}{p_1}} \ell_{p_1}^{M_j} & \xrightarrow{A} & B^\sigma_{p_1,q}(\Gamma) \\ {\scriptstyle id^j}\downarrow & & \downarrow{\scriptstyle id} \\ h_j^{\frac{1}{p_2}} \ell_{p_2}^{M_j} & \xleftarrow[B]{} & L_{p_2}(\Gamma) \end{array} \tag{7.58}$$

The operators A and B are defined as follows (where the source and target spaces are shown in the above diagram):

$$A(\{a_m\}_{m=1}^{M_j}) = \sum_{m=1}^{M_j} a_m \varphi(2^j(\gamma - \gamma_{j,m})), \qquad \gamma \in \Gamma \tag{7.59}$$

and

$$Bf = \left\{c_{j,m} h_j^{-1} \int_\Gamma f(\gamma)\psi(2^j(\gamma - \gamma_{j,m}))\, d\mu(\gamma)\right\}_{m=1}^{M_j}. \tag{7.60}$$

We interpret the right hand side of (7.59) as an atomic representation with (σ, p)-atoms. Therefore,

$$\|A(\{a_m\}_{m=1}^{M_j}) \mid B^\sigma_{p_1,q}(\Gamma)\| \leqslant c\sigma_j h_j^{\frac{1}{p}} \|\{a_m\}_{m=1}^{M_j} \mid \ell_{p_1}\|. \tag{7.61}$$

Therefore A is a linear and bounded operator with

$$\left\|A \mid \sigma_j h_j^{\frac{1}{p}} \ell_{p_1}^{M_j} \to B^\sigma_{p_1,q}(\Gamma)\right\| \leqslant c, \tag{7.62}$$

where c does not depend on $j \in \mathbb{N}_0$. As far as B is concerned, let us call $b_{j,m}$ the terms in brackets in (7.60). Then by virtue of Hölder's inequality with

$1/p_2 + 1/p_2' = 1$ we infer

$$\begin{aligned}|b_{j,m}|^{p_2} &\leqslant c h_j^{-p_2}\Big(\int_\Gamma |f(\gamma)||\psi(2^j(\gamma-\gamma_{j,m}))\,d\mu(\gamma)\Big)^{p_2}\\ &\leqslant c' h_j^{-p_2}\Big(\int_{\Gamma\cap B_{j,m}} |f(\gamma)|\,d\mu(\gamma)\Big)^{p_2} \leqslant c'' h_j^{-p_2}\int_{\Gamma\cap B_{j,m}} |f(\gamma)|^{p_2}\,d\mu(\gamma)\cdot h_j^{\frac{p_2}{p_2'}}\\ &\leqslant c'' h_j^{-1}\int_{\Gamma\cap B_{j,m}} |f(\gamma)|^{p_2}\,d\mu(\gamma).\end{aligned} \tag{7.63}$$

Hence, remembering that the balls $B_{j,m}$ are pairwise disjoint, we get

$$\big\|Bf \mid h_j^{\frac{1}{p_2}}\ell_{p_2}^{M_j}\big\| = h_j^{\frac{1}{p_2}}\Big(\sum_{m=1}^{M_j}|b_{j,m}|^{p_2}\Big)^{\frac{1}{p_2}} \leqslant c\|f \mid L_{p_2}(\Gamma)\| \tag{7.64}$$

(modification if $p_2 = \infty$).

Therefore, also B is a linear and bounded operator with

$$\big\|B \mid L_{p_2}(\Gamma) \to h_j^{\frac{1}{p_2}}\ell_{p_2}^{M_j}\big\| \leqslant c, \tag{7.65}$$

where c is independent of $j \in \mathbb{N}_0$. Now we factorize id^j through

$$\mathrm{id}^j = B \circ \mathrm{id} \circ A. \tag{7.66}$$

Notice that by (7.56) the above factorization is justified.

Then, by the above arguments, we have

$$e_k(\mathrm{id}^j) \leqslant c e_k(\mathrm{id}_\Gamma), \qquad k \in \mathbb{N}_0, \tag{7.67}$$

for a positive constant c independent of $j \in \mathbb{N}_0$. But this is the conclusion: exploiting 7.9 with the admissible choices

$$\mathfrak{p}_j = \sigma_j^{-1} h_j^{\frac{1}{p_1}-\frac{1}{p_2}} \qquad \text{and} \qquad M_j \sim h_j^{-1}, \tag{7.68}$$

we finally get

$$e_{h_j^{-1}} \geqslant c e_{2M_j}(\mathrm{id}^l) \geqslant c\sigma_j^{-1} h_j^{\frac{1}{p_1}-\frac{1}{p_2}} M_j^{\frac{1}{p_1}-\frac{1}{p_2}} \sim \sigma_j^{-1}. \tag{7.69}$$

Clipping together (7.68) and (7.69) we conclude

$$e_{h_j^{-1}}(\mathrm{id}) \geqslant c\sigma_j^{-1}, \qquad j \in \mathbb{N}_0, \tag{7.70}$$

which is what we claimed. ■

Remark 7.14. We preserve the notation and the assumptions of 7.12.

One may argue that the knowledge of $e_k(\mathrm{id})$ given only through the subsequence $\{e_{k_j}(\mathrm{id})\}_{j\in\mathbb{N}_0}$, where $k_j \sim h_j^{-1}$, $j \in \mathbb{N}_0$ is not sufficient to provide the complete description of the asymptotic behavior of the sequence $\{e_k(\mathrm{id})\}_{k\in\mathbb{N}_0}$. This is not the case: since $\{h_j^{-1}\}_{j\in\mathbb{N}_0}$, $\{\sigma_j\}_{j\in\mathbb{N}_0}$ and $\{\tau_j\}_{j\in\mathbb{N}_0}$ are all admissible sequences, we have

$$c_1\sigma_j^{-1}\tau_j \geqslant e_{k_j}(\mathrm{id}) \geqslant e_{k_j+1}(\mathrm{id}) \geqslant \cdots \geqslant e_{k_{j+1}}(\mathrm{id}) \geqslant c_2\sigma_j^{-1}\tau_j, \tag{7.71}$$

for some constants c_1 and c_2 independent of $j \in \mathbb{N}_0$.

Therefore, the knowledge of $e_{k_j}(\mathrm{id})$ is actually sufficient to describe completely the behavior of the whole sequence $\{e_k\}_{k\in\mathbb{N}}$.

Consequently, an explicit description of $e_k(\mathrm{id})$ can be obtained "solving" $k^{-1} = h_j$, i.e., "inverting" h_j. If j_k is such an essential inverse, say, $h_{j_k} \sim k^{-1}$, then

$$e_k(\mathrm{id}) \sim \sigma_{j_k}^{-1}\tau_{j_k}, \tag{7.72}$$

for all $k \in \mathbb{N}$.

We make clear with an explicit class of examples an application of the last consideration.

7.3. Some explicit cases

We restrict our attention to those h-sets $\Gamma \subset \mathbb{R}^n$ whose measure function h is of type

$$h(r) \sim r^d H(r), \qquad 0 < r \leqslant 1, \tag{7.73}$$

where $0 < d < n$ and H is a slowly varying function according to Definition 4.11. As we have pointed out in 4.12, we can assume without loss of generality that $r^d H(r)$ is a smooth monotone function. We refer to this special class of h-sets as to the class of *regular h-sets* and correspondingly measure functions h of type $h(r) \sim r^d H(r)$, with $0 < d < n$ and H slowly varying, are called *regular measure functions*.

By the property of slowly varying functions, if $h(r) \sim r^d H(r)$ we have

$$\dim_{\mathcal{H}} \Gamma = \liminf_{r\to 0} \frac{\log h(r)}{\log r} = \limsup_{r\to 0} \frac{\log h(r)}{\log r} = \dim_{\mathcal{P}} \Gamma = d, \tag{7.74}$$

and hence regular h-sets are dimension regular sets.

Notice also that by 4.10 any regular h-set fulfills the ball-condition, since in this case $\alpha_h = \beta_h = -d > -n$.

As an additional simplification, we restrict our attention to the following tuned spaces on a regular set Γ.

Definition 7.15. Let $h(r) \sim r^d H(r)$ be a regular measure function. Let $s \geqslant 0$, $a \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. Then we let

$$B_{p,q}^{(s,a)}(\Gamma) = B_{p,q}^{\sigma}(\Gamma) \ \left(= \mathrm{tr}_\Gamma\, B_{p,q}^{\sigma h_p}(\mathbb{R}^n)\right), \tag{7.75}$$

where $\sigma = \{2^{sj} H^a(2^{-j})\}_{j\in\mathbb{N}_0}$ and $h_p = \{2^{\frac{n-d}{p}j} H^{\frac{1}{p}}(2^{-j})\}_{j\in\mathbb{N}_0}$.

It is easy to show that actually $\beta_\sigma > 0$ for $s > 0$. Otherwise we agree always on

$$B^{(s,a)}_{p,q}(\Gamma) = L_p(\Gamma), \qquad \text{for } s = 0 \text{ and } a \in \mathbb{R}. \tag{7.76}$$

The exponent a represents an extra-tuning parameter for the usual Besov scale on Γ (i.e., with "scalar regularity"). The proposition concerning this concrete situation is the following.

Proposition 7.16. *Let Γ be a regular h-set in $\mathbb{R}^n$, where $h(r) \sim r^d H(r)$ for $0 < d < n$ and a slowly varying function H. Let*

$$0 < p_1, p_2, q_1, q_2 \leqslant \infty, \tag{7.77}$$

$$a_1, a_2 \in \mathbb{R} \qquad \text{and} \tag{7.78}$$

$$s_1 - s_2 > d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ . \tag{7.79}$$

Then the embedding

$$\text{id}: B^{(s_1,a_1)}_{p_1,q_1}(\Gamma) \to B^{(s_2,a_2)}_{p_2,q_2}(\Gamma) \tag{7.80}$$

is compact and for the related entropy numbers one has the explicit behaviour

$$e_k(\text{id}) \sim \left(kH^\sharp_d(k^{-1})^{-1}\right)^{-\frac{s_1-s_2}{d}} H^\sharp_d(k^{-1})^{a_1-a_2}, \qquad k \in \mathbb{N} \tag{7.81}$$

where $H^\sharp_d$ is the Bruijn conjugate of $H(r^d)$, according to (iii) of 4.12.

The proof of this assertion is easy: one has simply to use the Main Theorem 7.12, the inversion formulas stated in (iii) of 4.12 and the special form of the terms involved. So it seems reasonable to skip this proof and we provide instead some concrete examples.

• In the case where $H(r)$ is (equivalent to) an admissible function in the sense of the definition given by D. Edmunds and H. Triebel in [13] and quoted briefly in (4.13), one can easily see that $H^\sharp_d(r) \sim H(r)^{-1}$. Hence, the above estimation of entropy numbers (for the prescribed range of the parameters) results in

$$e_k(\text{id}) \sim \left(kH(k^{-1})\right)^{-\frac{s_1-s_2}{d}} H(k^{-1})^{a_2-a_1}, \tag{7.82}$$

which was established by D. Edmunds and H. Triebel in [13], [14] for $p_1, p_2 \geqslant 1$ and generalised to the full range of parameters by S. de Moura in [25].

• We consider now the case where $H(r) \sim \exp\{b|\log r|^\varkappa\}$, for $b \in \mathbb{R} \setminus \{0\}$ and $0 < \varkappa < 1$. This is a slowly varying function which is not admissible in the sense specified above. This is a more intricate example, since there is not a universal formula for the Brujin conjugate of H for all $0 < \varkappa < 1$: as a matter of fact as $\varkappa$ tends to 1 the related functions H tend to a forbidden border case which is not a slowly varying function. We list three cases and we refer to [1, p. 435] for details and proofs

$$H_d^\sharp(r) = \begin{cases} \exp\{-bd^{-\varkappa}|\log r|^{\varkappa}\}, & \text{for } 0 < \varkappa < 1/2, \\ \exp\{-bd^{-\varkappa}|\log r|^{\varkappa} + bd^{-\varkappa}\varkappa|\log r|^{2\varkappa-1}\}, & \text{for } 1/2 \leqslant \varkappa < 2/3, \\ \exp\{-bd^{-\varkappa}|\log r|^{\varkappa} + bd^{-\varkappa}\varkappa|\log r|^{2\varkappa-1} - \\ \quad - \frac{1}{2}bd^{-\varkappa}\varkappa(3\varkappa-1)|\log r|^{3\varkappa-2}\}, & \text{for } 2/3 \leqslant \varkappa < 3/4, \\ \dots \end{cases} \tag{7.83}$$

Hence, taking for simplicity only the case $0 < \varkappa < 1/2$, the above estimation of entropy numbers (for the prescribed range of the parameters) results in

$$e_k(\mathrm{id}) \sim \left(k \exp\{bd^{-\varkappa}|\log k|^{\varkappa}\}\right)^{-\frac{s_1-s_2}{d}} \exp\{bd^{-\varkappa(a_2-a_1)}|\log k|^{\varkappa}\}. \tag{7.84}$$

Remark 7.17. Also in this rather particular situation it is not always straightforward to calculate the Bruijn conjugate of a given slowly varying function. We refer to the already quoted monograph [1], especially to Appendix 5, for some methods of calculating the Bruijn conjugate of a given slowly varying function H. For instance, if $H(r)$ is the restriction of some holomorphic function $H(z)$ to the real axis, then we may assume $H(e^z) = \exp\{h(z)\}$, where h is holomorphic. Under a rather strong condition on h one obtains a series expansion of $h^\sharp$ (in additive notation), and hence an approximation of $H^\sharp$ ([1, Proposition A5.1]).

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation.* Cambridge Univ. Press, 1987.

[2] M. Bricchi and S. de Moura, *Complements on growth envelopes of spaces with generalised smoothness in the sub-critical case.* Jenaer Schriften zur Mathematik und Informatik, 04/02, 2002.

[3] M. Bricchi, *On some properties of (d,Ψ)−sets and related Besov spaces.* Jenaer Schriften zur Mathematik und Informatik, 99/31, 1999.

[4] M. Bricchi *On the relationship between Besov spaces $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$ and L_p spaces on a (d,Ψ)-set.* Jenaer Schriften zur Mathematik und Informatik, 00/13, 2000.

[5] M. Bricchi *Characterization of h-sets.* Proc. conf. Function Spaces Differential Operators and Nonlinear Analysis, Teistungen, 2002, to appear.

[6] M. Bricchi, *Existence and properties of h-sets.* Georgian Math. J. **9** (2002), 13–32.

[7] M. Bricchi, *Tailored Besov spaces and h-sets.* Math. Nachr., 2002, accepted.

[8] M. Bricchi, *Tailored function spaces and h-sets.* PhD thesis, University of Jena, 2002.

[9] B. Carl, *Entropy numbers, s-numbers and eigenvalue problems.* J. Funct. Analysis **41** (1981), 290–306.

[10] F. Cobos and D. L. Fernandez, *Hardy-Sobolev spaces and Besov spaces with a function parameter.* Proc. Lund conf., Springer Lect. Notes in Math., **1302** (1986), 158–170.

[11] B. Carl and I. Stephani, *Entropy, compactness and approximation of operators.* Cambridge Univ. Press, 1990.

[12] D. Edmunds and H. Triebel, *Function spaces, entropy numbers, differential operators.* Cambridge Univ. Press, 1996.

[13] D. Edmunds and H. Triebel, *Spectral theory for isotropic fractal drums.* C. R. Acad. Sci. Paris, 326, série I:1269–1274, 1998.

[14] D. Edmunds and H. Triebel, *Eigenfrequencies of isotropic fractal drums.* Operator Theory: Advances and Applications, 110:81–102, 1999.

[15] W. Farkas and H.-G. Leopold, *Characterisations of function spaces of generalised smoothness.* Jenaer Schriften zur Mathematik und Informatik, 23/01, 2001.

[16] M. L. Goldman, *A method of coverings for describing general spaces of Besov type.* Trudy Mat. Inst. Steklov **156** (1980, 47–81. English transl.: Proc. Steklov Institut Math. 1983, no. 2 (156).

[17] A. Jonsson and H. Wallin, *Function spaces on subsets of* $\mathbb{R}^n$. Math. Reports 2, part 1. London, Harwood Acad. Publ., 1984.

[18] G. A. Kalyabin, *Description of functions in classes of Besov-Lizorkin-Triebel type.* Trudy Mat. Inst. Steklov **156** (1980), 82–109. English transl.: Proc. Steklov Institut Math. 1983, no. 2 (156).

[19] G. A. Kalyabin and P. I. Lizorkin, *Spaces of functions of generalized smoothness.* Math. Nachr. **133** (1987), 7–32.

[20] H.-G. Leopold, *Embeddings for general weighted sequence spaces and entropy numbers.* Proc. Conf. Function Spaces Differential Operators and Nonlinear Analysis, Syöte, pages 170–186, 1999.

[21] H.-G. Leopold, *Embeddings and entropy numbers for general weighted sequence spaces: the non limiting-case.* Georgian Math. J. **4** (2000), 731–743.

[22] P. Mattila, *Geometry of sets and measures in euclidean spaces.* Cambridge Univ. Press, 1995.

[23] C. Merucci, *Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces.* Proc. Lund Conf., Springer Lect. Notes in Math. **1070** (1984), 183–201.

[24] S. Domingues de Moura, *Some properties of the spaces* $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ *and* $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$. Preprint, 1999. University of Coimbra.

[25] S. Domingues de Moura, *Function spaces of generalised smoothness.* Dissertationes Math. CCCXCVIII, 2001.

[26] C. A. Rogers, *Hausdorff measure.* Cambridge Univ. Press, 1970.

[27] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations.* Berlin, De Gruyter, 1996.

[28] H. Triebel, *Fractals and spectra.* Birkhäuser, Basel, 1997.

[29] H. Triebel, *The structure of functions.* Birkhäuser, Basel, 2001.

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