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ON THE ZEROS OF FUNCTIONS FROM THE EXTENDED SELBERG CLASS OF DEGREE 0

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Abstract: We give an improved upper bound for the error term in the Riemann-von Mangoldt formula for functions of degree 0 in the extended Selberg class. The bound is uniform for functions with no zeros in a certain half-plane. The first such bound was given by J. Kaczorowski and A. Perelli [3] in the proof that the extended Selberg class is a semigroup with factorization.
Keywords: Selber class, zeros of L-functions, Riemann-von Mangoldt formula.

1. Introduction

The Selberg class, denoted $\mathcal{S}$, and the extended Selberg class, $\mathcal{S}^\sharp$ consist of special Dirichlet series with functional equation. Both have been extensively studied in the recent years. We refer the reader to [2] or [1] for definitions and basic properties of $\mathcal{S}$ and $\mathcal{S}^\sharp$. Here we only recall them in the context of the class $\mathcal{S}_0^\sharp$, i.e. the functions of *degree* 0 in $\mathcal{S}^\sharp$. It is well known (cf. [1]) that $F \in \mathcal{S}_0^\sharp$ if and only if it is of the form

$$F(s) = \sum_{n|q_F} a_F(n)n^{-s}$$

with $a_F(n) = \frac{\omega n}{\sqrt{q_F}}\overline{a_F\left(\frac{q_F}{n}\right)}$, $n \mid q_F$, with q_F a positive integer and ω a complex number, $|\omega| = 1$, F not identically zero. The number q_F is uniquely determined and called the conductor of F. There exists a real number σ_F such that $F(s) \neq 0$ in the half-plane $\mathrm{Re}\, s > \sigma_F$. We denote by $\Omega(n)$ the number of prime divisors of n, counted with multiplicities.

We give an upper bound for the error term in the Riemann-von Mangoldt formula for F in terms of the conductor of F, uniform for all F with no zeros in a given half-plane. Let $N_F(T)$ denote the number of zeros ρ of F with $|\mathrm{Im}\, \rho| \leqslant T$ (all zeros are non-trivial for $F \in \mathcal{S}_0^\sharp$).

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Theorem. *If $F \in \mathcal{S}_0^\sharp$ and $F(s) \neq 0$ in the half-plane* $\operatorname{Re} s > \sigma_F$, *then*

$$N_F(T) = \frac{T}{\pi} \log q_F + O_{\sigma_F}(\log^2 q_F (\log\log q_F)^2)$$

uniformly for $T \geqslant 2$ and $F \in \mathcal{S}_0^\sharp$ with $a(1) = 1$, $q_f \geqslant 2$.

This is an improvement over the bound given by J. Kaczorowski and A. Perelli [3], who showed the above with $O_{\sigma_F}(\log^6 q_F)$. It was the main part of their proof that $\mathcal{S}^\sharp$ is a semigroup with factorization. The improvement is based on a better upper bound for the quantity $a(n, \delta_0)$ defined for $\delta_0 \geqslant 1$ and $n \in \mathbb{N}$ by $a(1, \delta_0) = \delta_0$ and

$$a(n, \delta_0) = \delta_0 + \sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{\substack{n_1 \cdots n_l = n \\ n_1, \ldots, n_l \geqslant 2}} a(n_1, \delta_0) \cdots a(n_l, \delta_0), \qquad n \geqslant 2,$$

an empty sum being equal to 0. It was shown in [3] that $a(n, \delta_0) \leqslant \delta_0^{\Omega(n)} 2^{\Omega(n)^3}$, $n \geqslant 2$. We show

Lemma. *If $\delta_0 \geqslant 1$ and $n \geqslant 2$, then we have*

$$\delta_0^{\Omega(n)} \exp\left(\tfrac{1}{7}\Omega(n) - 1\right) < a(n, \delta_0) < \delta_0^{\Omega(n)} \exp\left(3\Omega(n) \log \Omega(n)\right).$$

The proposition follows from the improved upper bound in the lemma in much the same way as in [3] so in fact we only need to prove the lemma. The lower bound shows that the upper bound cannot be made much lower. In fact with the argument given here and some computations it can be shown that

$$a(n, \delta_0) > \delta_0^{\Omega(n)} \exp\left(\Omega(n) \log \Omega(n)\right)$$

if $n \geqslant 60$ and n is a product of distinct primes. On the other hand we have

$$a(n, \delta_0) < \delta_0^{\Omega(n)} \exp\left(2\Omega(n) \log \Omega(n)\right)$$

if n is a prime power. The author was supported by the Foundation for Polish Science and by the Polish Research Committee (KBN grant No. 1P03A00826).

2. Proof of the lemma

It is easy to see that $a(n, \delta_0)$ depends only on δ_0 and the factorization type of n (the exponents in the factorization to prime powers). If p is a prime, $m = \Omega(n)$, $n = q_1 \ldots q_m$ with q_i primes (not necessarily distinct) and $p_1, \ldots, p_m$ distinct primes, then we show by induction on m that

$$a(p^m, \delta_0) \leqslant a(n, \delta_0) \leqslant a(p_1 \ldots p_m, \delta_0). \tag{1}$$

Indeed, this is trivial for $m=1$ and if it is true for all $m'<m$, then

$$
\begin{aligned}
a(p^m,\delta_0) &= \delta_0 + \sum_{l=2}^{m}\frac{1}{l}\sum_{\substack{m_1+\ldots+m_l=m\\ m_1,\ldots,m_l\geqslant 1}}\!\!\cdots\!\!\sum a(p^{m_1},\delta_0)\ldots a(p^{m_l},\delta_0)\\
&\leqslant \delta_0 + \sum_{l=2}^{m}\frac{1}{l}\sum_{\substack{m_1+\ldots+m_l=m\\ m_1,\ldots,m_l\geqslant 1}}\!\!\cdots\!\!\sum \prod_{i=1}^{l} a(q_{m_1+\ldots+m_{i-1}+1}\cdots q_{m_1+\ldots+m_i},\delta_0)\\
&\leqslant a(n,\delta_0).
\end{aligned}
$$

Similarly, if we map divisors of $p_1\ldots p_m$ onto those of n by $\sigma(p_{i_1}\ldots p_{i_k}) = q_{i_1}\ldots q_{i_k}$, then

$$
\begin{aligned}
a(n,\delta_0) &\leqslant \delta_0 + \sum_{l=2}^{m}\frac{1}{l}\sum_{\substack{n_1\ldots n_l=p_1\ldots p_m\\ n_1,\ldots,n_l\geqslant 2}}\!\!\cdots\!\!\sum a(\sigma(n_1),\delta_0)\ldots a(\sigma(n_l),\delta_0)\\
&\leqslant a(p_1\ldots p_m,\delta_0).
\end{aligned}
$$

Thus (1) is proved. Further note that $a(n,\delta_0)$ is, for a fixed $n\geqslant 2$, a polynomial of degree $\Omega(n)$ in δ_0, with nonnegative coefficients. Let $b(m)$ denote the leading term of $a(2^m,\delta_0)$ and $c(m)=a(p_1\ldots p_m,1)$, with $p_1,\ldots,p_m$ distinct primes, $m\in\mathbb{N}$. We have:

$$
\delta_0^{\Omega(n)}b(\Omega(n)) \leqslant a(n,\delta_0) \leqslant \delta_0^{\Omega(n)}c(\Omega(n)), \qquad n\geqslant 2.
$$

Moreover $c(1)=1$ and

$$
c(m) = 1+\sum_{l=2}^{m}\frac{1}{l}\sum_{\substack{m_1+\ldots+m_l=m\\ m_1,\ldots,m_l\geqslant 1}}\!\!\cdots\!\!\sum \frac{m!}{m_1!\ldots m_l!}c(m_1)\ldots c(m_l), \qquad m\geqslant 2,
$$

because, given $m_1,\ldots,m_l$ as above, we have $\frac{m!}{m_1!\ldots m_l!}$ choices of $n_1,\ldots,n_l$ with $n_1\ldots n_l=p_1\ldots p_m$ and $\Omega(n_i)=m_i$, $i=1,\ldots,l$. Hence, for $m\geqslant 2$ we have

$$
\begin{aligned}
\frac{c(m)}{m!} &= \frac{1}{m!}+\sum_{m_1=1}^{m-2}\frac{c(m_1)}{m_1!}\bigg(\frac{c(m-m_1)}{2(m-m_1)!}\\
&\qquad +\sum_{l\geqslant 3}\frac{l-1}{l}\cdot\frac{1}{l-1}\sum_{m_2+\ldots+m_l=m-m_1}\!\!\cdots\!\!\sum \frac{c(m_2)\ldots c(m_l)}{m_2!\ldots m_l!}\bigg)+\frac{1}{2}c(m-1)\\
&< \frac{3}{2}\sum_{m_1=1}^{m-1}\frac{c(m_1)}{m_1!}\frac{c(m-m_1)}{(m-m_1)!}.
\end{aligned}
$$

We note that $c(m) < m! \exp(2(m-1)\log m)$, $m \geqslant 1$. Indeed, it is true for $m = 1$ and if it is satisfied for all $m' < m$, then

$$\frac{c(m)}{m!} < \frac{3}{2}(m-1)\exp(2(m-2)\log m) < \exp(2(m-1)\log m).$$

The upper bound follows. The lower bound is obtained, in a very similar way, from

$$b(m) \geqslant \frac{7}{6}\sum_{m_1=1}^{m-2} b(m_1)b(m-m_1) + \frac{1}{2}b(m-1).$$

and direct computation of $b(m)$ for $m = 1, \ldots, 4$. ■

References

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