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ON A THREE-SPACE PROPERTY FOR LINDELÖF Σ-SPACES, (WCG)-SPACES AND THE SOBCZYK PROPERTY

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Dedicated to the memory of Susanne Dierolf

Abstract: Corson's example shows that there exists a Banach space E which is not weakly normal but E contains a closed subspace isomorphic to the Banach space $C[0,1]$ and such that the quotient space $E/C[0,1]$ is isomorphic to the weakly compactly generated Banach space $c_0[0,1]$. This applies to show the following two results:

(i) The Lindelöf property is not a three-space property.

(ii) The Lindelöf Σ-property is not a three-space property. In this note using the lifting property developed by Susanne Dierolf we present a very simple argument providing also (ii), see Theorem 1. This argument used in the proof applies also to show that under Continuum Hypothesis every infinite-dimensional topological vector space E which contains a dense hyperplane admits a stronger vector topology υ with the same topological dual and such that (E, υ) contains a dense non-Baire hyperplane. This partially answers a question of Saxon concerning Arias de Reyna-Valdivia-Saxon theorem.

A Banach space E has the *Sobczyk Property* if it contains an isomorphic copy of c_0 and every such a copy is complemented in E. The classical Sobczyk's theorem says that every separable Banach space has this property. We give an example of a $C(K)$-space E and its subspace Y isometric to c_0 such that E/Y is isomorphic to $c_0(\Gamma)$, with $\mathrm{card}(\Gamma) = 2^{\aleph_0}$, yet Y is uncomplemented in E. This complements Corson's example and shows that the Sobczyk Property (as well as the (WCG)-property, and the Separable Complementation Property) is not a three-space property.

In the last part we recall some facts (partially with a simpler presentation) concerning K-analytic, Lindelöf Σ and analytic locally convex spaces. Additionally, a few remarks concerning weakly K-analytic spaces are included.

Keywords: Lindelöf Σ-spaces, WCG-spaces, Banach spaces.

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1. Introduction

By a *three-space property* (in the category of topological vector spaces) we understand the following situation, see for example [40].

Suppose that E is a topological vector space (tvs) (either real or complex) and assume that $F \subset E$ is a closed vector subspace of E such that F and the quotient E/F have certain property $\mathcal{P}$. Does E have property $\mathcal{P}$?

Several important topological properties of tvs are a three-space property like separability, local boundedness, completeness, barelledness, metrizability, etc., see [41] and [40] and [9] and references.

A Hausdorff topological space X is a *Lindelöf* Σ*-space* (called also K-countably determined) if there is an upper semi-continuous (usco) map from a (non-empty) subset $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ with compact values in X whose union is X, see [36], [1], [29]. If the same holds for $\Sigma = \mathbb{N}^{\mathbb{N}}$, then X is called K*-analytic*. A continuous image of $\mathbb{N}^{\mathbb{N}}$ is called an *analytic* space.

Note that analytic $\Rightarrow$ K-analytic $\Rightarrow$ Lindelöf Σ $\Rightarrow$ Lindelöf, and none of the reverse implication is true in general. We refer to [1], [29, Proposition 1.2], and recent survey paper due to Tkachuk [50] for several well-known characterizations and results about of Lindelöf Σ-spaces.

The class of Lindelöf Σ-spaces is invariant with respect to images under compact-valued upper semi-continuous maps (hence, continuous maps, closed subspaces, perfect preimages), countable products and countable unions. Although even the square of a Lindelöf space need not be Lindelöf, any countable product of Lindelöf Σ-spaces is a Lindelöf space. This provides two important consequences following from Arkhangel'ski-Pytkeev result [1, Theorem II.1.1] and Orihuela [37, Theorem 3, Note 3], respectively.

Proposition 1. *If X is a Lindelöf Σ-space, then the space $C_p(X)$ of continuous real-valued maps on a completely regular Hausdorff space X has countable tightness and is angelic.*

Recall that a Fréchet space E, i.e. a metrizable and complete lcs, is a *weakly compactly generated space* ((WCG) shortly) if E admits a $\sigma(E,E')$-compact absolutely convex set whose linear span is dense in E. Every reflexive Banach space E is (WCG) and every (WCG) Fréchet space is weakly K-analytic [47], [30], i.e. the weak topology $\sigma(E,E')$ is K-analytic.

Although Lindelöf Σ-spaces enjoy good properties, from Corson's example, as stated in Abstract, it follows also that the Lindelöf Σ-space property for locally convex spaces is not a *three-space property*, i.e. there exists a lcs E (= the space E from the Abstract with the weak topology) which is not a Lindelöf Σ-space since it is not Lindelöf but E contains a closed analytic vector subspace F (= the space $C[0,1]$ with the weak topology is analytic) and the quotient E/F space is K-analytic (= $c_0[0,1]$ with the weak topology is K-analytic).

In this note, dedicated to Professor S. Dierolf, using a lifting procedure introduced by S. Dierolf from [11] and developing some arguments from [26] we provide in the first part much simpler approach yielding the same conclusion. We hope that this short work will supplement a pioneer work of W. Roelcke and S. Dierolf [40] concerning several problems related with a three-space property in the category of topological vector spaces.

We prove the following general

Theorem 1. *Let (E, ξ) be an infinite-dimensional separable Fréchet space. Then there exist on E two stronger locally convex topologies υ_1 and υ_2 such that (E, υ_i) is not a Lindelöf Σ-space for $i = 1, 2$, and $\xi = \inf\{\upsilon_1, \upsilon_2\}$ but*

(i) *(E, υ_i) contains a closed Baire metrizable separable subspace F_i;*
(ii) *The quotient space $(E/F_i, \upsilon_i/F_i)$ is linearly homeomorphic to the space $\ell^2(\aleph_1)$ endowed with the weak topology.*

Since $\ell^2(\aleph_1)$ is reflexive, the weak topology of $\ell^2(\aleph_1)$ is K-analytic, hence the spaces $(E/F_i, \upsilon_i/F_i)$ are K-analytic but not separable. Clearly each F_i is a Lindelöf Σ-space.

We do not know however if the constructed spaces (E, υ_i) are Lindelöf. Note however that the fact $\xi = \inf\{\upsilon_1, \upsilon_2\}$ does not guarantee in general that one of topologies υ_1, or υ_2 is Lindelöf. In fact, a similar argument used in the proof of Theorem 1 applies to show that for a complete normed separable topology ξ on an infinite-dimensional vector space E there exist two normed nonseparable topologies υ_1, υ_2 with $\xi = \inf\{\upsilon_1, \upsilon_2\}$.

The argument used above applies also to provide some observations related with still an open question if every tvs containing a dense hyperplane contains a dense non Baire hyperplane, see [39, Question 13.1.1] and [43]. Originally in [39] authors asked if every infinite-dimensional Banach space contains a dense non-Baire hyperplane.

Arias de Reina, see [39, Theorem 1.2.12] proved the following result: Under Martin's axiom every infinite-dimensional separable Banach space contains a dense hyperplane which is not a Baire space. This result has been generalized by Valdivia in [52] by showing that under Martin's axiom every separable Baire tvs contains a non Baire dense hyperplane if E admits a discontinuous linear functional. Finally Saxon in [43] obtained a complete answer in the category of locally convex spaces. He proved that Under $\mathfrak{c} - A$ axiom (which is essentially weaker than Martin's axiom) a lcs E admits a dense non-Baire hyperplane iff E admits a dense hyperplane.

Using Valdivia's result mentioned above we provide the following

Theorem 2. *Assume Continuum Hypothesis. Let (E, ξ) be an infinite-dimensional tvs which contains a dense hyperplane F. Then on E there exists a vector topology $\upsilon \geqslant \xi$ such that $(E, \xi)' = (E, \upsilon)'$ and (E, υ) contains a dense non-Baire hyperplane.*

In the third section we provide an example of a $C(K)$-space which shows the Sobczyk Property is not a three-space property. As a by-product of our result we obtain that the Separable Complementation Property ((SCP), for short) is not a three-space property either. We recall that a Banach space E has the (SCP) provided that every separable subspace Y_0 is included in a separable superspace $Y_1 \subset E$ such that Y_1 is already complemented in E. We shall also use the nontrivial fact that every (WCG)-space E has the *dens property*, i.e., the weak* density of E' equals the density of E; then we say that E is a DENS space (see, e.g., [53, Theorem 3.5 and p. 1762]). It is known that the implications below are strict [53, pp. 1762-1762]:

$$\text{(WCG)} \Rightarrow \text{(SCP)} \Rightarrow \text{Sobczyk Property} \qquad \text{and} \qquad \text{(WCG)} \Rightarrow \text{DENS}. \quad (1)$$

The source of the problem of *whether the Sobczyk Property is a three-space property* (poset at the end of the 90' by Drewnowski) is the result by Patterson [38]. She has proved that, in the non-separable $C(K)$-space (with K separable, yet non-metrizable) considered by Corson [10], every *isometric* copy of c_0 is complemented.

On the other hand, since $C(K)$ contains a copy of $C[0,1]$ with $C(K)/C[0,1]$ isomorphic to $c_0[0,1]$, it is natural to conjecture the above question has a positive answer; then every *isomorhic* copy of c_0 in $C(K)$ would be complemented.

The problem of complementability of copies of c_0 in spaces of the type $C(L)$, with L compact scattered, is studied also in a recent paper by Koszmider [28]. He proves that, under some additional assumptions, if L is a Mrówka space (its description is given below) then the only proper complemented subspaces of $C(L)$ are isomorphic to $C(L)$ or c_0.

In Theorem 5 below we show the above question has a negative answer, even for the class of $C(L)$-spaces studied by Koszmider.

Following [25] a lcs E is called *semi Baire-like* (shortly *sBL*) if given an increasing sequence $(A_n)_n$ of closed absolutely convex sets covering E, then A_m is absorbing in E for some $m \in \mathbb{N}$.

A lcs E is *barrelled* (ℓ^∞-barrelled) if every $\sigma(E',E)$-bounded set (sequence) in E' is equicontinuous.

Recall also that a topological space X has *countable tightness* if for every set $A \subset X$ and each $x \in \overline{A}$ there exists a countable subset $B \subset A$ whose closure contains x.

A topological space X is said to be *angelic* if every relatively countably compact set A in X is relatively compact and and for each $x \in \overline{A}$ there exists a sequence in A which converges to x, see [18].

2. Proofs of Theorems 1 and 2

The following result due to Talagrand [48] will be essential to prove Theorem 1.

Proposition 2. *Let (E,ξ) be a regular space which admits a stronger topology ϑ such that (E,ϑ) is a Lindelöf Σ-space. Then $d(E,\vartheta) \leqslant w(E,\xi)$, where $d(E)$ and $w(E)$ denote the density and the weight of E, respectively.*

We need also the following two useful facts, the first one due to S. Dierolf, see [40], [11], [12], and the other one has been proved by Kąkol and Saxon [25, Theorem 3.1]. For the sake of completeness we add short direct proofs.

Lemma 1. *Let ξ and ϑ be two vector topologies on a vector space E such that $\xi \leqslant \vartheta$. If F is a vector subspace of E such that $\xi|F = \vartheta|F$ and for the quotients we also have $\xi/F = \vartheta/F$, then $\xi = \vartheta$.*

Proof. Let U be a neighbourhood of zero in (E, ϑ). By hypothesis there exists a neighbourhood of zero V in (E, ξ) such that $(V - V) \cap F \subset U$, and $(U \cap V) + F$ is a neighbourhood of zero in (E, ξ). If

$$w \in V \cap [(U \cap V) + F],$$

then $w = x + y$ for some $x \in U \cap V$ and $y \in F$, so that

$$y = w - x \in (V - V) \cap F \subset U.$$

Thus $w \in U + U$. This means that in (E, ξ), the neighbourhood of zero $V \cap [(U \cap V) + F]$ is a subset of $U + U$. Therefore $\xi = \vartheta$. ∎

Lemma 2. *Let E be a lcs containing a closed metrizable barrelled subspace F such that E/F is sBL. Then E is sBL.*

Proof. Let $\mathfrak{F}$ be a basis of absolutely convex neighbourhoods of zero in E and let $(A_n)_n$ be a sequence in E as required. Since the space F is barrelled and metrizable, then F enjoys the property *Baire-like*, which means that every increasing sequence of absolutely convex closed subsets covering F contains a member which is neighbourhood of zero in F, see [42]. Hence there exists $m \in \mathbb{N}$ and $U \in \mathfrak{F}$ such that $3U \cap F \subset A_m$. Then there exists $k \in \mathbb{N}$ such that $\overline{q(U \cap A_k)}$ is absorbing in E/F, where $q : E \to E/F$ is the quotient map. The set

$$D := U \cap q^{-1}(\overline{q(U \cap A_k)})$$

is absorbing in E and since q is open, then $D \subset U \cap A_k + F + V \cap U$ for all $V \in \mathfrak{F}$. Hence

$$D \subset A_k + (3U) \cap F + V,$$

so $D \subset \overline{A_k + 3U \cap F} \subset \overline{A_k + A_m} \subset 2A_{k+m}$. ∎

We note also (for a convenience of the reader) that there exists a lcs which is not sBL but E contains a closed barrelled subspace F such that E/F is sBL, see [25, Example 4.1].

We are ready to prove Theorem 1

Proof. Let $(x_t)_{t \in T}$ be a Hamel basis of E. Consider a partition $(T_n)_n$ of T such that $T = \bigcup_n T_n$ and card $T =$ card T_n for all $n \in \mathbb{N}$. Set $G_n := \operatorname{lin}\{x_t : t \in \bigcup_{i=1}^n T_i\}$. Then $(G_n)_n$ covers E and

$$\dim E = \dim G_n = \dim(E/G_n) = 2^{\aleph_0}$$

for $n \in \mathbb{N}$. By the Baire category theorem there exists a dense Baire subspace $F_1 := G_m$ of E. Let σ be a locally convex topology on E/F_1 such that $(E/F_1, \sigma)$

is linearly homeomorphic with $\ell^2(\aleph_1)$ with the weak topology. Clearly $\xi/F_1 \leqslant \sigma$. Then there exists on E a coarsest locally convex topology υ_1 such that $\xi \leqslant \upsilon_1$ and

$$\upsilon_1/F_1 = \sigma, \qquad \xi|F_1 = \upsilon_1|F_1.$$

Note that the sets $U \cap q^{-1}(V)$, where U and V run over ξ- and σ-neighbourhoods of zero, respectively, form a basis of neighbourhoods of zero for υ_1, see [11], where $q : E \to E/F_1$ is the quotient map. Since (E, υ_1) is sBL by Lemma 2, then applying Lemma 2 again for $(G_n)_n$ we deduce that there exists $m \in \mathbb{N}$ such that $F_2 := G_m$ is υ_1-dense in E. Since $(G_n)_n$ is increasing we can choose F_2 (by the Baire category theorem) to be also Baire in ξ. Let υ_2 be a coarsest locally convex topology on E such that $\xi \leqslant \upsilon_2$ and the quotient topology

$$\upsilon_2/F_2 = \sigma, \qquad \xi|F_2 = \upsilon_2|F_2.$$

Clearly each F_i, $i = 1, 2$ is closed in (E, υ_i), respectively, and we have

$$\xi \leqslant \inf\{\upsilon_1, \upsilon_2\}, \qquad \xi|F_2 = \inf\{\upsilon_1, \upsilon_2\}|F_2 = \upsilon_2|F_2.$$

On the other hand the topologies

$$\xi/F_2 = \inf\{\upsilon_1, \upsilon_2\}/F_2 = \upsilon_1/F_2$$

are trivial. By Lemma 1 one gets that $\xi = \inf\{\upsilon_1, \upsilon_2\}$. Note also that υ_i, $i = 1, 2$, are non-separable since $\ell^2(\aleph_1)$ is non-separable. Finally observe that (E, υ_i), $i = 1, 2$, cannot be a Lindelöf Σ-space. This directly follows from Proposition 2 since $\xi \leqslant \upsilon_i$ and υ_i is not separable, $i = 1, 2$. ∎

A similar argument applies also to get the following variant of the Corson's example from the Abstract.

Theorem 3. *Let (E, ξ) be an infinite-dimensional separable Banach space. Then there exists on E a stronger normed topology ξ_1 such that (E, ξ_1) contains a closed subspace F which is a continuous image of ℓ^1 and E with the weak topology σ of (E, ξ_1) is not a Lindelöf Σ-space although $(E, \sigma)/F$ is isomorphic to the space $\ell^2(\aleph_1)$ endowed with the weak topology.*

Proof. By $\|.\|$ we denote a norm generating the topology ξ. Then there exists a sequence $(y_n)_n$ in E with $\sum_n \|y_n\| < \infty$ whose linear span is dense in E, and such that

$$(t_n) \in \ell^1 \qquad \wedge \qquad \sum_n t_n y_n = 0 \Rightarrow (t_n) = 0,$$

see [31]. Define a compact injective map

$$T : \ell^1 \to E, \qquad T(x) := \sum_n x_n y_n,$$

$x = (x_n) \in \ell^1$. Clearly $F := T(\ell^1)$ is a proper dense subspace. Note also that F is non-barrelled. Indeed, assume F is barrelled. Since T is injective and continuous, then F admits a finer topology γ such that (F, γ) is linearly homeomorphic with ℓ^1. By the closed graph theorem the identity map from barrelled F onto (F, γ) is continuous yielding the equality of γ with the original topology of F, a contradiction. Hence F is not barrelled. This combined with $\dim E = 2^{\aleph_o}$ and [39, Proposition 4.3.11] yields $\dim (E/F) = 2^{\aleph_0}$.

Let ϱ be the norm topology of $\ell^2(\aleph_1)$. Since the quotient space E/F carries the trivial topology and $\dim (E/F) = 2^{\aleph_0}$, then on E/F there exists a locally convex non-separable normed topology γ such that $\xi \leqslant \xi_1$ and $(E/F, \gamma)$ is linearly homeomorphic with $(\ell^2(\aleph_1), \varrho)$. As in the proof of Theorem 1 one gets a normed topology ξ_1 on E such that

$$\xi < \xi_1, \qquad \xi|F = \xi_1|F, \qquad \xi_1/F = \varrho,$$

note here that the property to have a normed topology is a three-space property, see [40]. Since $(F, \xi|F)$ is a continuous image of the space ℓ^1 and $\xi|F = \xi_1|F$, then also $(F, \xi_1|F)$ is a continuous image of ℓ^1. Hence $(F, \xi_1|F)$ is analytic. But the weak topology σ of (E, ξ_1) has a quotient (by F) isomorphic with the weak topology of $\ell^2(\aleph_1)$, and again Proposition 2 applies to get that (E, σ) is not a Lindelöf Σ- space but has a closed subspace and a Hausdorff quotient which analytic and K-analytic, respectively. ■

Problem 1. Let E be a lcs which contains a closed metrizable separable subspace F such that E/F is analytic. Is then E a Lindelöf Σ-space? Note that if F is even analytic, then E need not be analytic, see [26].

It is worth recalling here some positive result related with this subject, see [26]. First notice the following: If E is a topological vector space containing a subspace F such that F and E/F are separable Fréchet spaces, then E is separable Fréchet. This is clear, since separability, metrizability and completeness are a three-space property by [40]. Therefore, the following two cases are interesting to settle.

Let F and E/F be analytic and E metrizable. Assume that F or E/F are complete. Is E analytic?

Recall here that $(\star)$ a metrizable topological vector space E is analytic iff E has a compact resolution, see for example [5].

Theorem 4. *Let E be a metrizable tvs containing a closed subspace F such that F and E/F are analytic. If F is complete and locally convex, then E is analytic.*

Proof. Proof below is due to L. Drewnowski (private communication). Let G be the completion of E and let

$$Q : G \to G/F$$

be the quotient map. By a result of Michael [33], see also [3, Proposition 7.1] or [3, Corollary 7.1] for the case of Fréchet spaces, there is a continuous map $g : G/F \to G$ such that $Q \circ g$ is the identity map on G/F, i.e.

$$g(x + F) \in x + F$$

for each $x \in G$. Since the quotient $E/F \subset G/F$ is analytic, then the quotient space E/F admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Assume that $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in F.

We show that the compact sets

$$M_\alpha := g(K_\alpha) + A_\alpha$$

form a compact resolution on E for $\alpha \in \mathbb{N}^{\mathbb{N}}$. Indeed, first observe that $g(K_\alpha) \subset E$, so then each compact set M_α is contained in E. Fix $x \in E$. Since

$$g(x + F) \in x + F,$$

then there exists $y \in F$ such that

$$g(x + F) + y = x.$$

For some $\alpha \in \mathbb{N}^{\mathbb{N}}$ we note that $x + F \subset K_\alpha$ and $y \in A_\alpha$. This shows that $x \in M_\alpha$ and this proves that

$$E = \bigcup \{M_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}.$$

We proved that the space E (which metrizable and separable) has a compact resolution. Now we apply $(\star)$ to deduce that E is analytic. The proof is completed. ∎

We are ready to prove Theorem 2.

Proof. First assume that (E, ξ) is not a Baire tvs. Then no ξ-dense subspace of E is Baire. In that case set $\upsilon := \xi$.

Now assume that (E, ξ) is a Baire tvs. Similarly as in the proof of Theorem 1 one gets in E a dense Baire subspace F of codimension $2^{\aleph_0}$ (the Continuum Hypothesis is assumed). Fix $0 < p < 1$. Then the quotient space E/F admits a strictly finer separable and metrizable complete vector topology μ such that $(E/F, \mu)$ is isomorphic to the space $L^p[0, 1]$. It is well-known that the topological dual of $L^p[0, 1]$ equals $\{0\}$.

As in Theorem 1 there exists on E a stronger vector topology υ such that $\xi|F = \upsilon|F$ and $\upsilon/F = \mu$. By [52], or [43, Theorem 2] it follows that $(E/F, \mu)$ contains a dense non- Baire hyperplane. But then E contains a dense non Baire hyperplane as well. Indeed, let H be a υ/F-dense non Baire hyperplane in E/F. Then E contains a unique dense hyperplane L such that $L \supset H$ and $L/F = H$. Note that L is non-Baire as well. Since $(E/F, \mu)$ has no non-zero continuous linear functionals and F is ξ-dense, then (as easily seen) we have $(E, \xi)' = (E, \upsilon)'$. ∎

A family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of sets of a topological space E is called a compact resolution if K_α are compact in E covering E and $K_\alpha \subset K_\beta$ if $\alpha \leqslant \beta$. We proved in [23] that a Baire topological vector space admitting a compact resolution is a metrizable, complete and separable space. Note that every separable Fréchet space has a compact resolution. Indeed, for a countable and dense sequence $(x_n)_n$ in E set

$$K_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{n_k} B(x_j, k^{-1}),$$

where $B(x_j, k^{-1})$ is the closed ball in E with the center at point x_j and radius k^{-1} for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ and all $j, k \in \mathbb{N}$. Then $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution.

Having this in mind and results of Arians de Reina-Valdivia-Saxon one may ask if every separable Fréchet space admits a dense hyperplane with compact resolution.

A positive answer might provide another approach to solve the mentioned problem. Drewnowski in [14] being motivated by [23] proved however that a hyperplane which admits a compact resolution must be closed.

3. The Sobczyk property is not a three-space property

Let $\mathbf{N}$ denote the set of positive integers, and let $\mathcal{F}$ be a family of almost pairwise disjoint infinite subsets of $\mathbf{N}$, which is *maximal* with respect to almost disjointness (MAD family, or MADF for short).

Our example is based on a few properties of a locally compact topology defined on the set $\mathbf{N} \cup \mathcal{F}$: the topology has as a base all singletons $\{n\}$ for $n \in \mathbf{N}$ and all sets of the form $\{A\} \cup B$ where $A \in \mathcal{F}$ and B is a cofinite subset of A. This space is often denoted by $\Psi(\mathcal{F})$ (see [21]); its fundamental properties were described in the 1954 paper by Mrówka [34], and in 1977 Mrówka [35, pp. 86-90] studied further properties of $\Psi(\mathcal{F})$.

The properties of the Mrówka space $\Psi(\mathcal{F})$ which are useful for our purposes are collected in the lemma below (their proofs are given in [34, 35]).

Lemma 3. *Let $\mathcal{F}$ be a MADF in $\mathbf{N}$, and set $\Psi = \Psi(\mathcal{F})$. Then*

(i) *every singleton $\{n\}$, where $n \in \mathbf{N}$, is isolated in Ψ,*
(ii) *$\mathcal{F}$ is closed in Ψ and discrete,*
(iii) *the space Ψ is locally compact and noncompact,*
(iv) *$\mathbf{N}$ is an open and dense subset of Ψ, and so*
(v) *Ψ is separable and uncountable (i.e.,* $\mathrm{card}(\Psi) = \mathrm{card}(\mathcal{F}) = 2^{\aleph_0}$).

Let $K(\mathcal{F}) = \Psi \cup \{\infty\}$ be the one-point compactification of Ψ (see Lemma 3 (iii)). Because the succesive derived sets of $K(\mathcal{F})$ are of the form: $\mathcal{F} \cup \{\infty\}$, $\{\infty\}$, and $\emptyset$,

$$\text{the space } K(\mathcal{F}) \text{ is scattered.} \tag{2}$$

From part (v) of Lemma 3, from [46, Proposition 8.5.7] (=*a compact scattered space is metrizable iff it is countable*) and from (2) we obtain that

$$\text{the space } K(\mathcal{F}) \text{ is separable and not metrizable.} \tag{3}$$

Now we are ready to prove the main theorem of this section (here $\mathcal{F}$ still denotes a MADF in $\mathbf{N}$).

Theorem 5. *Set* $E := C(K(\mathcal{F}))$*, and* $Y = \{x \in E : x_{|\mathcal{F}\cup\{\infty\}} = 0\}$*. Then*

(i) Y *is isometric to* c_0*,*
(ii) E/Y *is isomorphic to* $c_0(\Gamma)$ *where* $\operatorname{card}(\Gamma) = 2^{\aleph_0}$*,*
(iii) E *is not a DENS space, and hence (by* (1)*) not a (WCG) space,*
(iv) Y *is not complemented in* E*.*

Consequently, the Sobczyk property is not a three-space property.

Proof. Let 1_n denote the characteristic function of the singleton $\{n\}$, where $n \in \mathbf{N}$.

Part (i). It is plain that

$$\overline{\operatorname{lin}}\{1_n : n \in \mathbf{N}\} \subset Y,$$

the former space isometric to c_0. For the reversed inclusion it is enough to show that, for every $\varepsilon > 0$ and every $x \in Y$, the set

$$D_\varepsilon := \{t \in K(\mathcal{F}) : |x(t)| \geqslant \varepsilon\}$$

is included in $\mathbf{N}$ and is finite. Since D_ε is closed in $K(\mathcal{F})$ and $x \in Y$, the first claim is obvious.

For the second one, assume, by way of contradiction, that D_ε is infinite. Hence D_ε is a closed infinite subset of Ψ. Now we use the following result included implicitly in the proof of [16, Lemma 5]: *For every infinite subset* A *of* $\mathbf{N}$ *the closure* $\overline{A}$ *in* Ψ *equals* $A \cup \mathcal{F}_A$*, where*

$$\mathcal{F}_A = \{B \in \mathcal{F} : B \cap A \text{ is infinite}\}.$$

Hence, since D_ε is closed,

$$\mathcal{F}_{D_\varepsilon} = D_\varepsilon \setminus D_\varepsilon = \emptyset.$$

This follows that for every $B \in \mathcal{F}$ the set $D_\varepsilon \cap B$ is finite. Hence D_ε is not in $\mathcal{F}$, and so $\mathcal{F} \cup \{D_\varepsilon\}$ is an almost disjoint family including $\mathcal{F}$ properly. This contadicts, however, maximality of $\mathcal{F}$. Consequently, every set D_ε is finite, as claimed. Finally, Y is isometric to c_0.

Part (ii). By Lemma 3 (ii), $\mathcal{F} \cup \{\infty\}$ is an Alexandroff compactification of the discrete space $\mathcal{F}$. This follows that the quotient space E/Y is isometric to

$$C(\mathcal{F} \cup \{\infty\}),$$

and it is known that the latter space is isomorphic to $c_0(\mathcal{F})$ (indeed, the subspace $C_0(\mathcal{F})$ of functions vanishing at ∞ is of codimension 1 in E and is isometric to $c_0(\mathcal{F})$). Now we apply Lemma 3 (v).

Part (iii). If E were a DENS space then, since $K(\mathcal{F})$ is separable, we have that

$$d(E) = w^* - d(E') = \aleph_0.$$

Thus, $E = C(K(\mathcal{F}))$ would be separable. Hence, the space $K(\mathcal{F})$ would be metrizable, but this contradicts property (3).

Part (iv). If Y were complemented in E then, by the just proved parts (i) and (ii) of our theorem, E would be isomorphic to $c_0(\mathcal{F})$. Thus, E would be a (WCG) space, which is impossible by part (iii) of our theorem. Therefore Y is not complemented in E. ■

From the above theorem we obtain another proof that (WCG) is not a three-space property. (Similarly, by the use of the DENS property, one obtains that neither (WCD) nor (WLD) - see [53, pp. 1760-1762] for their definitons and properties - is a three-space property; this, however, follows also from the Corson's example.) Moreover, the result below is yet another consequence of Theorem 5 and the first sequence of implications in (1).

Corollary 1. *The Separable Complementation Property is not a three-space property.*

Remark 1. The classical Mrówka space Ψ was applied successfully in studies of some problems of Banach spaces; see, e.g., [16, 21, 28]. In [17] we prove an analogue of Theorem 5 for a generalized Mrówka space and apply it for Banach spaces with the so-called Controlled Separable Projection Property.

4. Something more about weakly Lindelöf Σ-spaces

This section mainly recalls some already known facts about duality between tightness and Lindelöf property for lcs. We hope to present more light to some results of this type.

From Proposition 1 it follows that to know if a lcs E is $\sigma(E,E')$-angelic and $\sigma(E,E')$ has countable tightness it is enough to know if the weak dual $(E',\sigma(E',E))$ is a Lindelöf Σ-space. Indeed, this follows from Proposition 1 and the inclusion $(E,\sigma(E,E'))\subset C_p(E',\sigma(E',E))$. But then angelicity of $(E,\sigma(E,E'))$ yields the angelic property for any stronger topology ξ on E (not necessarily vector) by Fremlin angelic lemma, see again [18, Lemma 3.1]. This combined with Cascales [4, Corollary 1.1] allow us to say that to determine if (E,ξ) is K-analytic it is enough to find on (E,ξ) a compact resolution. In general every K-analytic space admits such a resolution but the converse fails, [4], [48].

On the other hand, if for a lcs E the weak topology $\sigma(E,E')$ has countable tightness, we know in general only that $(E',\sigma(E',E))$ is realcompact, i.e. $(E',\sigma(E',E))$ is homeomorphic to a closed subspace of a product of real lines. Indeed, by Corson criterium, see [51, p.137], it is enough to show that every linear functional f on E which is $\sigma(E,E')$-continuous on each $\sigma(E,E')$-closed separable vector subspace is continuous. Observe that the kernel $K:=f^{-1}(0)$ is closed in E. In fact, if $y\in\overline{K}$, then there is countable $D\subset K$ with $y\in\overline{D}$ (the closure in $\sigma(E,E')$). By assumption we have $f|\overline{\mathrm{lin}\,(D)}$ is $\sigma(E,E')$-continuous; hence $f(y)\in\overline{f(\mathrm{lin}\,(D))}\subset\overline{f(K)}=\{0\}$, so $y\in K$ and $f\in E'$.

In [24] we provided some sufficient conditions for a lcs E to have its weak dual $(E',\sigma(E',E))$ a Lindelöf Σ-space. In particular, if $C_p(X)$ admits a finer metrizable vector topology, then $C_p(X)$ has countable tightness iff its weak dual

$(C_p(X)', \sigma(C_p(X)', C_p(X))$ is a Lindelöf Σ-space. In general the Lindelöf Σ property of $(E', \sigma(E', E))$ can be described through the descriptive theory of sets as follows.

We showed in [24] the following

Proposition 3. *For a ℓ^∞-barrelled lcs E the following conditions are equivalent:*

(i) *$(E', \sigma(E', E))$ is a Lindelöf Σ-space;*
(ii) *$(E, \sigma(E, E'))$ has countable tightness and E is covered by a family $\{A_\alpha : \alpha = (a_n) \in \Sigma\}$ of sets for some $\Sigma \subset \mathbb{N}^\mathbb{N}$ such that each sequence $x_k \in C_{n_1,n_n,\ldots,n_k}$ is bounded in E, where $C_{n_1,n_n,\ldots,n_k} := \bigcup\{A_\beta = (b_n) \in \Sigma : a_n = b_n, n \leqslant k\}$.*

Proposition 3 at the first glance looks somewhat technical but it covers many concrete classes of topological vector spaces, for example each (df)-space $C_c(X)$ has its weak dual Lindelöf Σ as ℓ^∞-barrelled by [22, Corollary 3.3] and having a family of sets as in (ii): If $(S_n)_n$ is a fundamental sequence of bounded sets in $C_c(X)$ set $A_\alpha := \bigcap_n a_n S_n$ for $\alpha = (a_n) \in \mathbb{N}^\mathbb{N}$.

The following typical situation provides a nice motivation for Proposition 4 below covering a large class of locally convex spaces. Indeed, let E be a metrizable locally convex space and let $(U_n)_n$ be a decreasing basis of absolutely convex closed subsets of E. Let U_n° be the polar of U_n for each $Rn \in \mathbb{N}$. Then clearly each U_n° is $\sigma(E', E)$-compact and $E' = \bigcup_n U_n^\circ$. Then $(E', \sigma(E', E))$ is K-analytic as the countably union of compact sets. On the other hand, it is well-known by Kaplansky that $(E'\sigma(E, E'))$ has countable tightness.

One gets a general fact describing the link between countable tightness and K-analyticity via duality pairs. Namely, there is a large class of lcs (introduced by Cascales and Orihuela in [5] under the name *class* $\mathfrak{G}$) containing among the others all (LM)-spaces (hence all metrizable lcs) and dual metric spaces (hence (DF)-spaces) for which the following Proposition 4 due to Cascales, Kąkol and Saxon [7] holds.

First we recall [6] that a lcs E is said to be in *class* $\mathfrak{G}$ if there is a family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of its topological dual E' (called its $\mathfrak{G}$-representation) such that:

(a) $E' = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$,
(b) $A_\alpha \subset A_\beta$ when $\alpha \leqslant \beta$,
(c) in each A_α, sequences are equicontinuous,

Condition (c) implies that every set A_α is $\sigma(E', E)$-relatively countably compact. Therefore the weak dual $(E', \sigma(E', E))$ of a lcs E in class $\mathfrak{G}$ has a relatively countably compact resolution. The class $\mathfrak{G}$ contains (among the others) (LM)-spaces (hence (LF)-spaces), the dual metric spaces (hence (DF)-spaces), the space of distributions $D'(\Omega)$ and the space $A(\Omega)$ of the real analytic functions for open $\Omega \subset \mathbb{R}^\mathbb{N}$, see e.g. [8], [15]. The class $\mathfrak{G}$ is stable by taking subspaces, separated quotients, completions, countable direct sums and countable products [6].

Proposition 4. *If E is a lcs in class $\mathfrak{G}$, then $(E, \sigma(E', E'))$ has countable tightness iff $(E', \sigma(E', E))$ is K-analytic.*

This shows also that every lcs in class $\mathfrak{G}$ admits a family of sets as stated in (ii) of Proposition 3.

Example 1. Proposition 4 fails if E is not in class $\mathfrak{G}$.

Proof. *Step 1.* First notice that $C_p(X)$ belongs to class $\mathfrak{G}$ iff X is countable. Indeed, assume that $C_p(X)$ belongs to class $\mathfrak{G}$. As $C_p(X) \subset \mathbb{K}^X$ is a dense subspace, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is the scalar field, then $\mathbb{K}^X$ also belongs to class $\mathfrak{G}$ by applying [5]. But then as a Baire space in class $\mathfrak{G}$ must be metrizable by [7]. This shows that X is countable.

Step 2. Next assume that X is an uncountable Lindelöf P-space. Then any finite product X^n is Lindelöf and [1, Corollary II.1.5] applies to deduce that $C_p(X)$ has countable tightness.

Step 3. On the other hand, the weak dual

$$L_p(X) := (C_p(X)', \sigma(C_p(X)', C_p(X)))$$

is not K-analytic. Otherwise, as X is closed in $L_p(X)$, then X is K-analytic as well. We reached to the conclusion that X is an uncountable K-analytic space whose every compact subset is finite (the last property holds since X is a P-space, see [19]). But X as K-analytic admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. This yields that $X = \bigcup\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ and every compact set K_α finite and $K_\alpha \subset K_\beta$ whenever $\alpha \leqslant \beta$. Finally, by [1, Proposition IV.6.15] the space X is countable. ■

Note the following very general fact concerning class $\mathfrak{G}$ which can be used to get last Proposition 4. The first claim is the crucial basis point and has been proved in [24] and essentially depends on our previous work [15].

Proposition 5. *Let E be a lcs in class $\mathfrak{G}$. Then $(E', \sigma(E', E))$ is quasi-Suslin and $\upsilon(E', \sigma(E', E))$ is a K-analytic space, where υX means the realcompactification of a space X.*

Proof. Set

$$E'_\sigma := (E', \sigma(E', E)).$$

In [15] we proved that E'_σ is quasi-Suslin, i.e. there exists a set-valued map T from $\mathbb{N}^\mathbb{N}$ into X covering E' such that if $\alpha_n \to \alpha$ in $\mathbb{N}^\mathbb{N}$ and $x_n \in T(\alpha_n)$, then $(x_n)_n$ has a cluster point in $T(\alpha)$. Every $T(\alpha)$ is countably compact, so its closure $\overline{T(\alpha)}$ in $\upsilon E'_\sigma$ is compact. The map $\alpha \to \overline{T(\alpha)}$ is (usco), so

$$Z := \bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \overline{T(\alpha)}$$

is K-analytic. Since $E'_\sigma \subset Z \subset \upsilon E'_\sigma$, then $Z = \upsilon Z = \upsilon E'_\sigma$ is K-analytic. ■

As a consequence we note the following two result from [6].

Corollary 2. *Let E be a lcs in class $\mathfrak{G}$. Then every $\sigma(E, E')$- compact set $K \subset E$ is Talagrand compact, i.e. $C_p(K)$ is K-analytic.*

Proof. Since E'_σ is quasi-Suslin, it is web-compact in sense of [37] and then [6, Theorem 2] applies via the relation $(E, \sigma(E, E')) \subset C_p(E'_\sigma)$. ∎

As we have already noticed if a lcs E has countable tightness in the weak topology $\sigma(E, E')$, then E'_σ is realcompact, hence $E'_\sigma = \upsilon E'_\sigma$. This together with Proposition 5 provides an alternative approach to Proposition 4.

This might suggest also a question about another good sufficient conditions for $E \in \mathfrak{G}$ to have its weak dual E'_σ realcompact.

If E is a Banach space, then the Mackey dual $(E', \mu(E', E))$ is not metrizazble, except the case when E is reflexive. It is well-known that $(E', \mu(E', E))$ is a complete lcs. If B' is the dual unit ball in the dual E' of E, then one may expect that some cases (different from reflexivity in general case) may provide metrizability of $(B', \mu(E', E)|B')$.

In [44] Schlüchtermann and Wheeller introduced *strongly weakly compactly generated* ((SWCG) shortly) Banach spaces. They called a Banach space (SWCG) if the space $(B', \mu(E', E)|B')$ is metrizable, see also [45].

The following Theorem 6 (from [44, Theorem 2.1]) characterizes (SWCG) Banach spaces in term of some density condition. Theorem 6 shows also that every (SWCG) Banach space is (WCG).

In [44, Theorem 2.5] it is proved that every (SWCG) Banach space is weakly sequentially complete. Hence the space c_0 although is a (WCG) space is not (SWCG).

Theorem 6. *The following conditions are equivalent for a Banach space E with a closed unit balls $B \subset E$ and $B' \subset E'$.*

(i) $(B', \mu(E', E)|B')$ *is metrizable;*
(ii) *There exists a sequence $(K_n)_n$ of weakly compact absolutely convex subsets of E such that for every weakly compact set $L \subset E$ and every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $L \subset K_n + \epsilon B$;*
(iii) *There exists a weakly compact absolutely convex set $K \subset E$ such that for each weakly compact set $L \subset E$ and every $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $L \subset nK + \epsilon B$.*

Assume now that E is a separable (SWCG) Banach space. Then clearly the space $(E', \mu(E', E))$ is separable. Since $(E', \mu(E', E))$ is separable, then B' is separable as well. Indeed, let $\mathfrak{F}(E')$ be the set of all absolutely convex neighbourhoods of zero in $\mu(E', E)$ and let $U_m \in \mathfrak{F}(E')$, $m \in \mathbb{N}$, such that

$$(B' \cap [U_m + U_m])_m$$

is a basis of neighbourhoods of zero in B'. By separability there exists a countable set B_m such that

$$E' \subset B_m + U_m,$$

and then there exists in B' a countable subset C_m such that

$$B' \subset C_m + U_m + U_m.$$

Since $E' = \bigcup_n nB'$ and each nB' is metrizable separable and complete. Then the space $(E', \mu(E', E))$ is analytic. Therefore we have

Proposition 6. *Let E be a (SWCG) Banach space. Then $(E', \mu(E', E))$ is analytic iff E is separable.*

Let (S, Σ, μ) is a finite measure space. $L_1(\mu, E)$ denotes a Banach space of Bochner integrable functions on S into a Banach space E. In [44, Theorem 3.2] Schlüchtermann and Wheller studied a problem when $L_1(\mu, E)$ is (SWCG). Talagrand [49], see also Diestel [13], proved that $L_1(\mu, E)$ is (WCG) if E is a (WCG) Banach space.

If E is a separable Banach space, then the Mackey dual $(E', \mu(E', E))$ is a separable but the strong dual $(E', \beta(E', E))$ need not be separable. Clearly $(E', \beta(E', E))$ is analytic iff $(E', \beta(E', E))$ is separable.

Theorem 6 and Proposition 6 may suggest the following question:

Let E be a separable Banach space. Is it true that the Mackey dual $(E', \mu(E', E))$ of E is an analytic space?

The Mackey dual $(E', \mu(E', E))$ of a Banach space has been studied also in [45] and [27]. In [45] the authors proved among the others that if E is a separable (SWCG) Banach space, then $(E, \sigma(E, E'))$ (which is clearly analytic) is an $\aleph_0$-space, i.e., it has a countable pseudobase.

A collection $\mathcal{P}$ of subsets of a topological space E is called a *pseudobase* if for any open set $U \subset E$ and compact $K \subset U$ there exists $P \in \mathcal{P}$ with $K \subset P \subset U$. Recall also that every $\aleph_0$-space is separable and Lindelöf and every closed set is a G_δ-set, [32], [45, Theorem 4.1]. In [27] Kirk studied the Mackey dual for spaces $C(K)$ with compact K.

On the other hand, by Batt and Hiermeyer [2, 2.6], (see also [44], [45, p.274] and [45, Theorem 4.2]) there exists a separable Banach space E for which $(E, \sigma(E, E'))$ is not an $\aleph_0$-space. It is known also [32], [45, Theorem 4.1], that a regular topological space is both an $\aleph_0$-space and a k-space iff it is a quotient of a separable metric space. Therefore it seems to be natural to ask when for a Banach space E the space $(E, \sigma(E, E'))$ is a k-space.

Recall that a Hausdorff space X is a *k-space* if a set $A \subset X$ is closed in X iff $A \cap K$ is closed in K for each compact set $K \subset X$. We shall need the following fact due to Grothendieck [20, p.134].

Lemma 4. *Let $A \subset E'$ be $\mu(E', E)$-compact for a Banach space E. Then every $\sigma(E, E')$-convergent sequence in E converges uniformly on A.*

It turns out that for infinite-dimensional Banach spaces E the space $(E, \sigma(E, E'))$ is never a k-space.

Proposition 7. *If E is a Banach space for which $(E, \sigma(E, E'))$ is a k-space, then E is finite-dimensional.*

Proof. Let γ be the topology on E of uniform convergence on $\mu(E', E)$-compact sets. Clearly $\sigma(E, E') \leqslant \gamma$. Since $\sigma(E, E')$ and γ have the same sequentially compact sets by Lemma 4, then the both topologies have the same compact sets (recall that $\sigma(E', E)$ and γ are angelic). Assume that $(E, \sigma(E, E'))$ is a k-space, then we have $\sigma(E, E') = \gamma$.

Let $(x_n)_n$ be a null-sequence in the norm topology of E'. Since the set $\{0\} \cup \{x_n : n \in \mathbb{N}\}$ is $\mu(E', E)$-compact, then the sequence $(x_n)_n$ has finite-dimensional linear span. This yields that the dual space E' (hence also E) is finite-dimensional. ∎

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