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BOHR'S STRIPS FOR DIRICHLET SERIES IN BANACH SPACES

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Dedicated to the memory of Susanne Dierolf

Abstract: Each Dirichlet series $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$, with variable $s \in \mathbb{C}$ and coefficients $a_n \in \mathbb{C}$, has a so called Bohr strip, the largest strip in $\mathbb{C}$ on which D converges absolutely but not uniformly. The classical Bohr-Bohnenblust-Hille theorem states that the width of the largest possible Bohr strip equals 1/2. Recently, this deep work of Bohr, Bohnenblust and Hille from the beginning of the last century was revisited by various authors. New methods from different fields of modern analysis (e.g. probability theory, number theory, functional and Fourier analysis) allow to improve the Bohr-Bohnenblust-Hille cycle of ideas, and to extend it to new settings, in particular to Dirichlet series which coefficients in Banach spaces. We survey on various aspects of these new developments.

Keywords: Dirichlet series, power series, polynomials, Banach spaces.

1. The Bohr-Bohnenblust-Hille Theorem

One of the most famous functions ever is the Riemann zeta function given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{j=1}^{\infty} (1 - p_j^{-s})^{-1}$$

where $\operatorname{Re} s > 1$ and $2 = p_1 < p_2 < \ldots < p_j < \ldots$ is the sequence of prime numbers. To study this function it is natural, as H. Bohr did in 1913, to consider a general setting including it: Dirichlet series. A Dirichlet series is a series of the form $\sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ with coefficients a_n in the complex plane $\mathbb{C}$ and s a complex variable.

Dirichlet series can be seen as a particular case of a more general object: generalized Dirichlet series, that is series of the form $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, where the a_n's are complex numbers, the λ_n's are positive numbers and s is a complex variable. The theory of ordinary Dirichlet series and of power series in one variable can be seen as

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subtheories of the theory of general Dirichlet series. Clearly, letting $\lambda_n = \log n$ we have ordinary Dirichlet series, but $\lambda_n = n$ and the change of variable $e^{-s} = z$ leads to power series $\sum_{n=1}^{\infty} a_n z^n$. It is well known that the maximal sets of convergence of power series are disks in $\mathbb{C}$ and that there is essentially no difference between convergence, uniform convergence and absolute convergence of power series.
The situation is rather different when we go to Dirichlet series. In this case we have that if a Dirichlet series $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ converges (absolutely) for some s_0 then it converges (absolutely) for all s with $\operatorname{Re} s > \operatorname{Re} s_0$, hence the maximal sets of convergence are half–planes in $\mathbb{C}$. This gives, for each Dirichlet series D, three abscissas $\sigma_c(D)$, $\sigma_u(D)$ and $\sigma_a(D)$ that define the biggest half-planes in which the series converges, converges uniformly or converges absolutely. The limit function f that a Dirichlet series $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ defines on the half-plane $\operatorname{Re} s > \sigma_c(D)$ turns out to be holomorphic, and Bohr's aim was to determine the absolute and the uniform convergence abscissas $\sigma_a(D)$ and $\sigma_u(D)$ in terms of analytic properties of f. In case of the uniform convergence abscissa Bohr succeeded, solving the so called uniform convergence problem. He proved that $\sigma_u(D)$ defines exactly the biggest half-plane such that f is bounded on each strictly smaller half-plane (in other terms, the uniform convergence abscissa coincides with the abscissa of boundedness).

The absolute convergence problem leads to the question of determining the value of

$$S := \sup\{\sigma_a(D) - \sigma_u(D)\}\,,$$

where the supremum is taken with respect to all possible Dirichlet series $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$. The letter S stands for strip. Clearly, S is the width of the largest possible strip in $\mathbb{C}$ on which any given Dirichlet series $\sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ converges uniformly but not absolutely (note that Bohr denotes the number S by T).

The deep idea that Bohr had was to consider a simple one to one correspondence between Dirichlet series and formal power series in infinitely many variables. The correspondence is done in the following way: let p be the sequence of prime numbers $2 = p_1 < p_2 < \ldots < p_k < \ldots$. By the fundamental theorem of arithmetic any integer $n > 1$ has a unique decomposition

$$n = p^{\alpha} := p_1^{\alpha_1} \cdots p_k^{\alpha_k}\, p_{k+1}^{0} \cdots\,,$$

where α is a multi-index in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (i.e. a sequence in $\mathbb{N}_0$ that eventually becomes zero; the set of multi-indices is denoted by $\mathbb{N}_0^{(\mathbb{N})}$). In an ingenious fashion Bohr associated to each Dirichlet series $\sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ a formal power series in infinitely many variables:

$$\sum_{n=1}^{\infty} a_n \frac{1}{n^s} \leftrightsquigarrow \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha, \qquad \text{where } a_n = c_\alpha \text{ whenever } n = p^\alpha\,.$$

In that moment infinite dimensional holomorphy was giving its first steps and it was not even clear what the proper definition for a holomorphic function in infinitely many variables was. Hilbert suggested using monomial power series

[29, page 64] and this is in fact the approach that Bohr took. Nevertheless, it became clear later that this was not the right approach. If we denote by $\mathbb{D}$ the open unit disk of $\mathbb{C}$, it is well known that every holomorphic function $f : \mathbb{D}^n \to \mathbb{C}$ in n complex variables has a monomial series expansion; more precisely, for every f in $H(\mathbb{D}^n)$ we have $f(z) = \sum_{\alpha\in\mathbb{N}_0^n} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha$ for all $z \in \mathbb{D}^n$. From this fact it is easily deduced that for every $\mathbb{C}$-valued holomorphic function f defined on the open unit ball B_{ℓ_∞} of ℓ_∞ (by definition a complex-Fréchet differentiable function f on B_{ℓ_∞} with values in $\mathbb{C}$) there is a unique family $(c_\alpha(f))_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$ of scalars such that for all finite sequences z in B_{ℓ_∞} we have

$$f(z) = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha(f) z^\alpha . \tag{1.1}$$

This power series is called the monomial series expansion of f, and for every n and every multi-index $\alpha \in \mathbb{N}_0^n$ we have

$$c_\alpha(f) = \frac{\partial^\alpha f(0)}{\alpha!} .$$

Now a question arises: For which other than finite sequences in B_{ℓ_∞} does this series converge (i.e. converge absolutely as a net) for every holomorphic f on B_{ℓ_∞}? In order to deal with this question the following number is defined:

$$M = \sup\Big\{1 \leqslant p \leqslant \infty \mid \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f) z^\alpha| < \infty, \text{ for all } f \in H_\infty(B_{\ell_\infty}) \text{ and all } z \in \ell_p \cap B_{\ell_\infty}\Big\},$$

where $H_\infty(B_{\ell_\infty})$ stands for the Banach algebra of all bounded holomorphic functions on the open unit ball of ℓ_∞ endowed with the supremum norm. Again we are using a different notation than that used by Bohr and what is commonly used; our M is what Bohr called S. The reason is to provide with a coherent notation for all the aspects of the theory. Then M is a sort of measure for the set

$$\operatorname{dom} H_\infty(B_{\ell_\infty}) = \bigcap_{f\in H_\infty(B_{\ell_\infty})} \Big\{z \in B_{\ell_\infty} \mid \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f) z^\alpha| < \infty\Big\},$$

the so called set of monomial convergence of $H_\infty(B_{\ell_\infty})$. Showing the connection between Dirichlet series and power series in infinitely many variables Bohr proved in [5, Satz IX] that

$$S = \frac{1}{M} \tag{1.2}$$

and $M \geqslant 2$, and hence as a consequence that $S \leqslant \frac{1}{2}$. But he was not able to decide whether $M = 2$ or even ∞. It was clear to him that a better understanding of infinite dimensional holomorphy was needed: *'um dies Problem zu erledigen, ist* *ein tieferes Eindringen in die Theorie der Potenzreihen unendlich vieler Variabeln nötig, als es mir in §3 gelungen ist'*[1] [5, page 446]. He then started to study power series, beginning with power series in one variable. Working in this direction he obtained his celebrated *power series theorem*, [7, Theorem]:

Theorem 1.1 (Bohr's power series theorem). *If* $\left|\sum_{k=0}^{\infty} c_k z^k\right| \leqslant 1$ *whenever* $|z| \leqslant 1$, *then* $\sum_{k=0}^{\infty} |c_k z^k| \leqslant 1$ *when* $z \in \mathbb{C}$ *and* $|z| \leqslant 1/3$. *That is*

$$\sup_{z \in \frac{1}{3}\mathbb{D}} \sum_{k=0}^{\infty} |c_k z^k| \leqslant \sup_{z \in \mathbb{D}} \Big|\sum_{k=0}^{\infty} c_k z^k\Big|.$$

Furthermore, the number $\frac{1}{3}$ *is the best possible.*

On the other hand, an example given by Toeplitz in [39, page 622] showed that $M \leqslant 4$ and hence $S \geqslant \frac{1}{4}$. The final solution to this problem was given in an ingenious article by Bohnenblust and Hille in 1931 [3, Section 5].

Theorem 1.2 (Bohr-Bohnenblust-Hille theorem).

$$S = \frac{1}{2}.$$

Bohr brought power series in infinitely many variables into complex analysis in order to study Dirichlet series. Then his ideas were more or less forgotten, but it is interesting to remark that a problem in infinite dimensional holomorphy renewed interest in Bohr's work: in the late seventies Boland and Dineen in [8, Corollary 12] showed that the monomials on a nice nuclear space E form an absolute basis for the space of holomorphic functions $H(E)$, and they used an extension of Bohr's power series Theorem to several variables by Dineen and Timoney to answer the question if the existence of an absolute monomial basis implies nuclearity.

2. Bohr's strips, vector-valued

Our aim in this paper is to deal with vector valued Dirichlet series, that is, series $\sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ with coefficients a_n in a complex Banach space X. Dirichlet series, power series and Fourier series are closely related via generalized Dirichlet series. Since the mid 1970's there is a fruitful study of vector valued Fourier analysis begun by Kwapień, Bourgain, Burkholder, Pisier ... This study gave birth, for example, to the theory of type and cotype that is now in the core of modern Banach space theory. Due to this symmetry we feel that it is worthy to perform a study on vector valued Dirichlet series.

Again, like in the scalar valued case, Dirichlet series converge on half-planes, hence for a given Dirichlet series $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ with coefficients a_n in a complex

Banach space X the following abscissas of convergence are defined

$$\sigma_c(D) := \inf \left\{ \sigma \,|\, \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \text{ is convergent on } [\mathrm{Re}\, s > \sigma] \right\}$$

$$\sigma_u(D) := \inf \left\{ \sigma \,|\, \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \text{ is uniformly convergent on } [\mathrm{Re}\, s > \sigma] \right\}$$

$$\sigma_a(D) := \inf \left\{ \sigma \,|\, \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \text{ is absolutely convergent on } [\mathrm{Re}\, s > \sigma] \right\},$$

where $[\mathrm{Re}\, s > \sigma] = \{s \in \mathbb{C} | \mathrm{Re}\, s > \sigma\}$ and $\sigma \in \mathbb{R}$. When the Dirichlet series is nowhere convergent these three abscissas will be $+\infty$. Clearly, we have

$$\sigma_c(D) \leqslant \sigma_u(D) \leqslant \sigma_a(D).$$

As in the scalar case for all three abscissas the following Hadamard type formulas can be proved. The formulas for $\sigma_c(D)$ and $\sigma_a(D)$ in the scalar case were proved by Bohr in his PhD. thesis [4, Theorem VII and Theorem VIII, page 16]. In [27] analog formulas for $\sigma_c(D)$ (Theorem 7, page 6) and $\sigma_a(D)$ (Theorem 8, page 8) are given for generalized Dirichlet series and are presented as basic facts.

Proposition 2.1. *Let $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ be a Dirichlet series with coefficients in a Banach space X such that $\sum_{n=1}^{\infty} a_n$ is divergent. Then*

$$\sigma_c(D) = \limsup_{N\to\infty} \frac{\log\left(\left\|\sum_{n=1}^{N} a_n\right\|_X\right)}{\log N}$$

$$\sigma_a(D) = \limsup_{N\to\infty} \frac{\log\left(\sum_{n=1}^{N} \|a_n\|_X\right)}{\log N}$$

$$\sigma_u(D) = \limsup_{N\to\infty} \frac{\log\left(\sup_{t\in\mathbb{R}} \|\sum_{n=1}^{N} a_n n^{it}\|_X\right)}{\log N}$$

As Bohr did in the scalar case, for each Dirichlet series $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ with coefficients in a Banach space X **the uniform convergence problem** is considered: *To determine the uniform convergence abscissa $\sigma_u(D)$ in terms of analytic properties of the limit function*

$$f : [\mathrm{Re}\, s > \sigma_c(D)] \to X$$

of D. The answer turns out to be the same as in the scalar case, in particular it is independent of the specific geometry of the given Banach space X (see [17, Lemma 6]).

Theorem 2.2. *Let $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ be a Dirichlet series with coefficients in a Banach space X. Then the abscissa $\sigma_u(D)$ of uniform convergence equals the abscissa of boundedness, i.e.*

$$\sigma_u(D) = \inf \sigma\,,$$

where the infimum is taken over all $\sigma_c(D) < \sigma \in \mathbb{R}$ such that the holomorphic function defined by D on the half-plane $[\operatorname{Re} s > \sigma_c(D)]$ is bounded on the smaller half-plane $[\operatorname{Re} s \geqslant \sigma]$.

The proof is involved and needs a careful inspection of the arguments given for the scalar case. The crucial step again is that by Bohr's vision – now in the vector-valued case – each Dirichlet series $\sum_{n=1}^\infty a_n \frac{1}{n^s}$ with coefficients in X corresponds to a power series $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$ in infinitely many variables with coefficients in X, and vice versa:

$$\sum_{n=1}^{\infty} a_n \frac{1}{n^s} \leftrightsquigarrow \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha \qquad \text{with } a_n = c_\alpha \ \text{ for } \ n = p^\alpha\,.$$

The following lemma is less vage – in the scalar case sometimes quoted as Bohr's trick (see [36, page 44]); the more general vector valued case stated here is a simple consequence of the Hahn-Banach Theorem.

Lemma 2.3. *Let X be a Banach space and $a_1, \dots, a_N \in X$. Then*

$$\sup_{t\in\mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{it} \Big\|_X = \sup_{x\in\mathbb{D}^N} \Big\| \sum_{\alpha\in\Lambda_N} a_{p^\alpha} x^\alpha \Big\|_X,$$

where $\Lambda_N = \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} : 1 \leqslant p^\alpha \leqslant N\}$.

This 'trick' has many nice implications in the theory of Dirichlet series. For example it provides a very easy way of proving the following inequality.

Proposition 2.4. *Let X be a Banach space, $a_1, \dots, a_N \in X$ and two real numbers $\sigma_0 \leqslant \sigma$, then*

$$\sup_{\operatorname{Re} s\geqslant\sigma} \Big\| \sum_{n=1}^{N} a_n \frac{1}{n^s} \Big\|_X \leqslant \sup_{t\in\mathbb{R}} \Big\| \sum_{n=1}^{N} a_n \frac{1}{n^{\sigma_0+it}} \Big\|_X\,. \tag{2.1}$$

Which in turn, together with Theorem 2.2, allows to prove that the study of the uniform convergence of a Dirichlet series on a half-plane is equivalent to the study of uniform convergence of such a Dirichlet series on a vertical line.

Corollary 2.5. *Let $\sum_{n=1}^\infty a_n \frac{1}{n^s}$ be a Dirichlet series with values in a Banach space X, such that the series converges uniformly on a vertical line of abscissa σ_0. Then the Dirichlet series converges uniformly on the half-plane $[\operatorname{Re} s > r]$ for all $r > \sigma_0$.*

Given a complex Banach space X, we say that a mapping $f : B_{\ell_\infty} \to X$ is holomorphic if it is complex-Fréchet differentiable. We will denote by $H_\infty(B_{\ell_\infty}, X)$ the Banach space of all bounded holomorphic mappings from the open unit ball of ℓ_∞ into X. Next theorem is an important relative of Theorem 2.2 (see [17, Corollary 2]) in terms of infinite dimensional holomorphy. More precisely, in terms of monomial series representations of holomorphic mappings.

Theorem 2.6. *For each Dirichlet series $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ in a Banach space X we have*

$$\sigma_u(D) = \inf \sigma ,$$

where the infimum is taken over all $\sigma \in \mathbb{R}$ such that there is some $f \in H_\infty(B_{\ell_\infty}, X)$ which has $\sum_\alpha \frac{a_{p^\alpha}}{p^{\sigma\alpha}} z^\alpha$ as its monomial series representation.

The proof of the above theorem is obtained by using Theorem 2.2 and Bohr's local trick (Lemma 2.3) through the following lemma.

Lemma 2.7. *Let $\mathcal{D}$ be a family of scalar Dirichlet series $\sum_{n=1}^{\infty} a_n^j \frac{1}{n^s}$, $j \in J$. If $\sigma_0 \in \mathbb{R}$ satisfies that*

$$\sup_{j \in J} \sup_{N \in \mathbb{N}} \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n^j \frac{1}{n^{\sigma_0 + it}} \right| < \infty ,$$

then $\left\{ \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \frac{a_{p^\alpha}^j}{p^{\sigma\alpha}} z^\alpha \,|\, j \in J \right\}$ are the monomial expansions of a bounded subset of $H_\infty(B_{\ell_\infty})$ for all $\sigma > \sigma_0$.

Let us now consider **the absolute convergence problem**: *To determine the maximal width of the strip in $\mathbb{C}$ on which a Dirichlet series D converges uniformly but not absolutely in X. More generally: Given a non-zero bounded operator $v : X \to Y$ between Banach spaces, determine the maximal width of the strip in $\mathbb{C}$ on which D converges uniformly in X but its image $vD = \sum_{n=1}^{\infty} v(a_n) \frac{1}{n^s}$ does not converge absolutely in Y.*

Definition 2.8. *Let $v : X \to Y$ be a non-zero bounded operator between Banach spaces. Then*

$$S(v) := \sup\{\sigma_a(vD) - \sigma_u(D)\} ,$$

where the supremum is taken over all Dirichlet series $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ in X; if $v = id_X$, then we write $S(X)$ for $S(\mathrm{id}_X)$.

The operator point of view may at first glance seem artificial, but we will try to convince the reader later that it is quite meaningful (see the comment given before Theorem 5.6). We always have

$$\frac{1}{2} \leqslant S(v) \leqslant 1,$$

where the lower estimate follows from the scalar case. A question we are concerned with is to determine $S(v)$ for concrete bounded operators v or classes of bounded operators between Banach spaces.

In view of Bohr's vision the following definition turns out to be a sort of the counterpart of $S(v)$ in terms of infinite dimensional holomorphy.

Definition 2.9. *Let $v : X \to Y$ be a non-zero bounded operator between Banach spaces. Then*

$$M(v) := \sup p ,$$

where the supremum is taken over all $1 \leqslant p \leqslant \infty$ such that

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} ||v(c_\alpha(f))z^\alpha||_Y < \infty\,,$$

for all $f \in H_\infty(B_{\ell_\infty}, X)$ and all $z \in \ell_p \cap B_{\ell_\infty}$.

Following Bohr's techniques from the scalar case we obtain an operator version of (1.2), namely that $M(v)$ is the inverse of $S(v)$. The proof follows closely the proof of [17, Theorem 3] and uses as a particular feature the prime number Theorem.

Theorem 2.10. *For each non-zero bounded operator $v : X \to Y$ between Banach spaces we have*

$$S(v) = \frac{1}{M(v)}.$$

Then, estimates for $M(v)$ always mean estimates for $S(v)$ and vice versa; in particular we have

$$1 \leqslant M(v) \leqslant 2\,.$$

For each Banach space X the function space

$$\mathcal{H}^\infty(X)$$

by definition consists of all bounded and holomorphic functions $f : [\operatorname{Re} s > 0] \to X$ which on some half plane $[\operatorname{Re} s > \sigma]$ with $\sigma \geqslant 0$ have a pointwise representation as the limit function of a Dirichlet series $\sum_{n=1}^\infty a_n \frac{1}{n^s}$. Following some of the ideas in [6] that lead to the proof of Theorem 2.2, it is possible to see that in this definition we can actually take $\sigma = 0$. Theorem 2.2 leads to the following reformulation of the definition of $S(v)$.

Theorem 2.11. *Let $v : X \longrightarrow Y$ be a non-zero bounded operator between Banach spaces. Then for each $f = \sum_{n=1}^\infty a_n \frac{1}{n^s} \in \mathcal{H}^\infty(X)$ and each $\varepsilon > 0$ we have*

$$\sum_{n=1}^\infty \|v(a_n)\|_Y \frac{1}{n^{S(v)+\varepsilon}} < \infty\,.$$

Moreover $S(v)$ is the best exponent with this property.

3. Bohr's strips, the Banach space case

When we go to the vector-valued setting the geometry of the space happens to have an important role to play through the notion of cotype. Let us recall the defintition, details and properties can be found in [25, Chapter 11] or [40].

Let $2 \leqslant q < \infty$, a Banach space X has cotype q if there exists a constant $\kappa > 0$ such that for every choice of finitely many vectors $x_1, \dots, x_n \in X$,

$$\left(\sum_{k=1}^{n} \|x_k\|^q \right)^{\frac{1}{q}} \leqslant \kappa \left(\int_0^1 \| \sum_{k=1}^{n} r_k(t) x_k \|^2 dt \right)^{\frac{1}{2}} ,$$

where r_k stands for the k-th Rademacher function on $[0,1]$. As usual we write

$$\operatorname{Cot}(X) := \inf\{2 \leqslant q \leqslant \infty \,|\, X \text{ has cotype } q\} .$$

The following result and its consequences form the main results from [17].

Theorem 3.1 ([17], Theorem 1). *For each Banach space X we have*

$$S(X) = 1 - \frac{1}{\operatorname{Cot}(X)}.$$

From Theorem 2.11 we immediately deduce the following first consequence.

Corollary 3.2. *Let X be Banach space. Then for each $\sum_{n=1}^{\infty} a_n \frac{1}{n^s} \in \mathcal{H}_\infty(X)$ and each $\varepsilon > 0$ we have*

$$\sum_{n=1}^{\infty} \|a_n\| \frac{1}{n^{1 - \frac{1}{\operatorname{Cot} X} + \varepsilon}} < \infty ,$$

and $1 - \frac{1}{\operatorname{Cot} X}$ is the best exponent with this property.

For concrete X the theorem leads to concrete formulas. To see an example recall that

$$\operatorname{Cot}(\ell_q) = \begin{cases} 2 & \text{if } 1 \leqslant q \leqslant 2 \\ q & \text{if } 2 \leqslant q \leqslant \infty, \end{cases}$$

hence we have

Corollary 3.3 ([17], Corollary 3). *For each $1 \leqslant q \leqslant \infty$*

$$S(\ell_q) = \begin{cases} 1/2, & \textit{if } 1 \leqslant q \leqslant 2 \\ 1 - 1/q, & \textit{if } 2 \leqslant q \leqslant \infty. \end{cases}$$

Note that the preceding corollary implies that for every $t \in [\frac{1}{2}, 1]$ there is a Banach space X for which $t = S(X)$. We also mention a result which reflects the two extreme cases $S(X) = 1$ and $S(X) = 1/2$.

Corollary 3.4 ([17], Corollary 5). *Let X be an infinite dimensional Banach space, then*

(1) $S(X) = 1$ *if and only if X contains all Banach spaces ℓ_∞^n uniformly.*
(2) $S(X) = 1/2$ *if and only if X has cotype $2 + \varepsilon$ for every $\varepsilon > 0$.*

Finally we see that as a consequence of Theorem 2.10 all results quoted in this section can be formulated in terms of $M(X)$, i.e. in terms of infinite dimensional holomorphy. In particular we have that

$$M(X) = \frac{\mathrm{Cot}(X)}{\mathrm{Cot}(X) - 1},$$

and more concretely

$$M(\ell_p) = \begin{cases} 2 & \text{if } 1 \leqslant p \leqslant 2 \\ \frac{p}{p-1} & \text{if } 2 \leqslant p . \end{cases}$$

4. Bohnenblust–Hille inequalities, vector-valued

The starting point for Bohnenblust and Hille was the following result of Littlewood [33, Theorem 1].

Theorem 4.1. *Let $L : \ell_\infty \times \ell_\infty \to \mathbb{C}$ be continuous and bilinear; then for every n*

$$\left(\sum_{i,j=1}^{n} |L(e_i, e_j)|^{4/3} \right)^{3/4} \leqslant \sqrt{2} \sup_{x,y \in B_{\ell_\infty}} |L(x,y)|.$$

Moreover, the exponent $4/3$ is optimal.

Bohnenblust and Hille needed an m-linear version of this result to solve the absolute convergence problem. They got it in [3, Theorem I].

Theorem 4.2. *For each fixed $m \in \mathbb{N}$ there exists a constant $C_m > 0$ such that for every continuous m-linear function $L : \ell_\infty \times \cdots \times \ell_\infty \to \mathbb{C}$ the following inequality holds for every n*

$$\left(\sum_{i_1,\dots,i_m=1}^{n} |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \\ \leqslant C_m \sup_{x_1,\dots,x_m \in B_{\ell_\infty}} |L(x_1, \dots, x_m)|. \tag{4.1}$$

Moreover, the exponent $\frac{2m}{m+1}$ is optimal.

From the original proof of Bohnenblust and Hille one gets $C_m \leqslant m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}$. This result was overlooked for long time and re-discovered in the totally different frame of tensor products by Kaijser [30, Corollary 1.6] using techniques developed by Davie [11]. The new proof, using tensor product techniques, improves the constant and shows that $C_m \leqslant 2^{\frac{m-1}{2}}$.
As a matter of fact, what Bohnenblust and Hille needed to solve the absolute convergence problem was an inequality of this kind for power series; this leads to m-homogeneous polynomials. A (continuous) m-homogeneous polynomial between

Banach spaces is a mapping $P : X \to Y$ such that there exists a (continuous) m-linear mapping $L : X \times \cdots \times X \to Y$ satisfying

$$P(x) = L(x, \ldots, x). \tag{4.2}$$

Unless otherwise stated, all polynomials and multilinear mappings will be assumed to be continuous. Given an m-homogeneous polynomial P, there exists a unique symmetric m-linear mapping L satisfying (4.2). There are many expressions describing L in terms of P, but the most usual one called *the polarization formula* is the following (see [26, Corollary 1.7]):

$$L(x_1, \ldots, x_m) = \frac{1}{2^n n!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \ldots \varepsilon_m P(\varepsilon_1 x_1 + \cdots + \varepsilon_m x_m)\,.$$

Given an m-linear mapping $L : \ell_\infty \times \cdots \times \ell_\infty \to \mathbb{C}$, it can be represented by a matrix $\big(L(e_{i_1}, \ldots, e_{i_m})\big)_{i_1,\ldots,i_m=1,\ldots,\infty}$. On the other hand, the associated polynomial P is a holomorphic mapping and has a monomial expansion like (1.1) in which all coefficients $c_\alpha(P)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_k \neq m$ are zero. The coefficients of the polynomial and those of the associated symmetric m-linear mapping are closely related. Following [13, Section 2] (see also [16, Section 3] or [22, Lemma 5]) we consider the following three index sets

$$\begin{aligned} \mathcal{M}(m,n) &= \{1, \ldots, n\}^m \\ \mathcal{J}(m,n) &= \{\boldsymbol{i} = (i_1, \ldots, i_m) \in \mathcal{M}(m,n) \, : \, i_1 \leqslant \ldots \leqslant i_m\} \\ \Lambda(m,n) &= \{\alpha \in \mathbb{N}_0^n \, : \, |\alpha| = m\}. \end{aligned}$$

In $\mathcal{M}(m,n)$ we define the following equivalence relation: $\boldsymbol{i} \sim \boldsymbol{j}$ if there is a permutation $\pi \in \Pi_m$ such that $i_k = j_{\pi(k)}$ for all $k = 1, \ldots, m$. Clearly, the equivalence class $[\boldsymbol{i}]$ of a given index has at most $|\Pi_m| = m!$ elements; also $\mathcal{M}(m,n) = \bigcup_{\boldsymbol{i} \in \mathcal{J}(m,n)} [\boldsymbol{i}]$. On the other hand there is a one-to-one correspondence between $\mathcal{J}(m,n)$ and $\Lambda(m,n)$ defined in the following terms. If $\boldsymbol{i} \in \mathcal{J}(m,n)$ there is an associated multi-index $\alpha_{\boldsymbol{i}}$ given by $\alpha_r = |\{k \colon i_k = r\}|$ (i.e. α_1 is the number of 1's in $\boldsymbol{i}$, α_2 is the number of 2's, and so on ...). If $\alpha \in \Lambda(m,n)$ then we define $\boldsymbol{i}_\alpha = (1, \overset{\alpha_1}{\ldots}, 1, 2, \overset{\alpha_2}{\ldots}, 2, \ldots, n \overset{\alpha_n}{\ldots}, n) \in \mathcal{J}(m,n)$. Note that $\mathrm{card}[\boldsymbol{i}_\alpha] = m!/\alpha!$. Now, if we denote $a_{i_1,\ldots,i_m} = L(e_{i_1}, \ldots, e_{i_m})$ we have

$$c_\alpha(P) = \mathrm{card}[\boldsymbol{i}_\alpha] \; a_{\boldsymbol{i}_\alpha}\,. \tag{4.3}$$

Indeed,

$$\begin{aligned} \sum_{\alpha \in \Lambda(m,n)} c_\alpha z^\alpha &= \sum_{\boldsymbol{i} \in \mathcal{M}(m,n)} a_{i_1,\ldots,i_m} z_{i_1} \cdots z_{i_m} \\ &= \sum_{\boldsymbol{i} \in \mathcal{J}(m,n)} \sum_{\boldsymbol{j} \in [\boldsymbol{i}]} a_{\boldsymbol{j}} z_{\boldsymbol{j}} = \sum_{\boldsymbol{i} \in \mathcal{J}(m,n)} \mathrm{card}[\boldsymbol{i}] a_{\boldsymbol{i}} z_{\boldsymbol{i}}\,. \end{aligned}$$

The key point to go from Theorem 4.2 to an inequality on m-homogeneous polynomials is the well known polarization inequality (see e.g. [26, Proposition 1.8]).

Proposition 4.3. *For each Banach space X and each $m \in \mathbb{N}$ there exists a constant $c(X,m)$ such that, for every m-linear mapping L defined on X with associated m-homogeneous polynomial P, the following holds*

$$\sup_{x \in B_X} |P(x)| \leqslant \sup_{x_1,\dots,x_m \in B_X} |L(x_1,\dots,x_m)| \leqslant c(X,m) \sup_{x \in B_X} |P(x)|. \tag{4.4}$$

Moreover, $c(X,m) \leqslant \frac{m^m}{m!}$ for every Banach space X. The best constant $c(X,m)$ is called the 'polarization constant'.

The following polynomial version of the Bohnenblust-Hille inequality is easily deduced from Theorem 4.2 using (4.3) and (4.4).

Theorem 4.4. *For each m there exists a constant $\kappa_m > 0$ such that for every continuous m-homogeneous polynomial $P : \ell_\infty \to \mathbb{C}$ we have*

$$\left(\sum_{|\alpha|=m} |c_\alpha(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leqslant \kappa_m \sup_{x \in B_{\ell_\infty}} |P(x)|. \tag{4.5}$$

This is in fact a general situation: every time that we have an inequality like (4.1) for m-linear mappings it is possible to obtain an inequality like (4.5) for m-homogeneous polynomials. Furthermore [22, Lemma 5] shows that if the exponent is optimal for m-linear mappings, then it is also optimal for m-homogeneous polynomials.

The constant κ_m depends on the constant C_m in Theorem 4.2 and on the polarization constant $c(X,m)$. Then, clearly, a way to get better constants in Theorem 4.4 is to find good constants in the other two inequalities. Bohnenblust and Hille's proof gives $C_m \leqslant m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}$, and their bound for κ_m is

$$\kappa_m \leqslant (m!)^{\frac{m-1}{2m}} m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}} \frac{m^m}{m!} = 2^{\frac{m-1}{2}} \frac{m^{m+\frac{m+1}{2m}}}{(m!)^{\frac{m+1}{2m}}}.$$

As we already mentioned, the tensor proof of Kaijser gives a better bound for C_m. This gives

$$\kappa_m \leqslant (m!)^{\frac{m-1}{2m}} 2^{\frac{m-1}{2}} \frac{m^m}{m!} = 2^{\frac{m-1}{2}} \frac{m^m}{(m!)^{\frac{m+1}{2m}}}.$$

This improvement comes from having a better bound for C_m; another way to improve the bounds for κ_m is by finding better polarization constants. It is well known (see [28]; see also [35, 41]) that $c(\ell_\infty, m) \leqslant \frac{m^{m/2}(m+1)^{(m+1)/2}}{2^m m!}$. With this we have a new improvement

$$\kappa_m \leqslant (m!)^{\frac{m-1}{2m}} 2^{\frac{m-1}{2}} \frac{m^{\frac{m}{2}} (m+1)^{\frac{m+1}{2}}}{2^m m!} = \left(\sqrt{2}\right)^{m-1} \frac{m^{\frac{m}{2}} (m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}.$$

The factor $\sqrt{2}$ comes from the constant in the Khintchine's inequality (see [25, Theorem 1.10] or [14, Section 8.5]) that plays a fundamental role in the proof of Theorem 4.2. Queffélec uses in [36, Theorem III-1] the Khintchine-type inequality for Steinhaus random variables due to Sawa [37, Theorem A]. He proves Theorem 4.4 using the theory of p-Sidon sets and gets

$$\kappa_m \leqslant \left(\frac{2}{\sqrt{\pi}}\right)^{m-1} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}.$$

This constant is better than the previous one since $\frac{2}{\sqrt{\pi}} < \sqrt{2}$.

A major breakthrough in this topic has been given by Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip in [15, Theorem 1]. Using a different approach they show that the inequality in Theorem 4.4 is hypercontractive, that is, there exists a universal constant $\kappa > 0$ such that

$$\kappa_m \leqslant \kappa^m.$$

Actually their result is the following theorem.

Theorem 4.5. *Given an m-homogeneous polynomial $P : \mathbb{C}^N \longrightarrow \mathbb{C}$ defined as $P = \sum_{|\alpha|=m} c_\alpha z^\alpha$, then*

$$\left(\sum_{|\alpha|=m} |c_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leqslant \left(1 + \frac{1}{m-1}\right)^{m-1} \sqrt{m}(\sqrt{2})^{m-1} \sup_{z \in \mathbb{D}^N} \left|\sum_{|\alpha|=m} c_\alpha z^\alpha\right|. \quad (4.6)$$

Let us recall that a bounded operator between Banach spaces $v : X \to Y$ is said to be $(r,1)$-summing (see, e.g. [25, Chapter 10] or [40, § 11]) if there is a constant $C > 0$ such that for any N and any choice $x_1, \ldots, x_N \in X$ we have

$$\left(\sum_{i=1}^N \|v(x_i)\|_Y^r\right)^{\frac{1}{r}} \leqslant C \sup_{\gamma \in B_{X^*}} \sum_{i=1}^N |\gamma(x_i)|$$

or, equivalently, such that for every N and every linear bounded operator $L : \ell_\infty^N \to X$ we have

$$\left(\sum_{i=1}^N \|v(L(e_i))\|_Y^r\right)^{\frac{1}{r}} \leqslant C \sup_{z \in B_{\ell_\infty}^N} \|L(z)\|_X.$$

We compose now with m-linear mappings instead of linear bounded operators and say that a bounded operator $v : X \to Y$ is called $(\rho, 1)$*-summing of order m* if there is a constant $C_m > 0$ such that for every N and every m-linear mapping $L : \ell_\infty^N \times \cdots \times \ell_\infty^N \to X$ we have

$$\left(\sum_{i_1,\ldots,i_m=1}^N \|v(L(e_{i_1}, \ldots, e_{i_m}))\|_Y^\rho\right)^{1/\rho} \leqslant C_m \sup_{z_1,\ldots,z_m \in B_{\ell_\infty}^N} \|L(z_1, \ldots, z_m)\|_X. \quad (4.7)$$

The m-th Bohnenblust-Hille index of v, $BH_m(v)$, is defined to be the infimum over all ρ so that v is $(\rho, 1)$-summing of order m.

With this notation Theorem 4.2 says that $\mathrm{id}_{\mathbb{C}}$ is $(\frac{2m}{m+1}, 1)$-summing of order m and $BH^m(\mathrm{id}_{\mathbb{C}}) = \frac{2m}{m+1}$.

Again, using polarization, it is easy to prove that if a bounded operator v is $(\rho, 1)$-summing of order m, then there exists a constant $\kappa_m > 0$ such that for every m-homogeneous polynomial $P : \ell_\infty^N \to X$ with coefficients c_α the following holds:

$$\left(\sum_{\alpha \in \Lambda(m,N)} \|v(c_\alpha)\|_Y^\rho \right)^{1/\rho} \leqslant \kappa_m \sup_{z \in B_{\ell_\infty}^N} \|P(z)\|_X \,, \tag{4.8}$$

Even more, by [22, Lemma 5] if the exponent in (4.7) is optimal, then so also is the exponent in (4.8).

Dealing with the convergence problem for vector-valued Dirichlet series requires inequalities like (4.8) and this leads to the problem of determining for which ρ an $(r, 1)$-summing bounded operator is $(\rho, 1)$-summing of order m (or, in other words, to compute $BH_m(v)$; see Theorem 5.4). It is not difficult to prove that if a Banach space has cotype q, then id_X is $(q, 1)$-summing (a deep result by Talagrand [38] shows that in fact these two things are equivalent for $q > 2$). The following result from [9, Theorem 3.2] that is used in the proof of [17, Lemma 4] shows that id_X is also $(q, 1)$-summing of order m for every m.

Theorem 4.6. *For every Banach space X wiht cotype q and every m there exists a constant $C > 0$ such that for every continuous m-linear mapping $L : \ell_\infty \times \cdots \times \ell_\infty \to X$ the following holds:*

$$\left(\sum_{i_1, \ldots, i_m = 1}^{\infty} \|L(e_{i_1}, \ldots, e_{i_m})\|^q \right)^{1/q} \leqslant C \sup_{z_1, \ldots, z_m \in B_{\ell_\infty}} \|L(z_1, \ldots, z_m)\|.$$

Moreover, the exponent q is optimal.

We see that the exponent in (4.7) does not change for the identity of a cotype q space when we move from order 1 to order m, however this is not the case in general. The celebrated Bennett-Carl inequalities, obtained independently in [2] and [10, Theorems 1 and 2] show that the optimal value of r for the inclusion $\ell_p \hookrightarrow \ell_q$ to be $(r, 1)$-summing, is given by the formula $\frac{1}{r} = \frac{1}{2} + \frac{1}{p} - \max(\frac{1}{q}, \frac{1}{2})$. It is natural to ask the same question for order m; in this case we are forced to change the exponent.

Theorem 4.7 ([22], Theorem 1). *Given $m \in \mathbb{N}$ and $1 \leqslant p \leqslant q \leqslant \infty$, define*

$$\rho = \begin{cases} \frac{2m}{m + 2(\frac{1}{p} - \max\{\frac{1}{q}, \frac{1}{2}\})} & \text{if } p \leqslant 2 \\ p & \text{if } p \geqslant 2 \end{cases}.$$

Then there exists a constant $C > 0$ such that for every continuous m-linear mapping $L : \ell_\infty \times \cdots \times \ell_\infty \longrightarrow \ell_p$ the following holds:

$$\left(\sum_{i_1,\dots,i_m=1}^{\infty} \|L(e_{i_1},\dots,e_{i_m})\|_q^{\rho}\right)^{\frac{1}{\rho}} \leqslant C \sup_{z_1,\dots,z_m \in B_{\ell_\infty}} \|L(z_1,\dots,z_m)\|_p.$$

Moreover, ρ is the best possible.

A famous result by Kwapień shows that every bounded, linear operator $v : \ell_1 \to \ell_q$ is $(r,1)$-summing for $\frac{1}{r} = 1 - |\frac{1}{p} - \frac{1}{2}|$ [32, (1.1)] (see also [25, p. 208]). The corresponding result for order m is taken from [22, Corollary 9] and [19, Theorem 6.1], and gives some sort of Grothendiek-like result.

Theorem 4.8. *For m and $1 \leqslant q < \infty$ define*

$$\rho = \begin{cases} \frac{2m}{m+2-\frac{2}{q}}, & 1 \leqslant q \leqslant 2 \\ \frac{2m}{\frac{2m}{q}+1}, & 2 \leqslant q \leqslant \dfrac{2m}{m-1} \\ 2, & \dfrac{2m}{m-1} \leqslant q \leqslant \infty \end{cases}.$$

Then there is a constant $C > 0$ (that depends only on m and q) such that for every bounded operator $v : \ell_1 \to \ell_q$ and every continuous m-linear mapping $L : \ell_\infty \times \cdots \times \ell_\infty \to \ell_1$ we have

$$\left(\sum_{i_1,\dots,i_m=1}^{\infty} \|v(L(e_{i_1},\dots,e_{i_m}))\|_q^{\rho}\right)^{\rho} \leqslant C \sup_{z_1,\dots,z_m \in B_{\ell_\infty}} \|L(z_1,\dots,z_m)\|_1.$$

The inequalities in the preceding three results are included as particular cases of the following ([19, Theorem 5.1 and Corollary 5.2]), once one knows that the identity of a cotype q space is $(q,1)$-summing (for Theorem 4.6), the Bennett-Carl inequalities and that ℓ_q has cotype $\max(2,q)$ (for Theorem 4.7) and Kwapień's result and the cotype of ℓ_p spaces for Theorem 4.8.

Theorem 4.9. *Let $m \in \mathbb{N}$, Y a Banach space with cotype q and $v : X \to Y$ an $(r,1)$-summing bounded operator (with $1 \leqslant r \leqslant q$). Define*

$$\rho = \frac{qrm}{q + (m-1)r}.$$

Then there is a constant $C_m > 0$ such that for every continuous m-linear mapping $L : \ell_\infty \times \cdots \times \ell_\infty \to X$ the following holds:

$$\left(\sum_{i_1,\dots,i_m=1}^{\infty} \|v\big(L(e_{i_1},\dots,e_{i_m})\big)\|_Y^{\rho}\right)^{1/\rho} \leqslant C_m \sup_{z_1,\dots,z_m \in B_{\ell_\infty}} \|L(z_1,\dots,z_m)\|_X.$$

Problem 1. Are the exponents in Theorem 4.8 and Theorem 4.9 optimal? This would require to find examples of concrete m-linear mappings. By [22, Lemma 5], this would also imply that the exponent is optimal for polynomial inequalities.

Polynomial versions of these results, like Theorem 4.4, come as an immediate consequence of them using the relation between the coefficients of the polynomials and the associated m-linear mappings shown in (4.3) and the polarization inequality (4.4). This was proved in [22, Lemma 5], where the non-trivial part is to show that if the exponent is optimal in the m-linear case, then it is also optimal for the polynomial case.

Again, like in the scalar case, finding good constants in the inequalities becomes an important issue. If ρ is the exponent in the inequalities, κ_m denotes the constant in the polynomial inequality and C_m the constant in the m-linear case, then we have in general that $\kappa_m \leqslant (m!)^{1-1/r} c(X,m) C_m$. In certain cases we have that the polynomial inequality is, like in the scalar case, hypercontractive [18, Theorem 4.3].

Theorem 4.10. *Let Y be a q-concave Banach lattice, with $2 \leqslant q < \infty$, and $v : X \to Y$ an $(r,1)$-summing bounded operator with $1 \leqslant r \leqslant q$. Define*

$$\rho := \frac{qrm}{q + (m-1)r}.$$

Then there is a constant $\kappa > 0$ such that for every m-homogeneous polynomial $P : \ell_\infty^N \to X$ the following holds

$$\left(\sum_{\alpha \in \Lambda(m,N)} \| v(c_\alpha) \|_Y^\rho \right)^{1/\rho} \leqslant \kappa^m \sup_{z \in B_{\ell_\infty}} \| P(z) \|_X .$$

5. Bohr's strips for polynomials, vector-valued

Bohr proved in [6, Satz XII] that if a Dirichlet series only ranges on the prime numbers, then the abscissas of uniform and absolute convergence coincide (i.e. $\sigma_u = \sigma_a$ whenever the series is of the form $\sum_{p \text{ prime}} a_p \frac{1}{p^s}$). This was the starting point of Bohnenblust and Hille that went on this idea, considering Dirichlet series that range on natural numbers whose decomposition in prime numbers has only a prefixed number of primes. More precisely, if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the decomposition in prime factors, we write $\Omega(n) = \alpha_1 + \cdots + \alpha_k$; note that $\Omega(n) = |\alpha|$ if $n = p^\alpha$. We now consider Dirichlet series of the form $\sum_{\Omega(n)=m} a_n \frac{1}{n^s}$. We will call such a Dirichlet series an *m-homogeneous Dirichlet series*. Then what Bohnenblust and Hille did was to compute the maximal width of the strips when only m-homogeneous Dirichlet series are considered. Although not explicitly they consider for each fixed m

$$S_m := \sup\{\sigma_a(D) - \sigma_u(D)\} ,$$

where the supremum is taken over all m-homogeneous Dirichlet series $D = \sum_{\Omega(n)=m} a_n \frac{1}{n^s}$, and show that

$$S_m = \frac{m-1}{2m} .$$

This immediately implies $\frac{1}{2} \leqslant S$ and completes the proof of the Bohr–Bohnenblust–Hille theorem.

Definition 5.1. *Let $v : X \to Y$ be a non-zero bounded operator between Banach spaces and $m \in \mathbb{N}$. Then*

$$S_m(v) := \inf\{\sigma_a(vD) - \sigma_u(D)\},$$

where the infimum is taken over all m-homogeneous Dirichlet series $D = \sum_{\Omega(n)=m} a_n \frac{1}{n^s}$ in X and vD denotes the m-homogeneous Dirichlet series $\sum_{\Omega(n)=m} v(a_n) \frac{1}{n^s}$ in Y.

Obviously, for each m we have $S_m(v) \leqslant S(v)$.

Problem 2. Prove or disprove that $S(v) = \sup_m S_m(v)$.

Definition 5.2. *Let $v : X \to Y$ be a non-zero bounded operator between Banach spaces and $m \in \mathbb{N}$. Then*

$$M_m(v) := \sup p ,$$

where the supremum is taken over all $1 \leqslant p \leqslant \infty$ such that

$$\sum_{|\alpha|=m} ||v(c_\alpha(P)) z^\alpha||_Y < \infty ,$$

for all $P \in \mathcal{P}(^m\ell_\infty, X)$ and all $z \in \ell_p$.

For each m we have $M(v) \leqslant M_m(v)$. Toeplitz explicitly considered these M_m (although he used A_n) and proved that $M_2 \leqslant 4$ (see [39, page 420]).

Problem 3. Prove or disprove that $M(v) = \inf_m M_m(v)$.

The following homogeneous analog of Theorem 2.10 was obtained in [24, Theorem 4.1] by reproducing the techniques of Bohr.

Theorem 5.3. *For each non-zero bounded operator $v : X \to Y$ between Banach spaces and each m we have*

$$S_m(v) = \frac{1}{M_m(v)}.$$

In view of Theorem 5.3 and as it happens in the scalar case, in order to get estimates for $S_m(v)$ we will need Bohnenblust-Hille type inequalities like that in Theorem 4.2. Usually the exponent in the inequality gives a lower bound for $M_m(v)$ (hence an upper bound for $S_m(v)$) whereas the upper bounds for $M_m(v)$ are obtained by the fact that the exponent is optimal. This leads to the Bohnenblust-Hille index. We recall that given $1 \leqslant p \leqslant \infty$, the number p' stands for the conjugate of p i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 5.4 ([23], Proposition 6.1). *For each non-zero bounded operator $v : X \to Y$ between Banach spaces and each m we have*

$$\frac{1}{BH_1(v)'} = S_1(v) \leqslant S_m(v) \leqslant \frac{1}{BH_m(v)'}.$$

It remains an interesting open problem whether we in fact have equality in the second inequality.

Problem 4. Prove or disprove that $S_m(v) = \dfrac{1}{BH_m(v)'}$ for all $m > 1$.

The following counterpart of Theorem 3.1 is shown in [17, page 554].

Theorem 5.5.

$$S_m(X) = \begin{cases} \frac{m-1}{2m}, & \textit{if } \dim X < \infty \\ 1 - \frac{1}{\mathrm{Cot}(X)}, & \textit{if } \dim X = \infty. \end{cases}$$

The previous result can be viewed as a computation of $S_m(v)$ for $v = \mathrm{id}_X$, and shows in particular that in infinite dimensions Bohr's strips do not distinguish between arbitrary Dirichlet series and m-homogeneous Dirichlet polynomials. A careful study of this phenomenon for ℓ_p-spaces motivates our operator point of view. The next result from [23, Theorem 1.1] shows that if we look at inclusions between different ℓ_p spaces, then we in fact "find the polynomials back".

Theorem 5.6. *For each m and $1 \leqslant p \leqslant q \leqslant \infty$*

$$S_m(\mathrm{id} : \ell_p \hookrightarrow \ell_q) = \begin{cases} \dfrac{m - 2(1/p - \max\{1/q, 1/2\})}{2m}, & \textit{if } 1 \leqslant p \leqslant 2 \\ 1 - \dfrac{1}{p}, & \textit{if } 2 \leqslant p. \end{cases}$$

Moreover, for every $0 \leqslant \sigma \leqslant S_m(\mathrm{id} : \ell_p \hookrightarrow \ell_q)$ there exists an m-homogeneous Dirichlet D series in ℓ_p for which $\sigma_a^{\ell_q}(D) - \sigma_u^{\ell_p}(D) = \sigma$, where $\sigma_u^{\ell_p}(D)$ is the abscissa of uniform convergence of D in ℓ_p and $\sigma_a^{\ell_q}(D)$ is the abscissa of absolute convergence of D now considered as a Dirichlet series with coefficients in ℓ_q.

Again, since $S_m(v) \leqslant S(v)$ we see immediately that

Corollary 5.7. *For $1 \leqslant p \leqslant q \leqslant \infty$*

$$S(\mathrm{id} : \ell_p \hookrightarrow \ell_q) = \begin{cases} 1/2, & \textit{if } 1 \leqslant p \leqslant 2 \\ 1 - 1/p, & \textit{if } 2 \leqslant p \leqslant \infty. \end{cases}$$

Getting results for 'any' bounded operator is of course much more complicated and can only be done in certain special circumstances. In [19, Theorem 6.1] and [22, Corollary 9] we find the following analog to a famous result of Kwapień [32, (1.1)] (see also [25, page 208]).

Theorem 5.8. *Let $v : \ell_1 \longrightarrow \ell_p$ be a non-zero bounded operator with $1 \leqslant p < \infty$ and $m \in \mathbb{N}$. Then*

$$S_m(v) \leqslant \begin{cases} \frac{m-2(1-\frac{1}{p})}{2m}, & \text{if } 1 \leqslant p \leqslant 2 \\ \frac{2m(1-\frac{1}{p})-1}{2m}, & \text{if } 2 \leqslant p \leqslant \dfrac{2m}{m-1} \\ \frac{1}{2}, & \text{if } \dfrac{2m}{m-1} \leqslant p \leqslant \infty. \end{cases}$$

Problem 5. Obtain lower bounds in Theorem 5.8. This would imply finding good examples.

We trivially have

Corollary 5.9. *For each $v : \ell_1 \longrightarrow \ell_p$ a non-zero bounded operator with $1 \leqslant p < \infty$*

$$S(v) = \frac{1}{2}$$

As has already been remarked, we have $\frac{1}{2} \leqslant S(v)$. The fact that $S(v) \leqslant \frac{1}{2}$ follows from $S(v) \leqslant S(\mathrm{id}_{\ell_1}) = \frac{1}{2}$ (use that v maps absolutely convergent series into absolutely convergent ones and Corollary 3.3).

6. Bohr's strips, graduation of the vector-valued case

Maurizi and Queffélec [34, Theorem 2.4] observed that the maximal width S of Bohr's strip equals the infimum over all $\sigma \geqslant 0$ for which there exists a constant $C_\sigma \geqslant 1$ such that for each choice of $a_1, \ldots, a_N \in \mathbb{C}$ we have

$$\sum_{n=1}^{N} |a_n| \leqslant C_\sigma \, N^\sigma \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{-it} \right| .$$

This motivates the following definition.

Definition 6.1. *Let $v : X \to Y$ be a non-zero bounded operator. Given a natural number N, the N-th Queffélec number $Q_N(v)$ of v is the best constant $C \geqslant 1$ such that for all choices of $a_1, \ldots, a_N \in X$*

$$\sum_{n=1}^{N} \|v(a_n)\|_Y \leqslant C \sup_{t \in \mathbb{R}} \left\| \sum_{n=1}^{N} a_n n^{-it} \right\|_X .$$

We abbreviate $Q_N(X) = Q_N(id_X)$ and $Q_N = Q_N(\mathbb{C})$.

In these terms the above characterization of Maurizi and Queffélec has a straightforward extension to the vector-valued case given in [20] which shows that the Queffélec numbers $Q_N(v)$ in a sense graduate Bohr's strips.

Proposition 6.2. *Let $v : X \to Y$ be a non-zero bounded operator. Then*

$$S(v) = \inf\{\sigma \geqslant 0 \,|\, \text{there exists } C_\sigma \text{ such that } Q_N(v) \leqslant C_\sigma N^\sigma \text{ for all } N\}.$$

If this result is combined with the Hadamard type formulas for the abscissas of uniform and absolute convergence given in Proposition 2.1, then we obtain as a corollary the following formula from [20] which once again reflects the fact that the Queffélec numbers Q_N graduate $S(v)$.

Corollary 6.3.

$$S(v) = \limsup_{N\to\infty} \frac{\log Q_N(v)}{\log N}.$$

In the scalar case the following theorem gives the asymptotically correct order of Q_N, and it marks the endpoint of a long development started by Queffélec [36] in the mid nineties, continued by Queffélec and Konyagin [31] in 2002 and by de la Bretéche [12] in 2008. The final result was proved in 2009 by Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip [15, Theorem 3]; its proof uses the hypercontractivity of the Bohnenblust-Hille inequality.

Theorem 6.4. *For the N-th Queffélec number we have*

$$Q_N = \frac{\sqrt{N}}{e^{\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log N \log\log N}}}.$$

Let us indicate that results of this type have important consequences on Bohr's strips themselves. In view of the fact that for each Dirichlet series $\sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ the abscissa $\sigma_u(D)$ of uniform convergence equals the abscissa of boundedness (i.e. the infimum of those σ such that the analytic function represented by the Dirichlet series is bounded in $[\operatorname{Re} s \geqslant \sigma]$). The Bohr-Bohnenblust-Hille Theorem shows that the Dirichlet series defining a function $f \in \mathcal{H}^\infty(\mathbb{C})$ converges absolutely on the vertical line $[\operatorname{Re} s = 1/2 + \varepsilon]$ for every $\varepsilon > 0$, and that the number $1/2$ here is optimal. A deep result of Balasubramanian, Calado and Queffélec [1, Theorem 1.1] shows that such a Dirichlet series even converges absolutely on the vertical line $[\operatorname{Re} s = 1/2]$. But the preceding theorem allows to say even more; it adds a level of precision that enables us to extract more precise information about the absolute values $|a_n|$ than what is obtained from the solution of Bohr's absolute convergence problem.

Corollary 6.5 ([15], Corollary 2). *The supremum of the set of real numbers c such that for every Dirichlet series $\sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ in $\mathcal{H}^\infty(\mathbb{C})$*

$$\sum_{n=1}^{\infty} |a_n|\, e^{c\sqrt{\log n \log\log n}} \frac{1}{n^{\frac{1}{2}}} < \infty,$$

equals $1/\sqrt{2}$.

What about the asymptotics of Queffélec number in the vector-valued case? Note first that we may deduce from the preceding theorem that for each non-zero bounded operator $v : X \to Y$ we have

$$\frac{\sqrt{N}}{e^{\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log N \log\log N}}} \leqslant Q_N(v) \leqslant N\,, \tag{6.1}$$

where the left side is attained for the case $v = \mathrm{id}_{\mathbb{C}}$ and the right side for $v = \mathrm{id}_{\ell_\infty}$.

The following result from [20] (see also [21, Theorem 6.6]) gives the asymptotic growth of $Q_N(X)$. It is a counterpart of Theorem 6.4, and shows that we have to carefully distinguish between the finite dimensional case and infinite dimensional case. The notation $a_N \prec b_N$ means that there exists a universal constant $B > 0$ such that $a_N \leqslant B b_N$ for all N and $a_N \asymp b_N$ means $a_N \prec b_N$ and $b_N \prec a_N$.

Theorem 6.6. *Let X be a Banach space. Then with constants depending only on X we have:*

(1) *For finite dimensional X*

$$Q_N(X) = \frac{\sqrt{N}}{e^{(\frac{1}{\sqrt{2}}+o(1))\sqrt{\log N \log\log N}}}\,;$$

(2) *For infinite dimensional X and any ε*

$$N^{1-\frac{1}{\mathrm{Cot}(X)}} \prec Q_N(X) \prec N^{1-\frac{1}{\mathrm{Cot}(X)+\varepsilon}}\,.$$

Problem 6. Is it possible to take $\varepsilon = 0$ in Theorem 6.6 provided that $\mathrm{Cot}(X)$ is attained?

For ℓ_p-spaces we have the following result from [20] (see also [21, Corollary 6.7]).

Corollary 6.7.

(1) $N^{\frac{1}{2}} \prec Q_N(\ell_p) \prec N^{1-\frac{1}{2+\varepsilon}}$ *if* $p < 2$.
(2) $N^{\frac{1}{p}} \asymp Q_N(\ell_p)$ *if* $p \geqslant 2$.

Problem 7. Does Corollary 6.7 hold for $\varepsilon = 0$ and $1 \leqslant p < 2$?

For the embedding $\mathrm{id} : \ell_p \hookrightarrow \ell_q$ in [20] (see also [21, Theorem 6.8]) the following upper estimates are proved.

Theorem 6.8. *Let $1 \leqslant p < q < \infty$. Then with constants depending only on p, q we have:*

$$Q_N(\mathrm{id} : \ell_p \hookrightarrow \ell_q) \prec \begin{cases} \dfrac{\sqrt{N}}{e^{\left(\sqrt{\frac{1}{p}-\frac{1}{q}}+o(1)\right)\sqrt{\log N \log\log N}}}, & \text{if } p < 2 \\ N^{1-\frac{1}{p}}, & \text{if } p \geqslant 2. \end{cases}$$

From (6.1) we conclude that this result for $p = 1$ and $q = 2$ is optimal.

Corollary 6.9.

$$Q_N(\mathrm{id} : \ell_1 \hookrightarrow \ell_2) = \frac{\sqrt{N}}{e^{\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log N \log\log N}}}.$$

Problem 8. Prove or disprove that the upper bounds in Theorem 6.8 are asymptotically optimal.

Again, like in Theorem 5.8, if we replace the embeddings $\mathrm{id} : \ell_1 \hookrightarrow \ell_q$ by arbitrary bounded operators $v : \ell_1 \to \ell_q$ we have

Theorem 6.10 ([20], also [21], Theorem 6.9). *Let $v : \ell_1 \to \ell_q$ be a non-zero bounded operator and $1 \leqslant q \leqslant 2$. Then*

$$Q_N(v) \leqslant \frac{\sqrt{N}}{e^{\left(\sqrt{1-\frac{1}{q}}+o(1)\right)\sqrt{\log N \log\log N}}}.$$

Problem 9. Find lower bounds in Theorem 6.10.

In this same spirit, but working with m-homogeneous Dirichlet series, [1, Theorem 1.4] and [34, Theorem 3.1] give

Theorem 6.11. *For every m-homogeneous Dirichlet series $\sum_{\Omega(n)=m} a_n \frac{1}{n^s} \in \mathcal{H}^\infty(\mathbb{C})$ we have*

$$\sum_{\Omega(n)=m} |a_n| (\log n)^{\frac{m-1}{2}} \frac{1}{n^{\frac{m-1}{2m}}} < \infty\,.$$

Moreover, the exponent $\frac{m-1}{2}$ in the log*-term is optimal.*

This motivates to introduce the m-homogeneous Queffélec number of a bounded operator $v : X \to Y$ (denoted by $Q_{N,m}(v)$) as the best constant $C \geqslant 1$ such that for every m-homogenous Dirichlet series $\sum_{\Omega(n)=m} a_n \frac{1}{n^s}$ we have that

$$\sum_{n=1}^{N} \|v(a_n)\|_Y \leqslant C \sup_{t\in\mathbb{R}} \left\| \sum_{n=1}^{N} a_n n^{-it} \right\|_X .$$

The study of such numbers in the vector-valued case is a work in progress; for more information see [20].

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