

ON THE SPLITTING RELATION FOR FRÉCHET–HILBERT SPACES

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Dedicated to the memory of Susanne Dierolf

Abstract: A shorter proof is given for a theorem of Domański and Mastyo characterizing the pairs (E, F) of Fréchet–Hilbert spaces with the property that every exact sequence $0 \longrightarrow F \longrightarrow G \longrightarrow E \longrightarrow 0$ of Fréchet–Hilbert spaces splits. The results on acyclicity of inductive spectra of metrizable locally convex spaces which we use are also presented with proofs.

Keywords: Fréchet–Hilbert space, exact sequence, splitting condition, inductive spectrum, acyclic.

In their paper [3] Domański and Mastyo showed that the splitting condition (S) (see definition in Section 3) characterizes the pairs (E, F) of Fréchet–Hilbert spaces with the property that every exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow E \longrightarrow 0 \tag{1}$$

of Fréchet–Hilbert spaces splits. The crucial part had been to show the sufficiency of (S) and this had been open for quite a while. In the present paper we give a simplified proof for the sufficiency of (S).

It had been known that (S) characterizes the pairs (E, F) of Fréchet spaces such that every exact sequence (1) splits, under the assumption that one of the spaces is nuclear or both spaces are Köthe sequence spaces (for the latter see Krone–Vogt [6]). In the case of one of the spaces being nuclear, necessity of (S) and sufficiency of a stronger splitting condition, which had been proposed by Apiola [1], had been shown in Vogt [13], sufficiency of (S) in this case was shown in Frerick [4] and Frerick–Wengenroth [5]. In the present paper we use the equivalent but more elegant formulation of (S) given by Langenbruch [7].

The core of the present paper is Section 1, where we give a short proof of the basic result in [3]. The idea of the proof of Lemma 1.1 is similar to the proof of [3, Theorem 3.1] and based on ideas from Ovchinnikov [9]. We believe that also

the subsequent presentation of the proof of the characterization deserves some interest. As we use the characterization of acyclicity of certain inductive spectra we include an excursion on this topic which might also be of interest.

In Section 2 we present the essential results, as far as we need them, on the characterization of acyclic and weakly acyclic inductive spectra of metrizable locally convex spaces with full proofs. The presentation is in the spirit of [14], [17] where a version of (P) appeared as a necessary condition. The first characterization of acyclicity in terms of a condition (M) had been given by Palamodov [10] and Retakh [12], it was essentially improved by Wengenroth [19]. His argument has in consequence also the sufficiency of (P) and appears here (in different language) in Lemma 2.2. Of course, this is directly connected to the proofs of the characterization of $\text{Proj}^1 = 0$ for a projective spectrum of (LB)-spaces as given in [5] and [2].

In Section 3 we give a direct and self contained proof of the sufficiency part of the splitting theorem in the Fréchet-Hilbert case, the part which had been open and was shown in [3], and in Section 4 we give a proof of the complete characterization of Fréchet-Hilbert splitting pairs. Here we don't try to be self-contained.

We use standard terminology and results of the theory of locally convex spaces, see e.g. [8]. For the homological background see [20].

1. Fundamental lemma

A pair (X_0, X_1) of Hilbert spaces which are both subspaces of a linear space V we call a *couple of Hilbert spaces*. In particular, $X_0 \cap X_1$ is then well defined. The following is our version of [3, Theorem 3.1].

Lemma 1.1. *Let $s, t > 0$ and $(X_0, X_1), (Y_0, Y_1)$ be two couples of Hilbert spaces. On $X = X_0 \cap X_1$ and $Y = Y_0 \cap Y_1$ we assume to have Hilbert norms such that*

$$\|y\|_Y \|x\|_X \leq s \|y\|_{Y_0} \|x\|_{X_0} + t \|y\|_{Y_1} \|x\|_{X_1}$$

for all $x \in X, y \in Y$. Then we have

$$\|u\|_{X \otimes_\pi Y} \leq s \|u\|_{Y_0 \otimes_\pi X_0} + t \|u\|_{Y_1 \otimes_\pi X_1}$$

for all $u \in Y \otimes X$.

Proof. If $u = \sum_{\nu=1}^n y_\nu \otimes x_\nu$ then we may assume that the y_ν as well as the x_ν are linearly independent. The various tensor norms can be calculated in $\text{span}\{y_1, \dots, y_n\} \otimes \text{span}\{x_1, \dots, x_n\}$ equipped with the respective norms. Therefore we may assume that $X = X_0 = X_1$ and $Y = Y_0 = Y_1$, both of the same finite dimension.

u defines linear maps $T_j = Y_j \rightarrow X_j, j = 0, 1$ by

$$T_j y = \sum_{\nu=1}^n \langle y, y_\nu \rangle_{Y_j} x_\nu.$$

By the above remarks we may assume T_j invertible.

Let $T_j = U_j \circ |T_j|$ be the polar decomposition of T_j . We set $A_j = U_j \circ_+ \sqrt{|T_j|}$ and $B_j =_+ \sqrt{|T_j|}$. On X we define scalar products by

$$(x, y)_j := \langle A_j^{-1}x, A_j^{-1}y \rangle_{Y_j}$$

with norm $|\cdot|_j$ and we choose an orthonormal basis e_1, \dots, e_n in X with respect to $(\cdot, \cdot)_0$ which is orthogonal with respect to $(\cdot, \cdot)_1$. We set $e_\nu^0 = e_\nu$ and $e_\nu^1 = \frac{1}{|e_\nu|_1} e_\nu$. Then we have

$$T_j y = \sum_{\nu=1}^n (T_j y, e_\nu^j)_j e_\nu^j = \sum_{\nu=1}^n \langle B_j y, A_j^{-1} e_\nu^j \rangle_{Y_j} e_\nu^j = \sum_{\nu=1}^n \langle y, B_j A_j^{-1} e_\nu^j \rangle_{Y_j} e_\nu^j$$

for all $y \in Y$ and therefore

$$u = \sum_{\nu=1}^n (B_0 A_0^{-1} e_\nu) \otimes e_\nu = \sum_{\nu=1}^n \left(\frac{1}{|e_\nu|_1^2} B_1 A_1^{-1} e_\nu \right) \otimes e_\nu.$$

This implies $B_0 A_0^{-1} e_\nu = \frac{1}{|e_\nu|_1^2} B_1 A_1^{-1} e_\nu =: g_\nu$ for all ν .

By use of our assumption we have

$$\begin{aligned} \|u\|_{X \otimes_\pi Y} &\leq \sum_{\nu=1}^n \|g_\nu\|_Y \|e_\nu\|_X \\ &\leq s \sum_{\nu=1}^n \|g_\nu\|_{Y_0} \|e_\nu\|_{X_0} + t \sum_{\nu=1}^n \|g_\nu\|_{Y_1} \|e_\nu\|_{X_1}. \end{aligned}$$

We set $f_\nu^j = A_j^{-1} e_\nu^j$. Then f_1^j, \dots, f_n^j is an orthonormal basis of Y_j and we obtain, using that U_j is unitary,

$$\begin{aligned} \sum_{\nu=1}^n \|g_\nu\|_{Y_j} \|e_\nu\|_{X_j} &= \sum_{\nu=1}^n \|B_j f_\nu^j\|_{Y_j} \|A_j f_\nu^j\|_{X_j} \\ &= \sum_{\nu=1}^n \|B_j f_\nu^j\|_{Y_j}^2 = \sigma(B_j)^2 = \nu(T_j) \\ &= \|u\|_{Y_j \otimes_\pi X_j}. \end{aligned}$$

Here $\sigma(\cdot)$ denotes the Hilbert-Schmidt norm and $\nu(\cdot)$ denotes the nuclear norm. ■

2. Acyclic inductive spectra

Let X be linear space, $X_1 \subset X_2 \subset X_3 \subset \dots$ an increasing sequence of linear spaces with $\bigcup_{n=1}^\infty X_n = X$. We assume that every X_n is equipped with an increasing sequence $\| \cdot \|_{n,k}$, $k \in \mathbb{N}$ of seminorms, such that $\| \cdot \|_{n,k} \geq \| \cdot \|_{n+1,k}$ for all n, k . Without restriction of generality we may assume $2 \| \cdot \|_{n,k} \leq \| \cdot \|_{n,k+1}$ for all n, k .

We define a map $\tau : \bigoplus_{n=1}^{\infty} X_n \longrightarrow \bigoplus_{n=1}^{\infty} X_n$ by $\tau : (u_n)_{n \in \mathbb{N}} \mapsto (u_n - u_{n-1})_{n \in \mathbb{N}}$, where $u_0 := 0$. Then we have $R(\tau) = \{(v_n)_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} v_n = 0\}$, the sums, of course, being finite. Obviously τ is injective. Due to our assumptions, the seminorms of the form $\sum_{n=1}^{\infty} \|u_n\|_{n,m(n)}$ are a fundamental system of seminorms on $\bigoplus_{n=1}^{\infty} X_n$.

The *inductive spectrum* $\mathcal{X} : X_1 \subset X_2 \subset X_3 \subset \dots$ is called *acyclic* if τ is topologically injective, i.e. $\tau^{-1} : R(\tau) \longrightarrow \bigoplus_{n=1}^{\infty} X_n$ is continuous. It is called *weakly acyclic* if τ^{-1} is weakly continuous. If \mathcal{X} is (weakly) acyclic then also every equivalent spectrum is (weakly) acyclic. The spectrum $\tilde{X}_1 \subset \tilde{X}_2 \subset \tilde{X}_3 \subset \dots$ is equivalent, if for every n there is N such that $X_n \subset \tilde{X}_N$ and $\tilde{X}_n \subset X_N$ with continuous imbeddings.

Definition 2.1. \mathcal{X} satisfies condition (P) if the following holds:

$$\forall \mu \exists k \forall K \exists \nu \forall m, \varepsilon > 0 \exists N, S \forall u \in X_\mu : \|u\|_{k,m} \leq \varepsilon \|u\|_{\mu,\nu} + S \|u\|_{K,N}.$$

We wish to show that (P) is equivalent to acyclicity. While necessity of (P) is relatively easy the crucial point is sufficiency of (P).

Let (P) be satisfied. First we replace $\|\cdot\|_{n,k}$ by $2^k \|\cdot\|_{n,k}$. Then, by inductively choosing subsequences of the n and k , we get an equivalent inductive spectrum $\tilde{\mathcal{X}} = (X_{n_\nu})_{\nu \in \mathbb{N}}$ which satisfies the following assumptions:

1. $2\|\cdot\|_{n,m} \leq \|\cdot\|_{n,m+1}$ for all n and m ;
2. For every $n > 1, m \in \mathbb{N}$ and $\varepsilon > 0$ there is $N = N(n, m, \varepsilon)$ with

$$\|u\|_{n,m} \leq \varepsilon (\|u\|_{n-1,n-1} + \|u\|_{n+1,N}) \tag{2}$$

for all $u \in X_{n-1}$;

3. $N(n, n, 1/4) = n + 1$.

We will use the following notation: Let $\|\cdot\|_j, j = 1, \dots, m$ be seminorms on subspaces X_j of a linear space X then we set on $X_1 + \dots + X_m$

$$\bigwedge_{j=1}^m \|u\|_j := \inf \left\{ \sum_{j=1}^m \|u_j\|_j : u_j \in X_j \text{ for all } j \text{ and } u = \sum_{j=1}^m u_j \right\}.$$

We will need the following simple lemma:

Lemma 2.1. Let $\|\cdot\|_1, \|\cdot\|'_2 \leq \|\cdot\|_2, \|\cdot\|_3$ be seminorms on linear spaces $X_1 \subset X_2 \subset X_3$, with

$$\|\cdot\|'_2 \leq \|\cdot\|_1 + \|\cdot\|_3 \quad \text{on } X_1 \quad \text{and} \quad \|\cdot\|_3 \leq \|\cdot\|_2 \quad \text{on } X_2.$$

Then we have

$$\|\cdot\|'_2 \leq 2 \bigwedge_{j=1}^2 \|\cdot\|_j + \|\cdot\|_3 \quad \text{on } X_2.$$

Proof. For $u = u_1 + u_2$, $u_j \in X_j$, we obtain

$$\begin{aligned} \|u\|'_2 &\leq \|u_2\|'_2 + \|u_1\|_1 + \|u_1\|_3 \\ &\leq \|u_2\|'_2 + \|u_1\|_1 + \|u\|_3 + \|u_2\|_3 \\ &\leq 2(\|u_1\|_1 + \|u_2\|_2) + \|u\|_3. \end{aligned}$$

For the first inequality we used the first assumption, for the third one $\|u_2\|'_2 \leq \|u_2\|_2$ and the second assumption. ■

Lemma 2.2. *Let \mathcal{X} satisfy assumptions 1., 2., 3. above. Then for every $n > 1$, $m \in \mathbb{N}$ and $\varepsilon > 0$ there is $M = M(n, m, \varepsilon)$ with*

$$\|u\|_{n,m} \leq \varepsilon \left(\bigwedge_{k=1}^{n-1} \|u\|_{k,k+1} + \|u\|_{n+1,M} \right)$$

for all $u \in X_{n-1}$. Moreover we may choose $M(n, n, 1) = n + 1$.

Proof. We proceed by induction on n . For $n = 2$ this follows from the assumption. So we assume the assertion for $n - 1$ and this implies

$$\|u\|_{n-1,n-1} \leq \bigwedge_{k=1}^{n-2} \|u\|_{k,k+1} + \|u\|_{n,n}, \quad u \in X_{n-2}.$$

From Lemma 2.1 we obtain

$$\|u\|_{n-1,n-1} \leq 2 \bigwedge_{k=1}^{n-1} \|u\|_{k,k+1} + \|u\|_{n,n}, \quad u \in X_{n-1}.$$

For given $m \geq n$ and $\varepsilon \leq 1/2$ we insert this in the inequality 2. This yields

$$\|u\|_{n,m} \leq \varepsilon \left(2 \bigwedge_{k=1}^{n-1} \|u\|_{k,k+1} + \|u\|_{n,n} + \|u\|_{n+1,N} \right), \quad u \in X_{n-1}.$$

Putting $\varepsilon\|u\|_{n,n} \leq \frac{1}{2}\|u\|_{n,m}$ to the left and multiplying by 2 we obtain

$$\|u\|_{n,m} \leq 4\varepsilon \bigwedge_{k=1}^{n-1} \|u\|_{k,k+1} + 2\varepsilon\|u\|_{n+1,N}, \quad u \in X_{n-1}.$$

Since $0 < \varepsilon \leq 1/2$ was arbitrary this completes the induction in the general case. The special case then follows from $N(n, n, 1/4) = n + 1$. ■

Theorem 2.1. \mathcal{X} is acyclic if and only if it satisfies (P).

Proof. First we prove sufficiency of (P). We may assume \mathcal{X} to be of the special form as before. We have to show that $\tau^{-1} : R(\tau) \rightarrow \bigoplus_{k \in \mathbb{N}} X_k$ is continuous. Since $\tau^{-1}(u_k)_{k \in \mathbb{N}} = \left(\sum_{k=1}^n u_k \right)_{n \in \mathbb{N}}$, this means that for every sequence $(m(n))_{n \in \mathbb{N}}$ we should find a sequence $(M(n))_{n \in \mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} \left\| \sum_{k=0}^n u_k \right\|_{n, m(n)} \leq \sum_{n=1}^{\infty} \|u_n\|_{n, M(n)}$$

for all $(u_n)_{n \in \mathbb{N}} \in R(\tau)$.

So let $u_n \in X_n$ be a sequence, $m \in \mathbb{N}$ and $M = M(n, m, \varepsilon)$ as in Lemma 2.2 which we will use for the second estimate of the following. We may assume $M \geq m$.

$$\begin{aligned} \left\| \sum_{k=0}^n u_k \right\|_{n, m} &\leq \|u_n\|_{n, m} + \left\| \sum_{k=0}^{n-1} u_k \right\|_{n, m} \\ &\leq \|u_n\|_{n, m} + \varepsilon \left(\sum_{k=1}^{n-1} \|u_k\|_{k, k+1} + \left\| \sum_{k=0}^{n-1} u_k \right\|_{n, M} \right) \\ &\leq \|u_n\|_{n, m} + \varepsilon \left(\sum_{k=1}^{n-1} \|u_k\|_{k, k+1} + \|u_n + u_{n+1}\|_{n+1, M} \right. \\ &\quad \left. + \left\| \sum_{k=0}^{n+1} u_k \right\|_{n+1, M} \right) \\ &\leq 2\|u_n\|_{n, M} + \|u_{n+1}\|_{n+1, M} + \varepsilon \sum_{k=1}^{n-1} \|u_k\|_{k, k+1} \\ &\quad + \varepsilon \left\| \sum_{k=0}^{n+1} u_k \right\|_{n+1, M}. \end{aligned}$$

We use it to choose inductively a strictly increasing sequence $M(n) \geq m(n) + 1$ of integers, such that

$$\begin{aligned} \left\| \sum_{k=0}^n u_k \right\|_{n, M(n)} &\leq 2\|u_n\|_{n, M(n+1)} + \|u_{n+1}\|_{n+1, M(n+1)} \\ &\quad + 2^{-n} \sum_{k=1}^{n-1} \|u_k\|_{k, k+1} + 1/2 \left\| \sum_{k=0}^{n+1} u_k \right\|_{n+1, M(n+1)}. \end{aligned}$$

Summing up over n we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \left\| \sum_{k=0}^n u_k \right\|_{n, M(n)} &\leq 3 \sum_{n=1}^{\infty} \|u_n\|_{n, M(n+1)} + \sum_{k=1}^{\infty} \|u_k\|_{k, k+1} \\ &\quad + 1/2 \sum_{n=1}^{\infty} \left\| \sum_{k=0}^{n+1} u_k \right\|_{n+1, M(n+1)}. \end{aligned}$$

Putting the last term to the left and multiplying by 2 we get

$$\sum_{n=1}^{\infty} \left\| \sum_{k=0}^n u_k \right\|_{n, m(n)} \leq \sum_{n=1}^{\infty} \left\| \sum_{k=0}^n u_k \right\|_{n, M(n)} \leq 8 \sum_{n=1}^{\infty} \|u_n\|_{n, M(n+1)}$$

which completes the proof of sufficiency.

To prove necessity of (P) we assume that (P) does not hold. Then there exists μ , such that for all k there are $m(k), K(k) \geq k, \varepsilon(k) > 0$ such that for every $N, S > 0$ there is $u \in X_\mu$ with

$$\|u\|_{k, m(k)} > \varepsilon(k) \|u\|_{\mu, k} + S \|u\|_{K(k), N}.$$

Continuity of τ^{-1} yields sequences $n(k)$ and $D(k)$ such that

$$\sum_{k=1}^{\infty} \frac{k}{\varepsilon(k)} \|u_k\|_{k, m(k)} \leq \sum_{k=1}^{\infty} D(k) \|u_k - u_{k-1}\|_{k, n(k)}.$$

For $u \in X_\mu$ we set $u_\mu = \dots = u_{K(k)} = u, u_k = 0$ otherwise and obtain

$$k \|u\|_{k, m(k)} \leq \varepsilon(k) D(\mu) \|u\|_{\mu, n(\mu)} + \varepsilon(k) D(K(k)) \|u\|_{K(k), n(k)}.$$

for all $u \in X_\mu$ and $k \geq \mu$. We choose k so large that $k \geq D(\mu), k \geq n(\mu)$ and obtain a contradiction. ■

Remark. In fact we showed even necessity of the following condition, which is formally stronger than (P):

$$\forall \mu \exists k, \nu \forall m, K, \varepsilon > 0 \exists N, S \forall u \in X_\mu^* : \|u\|_{k, m} \leq \varepsilon \|u\|_{\mu, \nu} + S \|u\|_{K, N}.$$

This means, of course, that both conditions are equivalent. In fact, the quantifiers in the stronger version of (P) are those of the earlier version (P_2^*) in [14, 17]. The version we are using in our definition is due to Langenbruch [7].

Lemma 2.3. *As the proof of necessity shows, to show (P) it is enough that τ^{-1} is continuous on every countably dimensional subspace of X .*

For the case of weak acyclicity we will need only the necessity part.

Proposition 2.1. *If \mathcal{X} is weakly acyclic then it satisfies the following condition (WP):*

$$\forall \mu \exists k \forall K \exists \nu \forall m \exists N, S \forall u \in X_\mu : \|u\|_{k,m} \leq S(\|u\|_{\mu,\nu} + \|u\|_{K,N}).$$

Proof. In precisely the same way as in the necessity proof of Theorem 2.1 (write down contraposition, apply to same element as in this proof, will contradict continuity of y with respect to topology induced by $\bigoplus_{k=1}^\infty X_k$) we show

$$\forall \mu \exists k \forall K \exists \nu \forall y \in X_k^* \exists N, S \forall u \in X_\mu : |y(u)| \leq S(\|u\|_{\mu,\nu} + \|u\|_{K,N}).$$

The result then follows by a simple application of Baire’s theorem to the canonical Banach subspaces of X_k^* . ■

3. Splitting theorem, rapid access

Let E and F be Fréchet–Hilbert spaces, $\| \cdot \|_1 \leq \| \cdot \|_2 \leq \dots$ fundamental systems of Hilbert-seminorms and E_k, F_k the respective local Hilbert spaces with linking maps $j_{k+1}^k : F_{k+1} \rightarrow F_k$ (in the case of F).

We set $X_n = E \otimes_\pi F_n^*$. Then X_n is locally convex, metrizable with the seminorms $\| \cdot \|_{n,k} = \| \cdot \|_{E_k \otimes F_n^*}$. We obtain an inductive spectrum \mathcal{X} .

Definition 3.1. *(E, F) satisfies condition (S) if the following holds:*

$$\begin{aligned} \forall \mu \exists k \forall K \exists \nu \forall m, \varepsilon > 0 \exists N, S \forall x \in E, y \in F_\mu^* : \\ \|x\|_m \|y\|_k^* \leq \varepsilon \|x\|_\nu \|y\|_\mu^* + S \|x\|_N \|y\|_K^*. \end{aligned}$$

Lemma 3.1. *\mathcal{X} satisfies (P) if and only if (E, F) satisfy (S).*

Proof. The *if*-part is an immediate consequence of Lemma 1.1, the *only if*-part is trivial. ■

We define a linear map $\sigma : \prod_n L(E, F_n) \rightarrow \prod_n L(E, F_n)$ by $\sigma(\varphi_n)_{n \in \mathbb{N}} = (j_{n+1}^n \circ \varphi_{n+1} - \varphi_n)_{n \in \mathbb{N}}$ and observe that $\sigma = \tau^*$ where τ is the map defined in the previous section applied to \mathcal{X} as defined before. Here we identified $L(E, F_n) = (E \otimes_\pi F_n^*)^*$. From Lemma 3.1 and Theorem 2.1 we obtain:

Lemma 3.2. *If (E, F) satisfy (S) then σ is surjective.*

A standard argument now leads to our main result, the sufficiency part of Theorem 4.2 in [3].

Theorem 3.1. *If (E, F) satisfy (S), then every exact sequence*

$$0 \longrightarrow F \longrightarrow G \xrightarrow{q} E \longrightarrow 0$$

of Fréchet–Hilbert spaces splits.

Proof. We may assume $F = \ker q$ and choose a fundamental system of hilbertian seminorms on G . Then the trace, resp. quotient seminorms define a fundamental system of hilbertian seminorms on F and E and we get for every n an exact sequence

$$0 \longrightarrow F_n \longrightarrow G_n \xrightarrow{q_n} E_n \longrightarrow 0$$

of Hilbert spaces which splits. Again we may assume $F_n = \ker q_n$. Let r_n be a right inverse for q_n then $a_n := j_{n+1}^n \circ q_{n+1} - q_n$ defines a map in $L(E, F_n)$. By Lemma 3.2 we get a sequence $\varphi_n \in L(E, F_n)$, $n \in \mathbb{N}$, with $a_n = j_{n+1}^n \circ \varphi_{n+1} - \varphi_n$ for every n . Then $R_n = r_n - \varphi_n$ satisfies $j_{n+1}^n \circ R_{n+1} = R_n$ for every n , hence defines a map $R \in L(E, G)$ which is easily seen to be a right inverse for q . ■

4. Splitting theorem, complete version

To obtain the complete characterization of Fréchet–Hilbert splitting pairs we need analogous to (WP) the weaker version of (S).

Definition 4.1. (E, F) satisfies condition (WS) if the following holds:

$$\forall \mu \exists k \forall K \exists \nu \forall m \exists N, S \forall x \in E, y \in F_\mu^* : \\ \|x\|_m \|y\|_k^* \leq S(\|x\|_\nu \|y\|_\mu^* + \|x\|_N \|y\|_K^*).$$

We first state the well known fact that in most relevant cases (S) and (WS) are equivalent.

Proposition 4.1. *If E is a proper (not normable) Fréchet space then (S) is equivalent to (WS).*

We consider exact sequences of Fréchet–Hilbert spaces

$$0 \longrightarrow F \longrightarrow G \xrightarrow{q} H \longrightarrow 0 \tag{3}$$

and assume that E is an infinite dimensional Fréchet–Hilbert space. Then we ask when the induced sequence

$$0 \longrightarrow L(E, F) \longrightarrow L(E, G) \xrightarrow{q_*} L(E, H) \longrightarrow 0 \tag{4}$$

is exact which means, when is q_* is surjective.

The equivalence 1 \Leftrightarrow 2. in the following Theorem is Theorem 4.2 in [3].

Theorem 4.1. *The following are equivalent:*

1. (E, F) satisfies (S).
2. Every exact sequence (1) of Fréchet–Hilbert spaces splits.
3. For any exact sequence (3) the exact sequence (4) is exact.
4. for any exact sequence (3) every equicontinuous set in $L(E, H)$ can be lifted in (4) to an equicontinuous set in $L(E, G)$.

Proof. 1. \Rightarrow 4.: By Lemma 3.1 we get condition (P) for \mathcal{X} , defined as in Section 3. The argument leading to Lemma 3.2 also shows that σ lifts products of equicontinuous sets in $L(E, F_n)$ into products of equicontinuous sets. In the proof of Theorem 3.1 an equicontinuous set in $L(E, F)$ leads to a product of equicontinuous sets in $L(E, F_n)$.

4. \Rightarrow 3. and 3. \Rightarrow 2. are obvious. 2. \Rightarrow 3. is well known.

3. \Rightarrow 1.: 3. implies that \mathcal{X} is weakly acyclic and therefore, by Proposition 2.1, satisfies (WP) which implies (WS). If E is a proper Fréchet space then Proposition 4.1 gives the result. If E is a Hilbert space then, due to Lemma 2.3, we may assume E and F to be separable. As a sample sequence (3) we use the canonical resolution

$$0 \longrightarrow F \longrightarrow \prod_n F_n \xrightarrow{\sigma} \prod_n F_n \longrightarrow 0.$$

Any bounded set in $\prod_n F_n$ is contained in a Hilbert ball B which generates a separable Hilbert space H_B . There is an isomorphism from E onto H_B which can be lifted under σ , hence B can be lifted under σ to a bounded set. This means that the spectrum $F_1^* \subset F_2^* \subset \dots$ is acyclic and the necessity part of Theorem 2.1 yields the result. Notice that in (S) we have $\|x\|_m = \|x\|_\nu = \|x\|_N$. \blacksquare

References

- [1] H. Apiola, *Characterization of subspaces and quotients of nuclear $L_f(\alpha, \infty)$ -spaces*, Compositio Math. **50** (1983), 65–81.
- [2] R.W. Braun and D. Vogt, *A sufficient condition for $\text{Proj}^1 \mathcal{X} = 0$* , Michigan Math. J. **44** (1997), 149–156.
- [3] P. Domański and M. Mastyło, *Characterization of splitting for Fréchet–Hilbert spaces via interpolation*, Math. Ann. **339** (2007), 317–340.
- [4] L. Frerick, *A splitting theorem for nuclear Fréchet spaces*, in: Functional Analysis, proceedings of the first Workshop at Trier University, S. Dierolf, S. Dineen, P. Domański, eds.
- [5] L. Frerick and J. Wengenroth, *A sufficient condition for vanishing of the derived projective limit functor*, Arch. Math. **67** (1996), 296–301.
- [6] J. Krone and D. Vogt, *The splitting relation for Köthe spaces*, Math. Z. **190** (1985), 387–400.
- [7] M. Langenbruch, *Characterization of surjective partial differential operators on spaces of real analytic functions*, Studia Math. **162** (2004), 53–96.
- [8] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford 1997.
- [9] V.I. Ovchinnikov, *Coherently nuclear operators in pairs of Hilbert spaces*, Math. Zametki **63** (1998), 866–872 (in Russian); English transl. in Math. Notes **63**(6), (1998) 764–769.
- [10] V.P. Palamodov, *Homological methods in the theory of locally convex spaces* (Russian), Usp. Mat. Nauk **26** 1 (1971), 3–66, English transl. Russian Math. Surveys **26** 1(1971), 1–64.

- [11] M. Poppenberg and D. Vogt: *A tame splitting theorem for exact sequences of Fréchet spaces*, Math. Z. **219** (1995), 141–161.
- [12] V.S. Retakh, *Subspaces of a countable inductive limit*, Dokl. Akad. Nauk SSSR **194** (1970), No. 6, English transl. *Soviet Math. Dokl.* **11** (1970), 1384–1386.
- [13] D. Vogt, *On the functor $Ext^1(E, F)$ for Fréchet spaces*, Studia Math. **85** (1987), 163–197.
- [14] D. Vogt, *Lectures on projective spectra of (DF) -spaces*, Seminar lectures, AG Funktionalanalysis, Düsseldorf/Wuppertal, 1987.
- [15] D. Vogt, *Topics on projective spectra of (LB) -spaces*, in: T. Terzioğlu (ed.), *Advances in the Theory of Fréchet Spaces*, NATO Adv. Sci. Inst. Ser. C 287, Kluwer, 1989, 11–27.
- [16] D. Vogt, *Interpolation of nuclear operators and a splitting theorem for exact sequences of Fréchet spaces*, preprint
- [17] D. Vogt, *Regularity properties of (LF) -spaces*, in: Progress in Functional Analysis, K.D. Bierstedt, J. Bonnet, J. Horváth, M. Maestre (Eds.), Elsevier (1992), 57–84.
- [18] D. Vogt and M. J. Wagner, *Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau*, Studia Math. **67** (1980), 225–240.
- [19] J. Wengenroth, *Acyclic inductive spectra of Fréchet spaces*, Studia Math. **120** (1996), 247–257.
- [20] J. Wengenroth, *Derived functors in functional analysis*, Lect. Notes Math. **1810**, Springer, Berlin 2003.

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