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THE PHRAGMÉN LINDELÖF CONDITION FOR EVOLUTION FOR QUADRATIC FORMS

Chiara Boiti, Reinhold Meise

Dedicated to the memory of our friend and colleague Susanne Dierolf.

Abstract: Let $P \in \mathbb{C}[\tau, \zeta_1, \dots, \zeta_n]$ be a quadratic polynomial for which the τ-variable is non-characteristic. We characterize when the zero-variety $V(P)$ of P satisfies the Phragmén-Lindelöf condition $\mathrm{PL}(\omega)$ or equivalently when the pair $(\mathbb{R}^n_x, \mathbb{R}_\tau \times \mathbb{R}^n_x)$ is of evolution in the class $\mathcal{E}_\omega$ for the partial differential operator $P(D)$ with symbol P.

Keywords: Phragmén-Lindelöf conditions, ultradifferentiable functions, differential equations of evolution

1. Introduction

Let $\omega : \mathbb{C}^k \times \mathbb{C}^n \to [0, \infty[$ be a weight function like $\omega(\tau, \zeta) := |\tau|^{\alpha_1} + |\zeta|^{\alpha_2}$ for $0 < \alpha_1, \alpha_2 < 1$. We say that an algebraic variety V in $\mathbb{C}^k \times \mathbb{C}^n$ satisfies the Phragmén-Lindelöf condition $\mathrm{PL}(\omega)$ of evolution if there exists $A > 0$ such that each plurisubharmonic function u on V which satisfies the estimates

$$u(\tau, \zeta) \leqslant |\operatorname{Im} \tau| + |\operatorname{Im} \zeta| + \omega(\tau, \zeta)$$
$$u(\tau, \zeta) \leqslant O(|\operatorname{Im} \zeta| + \omega(\tau, \zeta) + 1)$$

on V already satisfies

$$u(\tau, \zeta) \leqslant A(|\operatorname{Im} \zeta| + \omega(\tau, \zeta) + 1), \qquad (\tau, \zeta) \in V.$$

The significance of $\mathrm{PL}(\omega)$ for linear partial differential operators was shown by Boiti and Nacinovich in [4] and [5] and we refer to our paper [3] for a detailed discussion. The algebraic curves in $\mathbb{C}_\tau \times \mathbb{C}^n_\zeta$ which satisfy $\mathrm{PL}(\omega)$ were characterized in [2] in terms of their Puiseux series expansion.

The main aim of the present paper is to characterize the algebraic hypersurfaces $V(P) := \{(\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^n : P(\tau, \zeta) = 0\}$ in $\mathbb{C}_\tau \times \mathbb{C}^n_\zeta$ that satisfy $\mathrm{PL}(\omega)$ for quadratic polynomials P for which the τ-variable is non-characteristic. To achieve this characterization we first show that for a homogeneous non-constant

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polynomial $P \in \mathbb{C}[\tau, \zeta_1, \ldots, \zeta_n]$ for which the τ-variable is non-characteristic, its zero-variety $V(P)$ satisfies $\mathrm{PL}(\omega)$ if and only if P is hyperbolic for $N = (1, 0, \ldots, 0)$. Moreover, we show that for $P \in \mathbb{C}[\tau, \zeta_1, \ldots, \zeta_n]$ for which the τ-variable is non-characteristic, $V(P)$ satisfies $\mathrm{PL}(\omega)$ only if for the principal part P_m of P the variety $V(P_m)$ satisfies $\mathrm{PL}(\omega)$. The latter condition implies that, up to a complex constant factor, P_m has real coefficients. If P as above has degree 2 and satisfies $\mathrm{PL}(\omega)$ then it is therefore no restriction to assume that its principal part P_2 has real coefficients. This means that P has the form

$$P(\tau, \zeta) = \tau^2 + 2\tau l(\zeta) + Q(\zeta) + 2a\tau + L(\zeta) + C,$$

where l is a real linear form, Q is a real quadratic form, L is a complex linear form, and a, C are complex numbers. Using arguments from the proof of Meise, Taylor, and Vogt [9], Lemma 3, we then show in Lemma 14 that there exist a real linear form λ, $0 \leqslant m \leqslant n$, a complex linear form $\Lambda_0(z) = \sum_{j=m+1}^{n} l_j z_j$, $C_0 \in \mathbb{C}$ and, if $m \neq 0$, a quadratic form $D(z) := \sum_{j=1}^{m} d_j z_j^2$ with $d_j \neq 0$ for $1 \leqslant j \leqslant m$, such that for

$$P_0(\tau, z) := (\tau + \lambda(z))^2 + D(z) + \Lambda_0(z) + C_0$$

the variety $V(P)$ satisfies $\mathrm{PL}(\omega)$ if and only if $V(P_0)$ satisfies $\mathrm{PL}(\omega)$. The desired characterization is therefore contained in the following theorem.

Main Theorem 1. *Assume that P_0 is defined as above and let $\omega(\tau, \zeta) := \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ be a given weight function (see Definition 2 for σ_α). Then the following assertions are equivalent:*

(1) *$V(P_0)$ satisfies* $\mathrm{PL}(\omega)$.
(2) *$V(P_0)$ is hyperbolic for ω.*
(3) *D is negative semidefinite and one of the following conditions holds:*
  (3.a) $\Lambda_0 \equiv 0$.
  (3.b) *$\Lambda_0 \not\equiv 0$, there exists $\xi \in \{0\} \times \mathbb{R}^{n-m}$ such that $\Lambda_0(\xi) \neq 0$ and $\lambda(\xi) = 0$, and $\alpha_2 \geqslant 1/2$.*
  (3.c) *$\Lambda_0 \not\equiv 0$, for each $\xi \in \{0\} \times \mathbb{R}^{n-m}$ we have that $\lambda(\xi) \neq 0$ whenever $\Lambda_0(\xi) \neq 0$, and* $\max\{\alpha_1, \alpha_2\} \geqslant 1/2$.

2. Proof of the Main Theorem

Definition 2. *For $0 \leqslant \alpha < 1$, the weight function $\sigma_\alpha : \mathbb{R} \to [0, +\infty[$ is defined by*

$$\sigma_\alpha(t) = \begin{cases} |t|^\alpha & \text{if } 0 < \alpha < 1 \\ \log(1 + |t|) & \text{if } \alpha = 0. \end{cases} \tag{1}$$

We split $\mathbb{R}^N \simeq \mathbb{R}_t^k \times \mathbb{R}_x^n$, set $\theta = (\tau, \zeta) \in \mathbb{C}^k \times \mathbb{C}^n$ for the dual coordinates of $z = (t, x) \in \mathbb{R}^k \times \mathbb{R}^n$, and denote by ω the plurisubharmonic function

$$\omega(\tau, \zeta) = \omega_1(\tau) + \omega_2(\zeta) := \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|) \qquad \text{for } (\tau, \zeta) \in \mathbb{C}^k \times \mathbb{C}^n, \tag{2}$$

where $\alpha_j \in [0, 1[$ for $j = 1, 2$. Also ω will be called a weight function. Here and in the following we shall assume that $\alpha_2 = 0$ implies $\alpha_1 = 0$.

Definition 3. *For a weight function ω as in (2) we define $\mathcal{E}_\omega(\mathbb{R}^N)$ as $\mathcal{E}(\mathbb{R}^N)$ if $\alpha_1 = \alpha_2 = 0$, as*

$$\mathcal{E}_\omega(\mathbb{R}^N) := \{f \in \mathcal{E}(\mathbb{R}^N) : \forall K \subset\subset \mathbb{R}^N \ \forall \epsilon > 0 \ \forall \beta \in \mathbb{N}_0^k \ \exists c > 0 \ \forall \gamma \in \mathbb{N}_0^n : \sup_K |D_t^\beta D_x^\gamma f(t,x)| \leqslant c\epsilon^{|\gamma|}(\gamma!)^{1/\alpha_2}\}$$

if $\alpha_1 = 0$ and $\alpha_2 \neq 0$, and as

$$\mathcal{E}_\omega(\mathbb{R}^N) := \{f \in \mathcal{E}(\mathbb{R}^N) : \forall K \subset\subset \mathbb{R}^N \ \forall \epsilon > 0 \ \exists c > 0 \ \forall \beta \in \mathbb{N}_0^k, \ \gamma \in \mathbb{N}_0^n : \sup_K |D_t^\beta D_x^\gamma f(t,x)| \leqslant c\epsilon^{|\gamma|+|\beta|}(\beta!)^{1/\alpha_1}(\gamma!)^{1/\alpha_2}\}$$

if $\alpha_1 > 0$ and $\alpha_2 > 0$. Endowed with their natural locally convex topologies, these spaces are nuclear Fréchet spaces.

Definition 4. *Let V be an algebraic variety in $\mathbb{C}^N$. A function $u : V \to [-\infty, \infty[$ is called plurisubharmonic if it is locally bounded above, plurisubharmonic in the usual sense on V_{reg}, the set of all regular points of V, and satisfies*

$$u(z) = \limsup_{\zeta \in V_{\text{reg}}, \zeta \to z} u(\zeta)$$

at the singular points of V. By $\text{PSH}(V)$ we denote the set of all functions that are plurisubharmonic on V.

Definition 5. *Let V be an algebraic variety in $\mathbb{C}_\tau^k \times \mathbb{C}_\zeta^n$ and let ω be a weight function. We say that V satisfies $\text{PL}(\omega)$ if there exists $A > 0$ such that for each $u \in \text{PSH}(V)$ which for some $\alpha_u \geqslant 1$ satisfies*

(α) $u(\tau,\zeta) \leqslant |\operatorname{Im}\tau| + |\operatorname{Im}\zeta| + \omega_1(\tau) + \omega_2(\zeta)$, $(\tau,\zeta) \in V$
(β) $u(\tau,\zeta) \leqslant \alpha_u(|\operatorname{Im}\zeta| + \omega_1(\tau) + \omega_2(\zeta) + 1)$, $(\tau,\zeta) \in V$

also satisfies

(γ) $u(\tau,\zeta) \leqslant A(|\operatorname{Im}\zeta| + \omega_1(\tau) + \omega_2(\zeta) + 1)$, $(\tau,\zeta) \in V$.

Definition 6. *For V and ω as in Definition 5 we say that V is hyperbolic for ω if there exists $C > 0$ such that*

$$|\operatorname{Im}\tau| \leqslant C(|\operatorname{Im}\zeta| + \omega_1(\tau) + \omega_2(\zeta) + 1), \qquad (\tau,\zeta) \in V.$$

Remark 7. If an algebraic variety V in $\mathbb{C}_\tau^k \times \mathbb{C}_\zeta^n$ is hyperbolic for some weight function ω then V satisfies $\text{PL}(\omega)$. Note that the converse implication does not hold as Example 8 below shows.

Example 8. Let

$$V = \{(\tau,\zeta) \in \mathbb{C}^2 : \ \tau^2 = \zeta^3\}.$$

Since $\zeta = \tau^{2/3}$ we are in case (2) (iii) of Theorem 2 of [3] with $p = 2$, $q = 3$, and $G_2^q = 1$. Therefore, V satisfies $\text{PL}(\omega)$ for $\omega(\tau,\zeta) = \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ if and only if $\alpha_1 \geqslant 2/3$.

However, V is not hyperbolic for any weight function ω. Indeed, from $\tau = \zeta^{3/2}$ we get, for $\zeta_R = -R$:

$$|\operatorname{Im}\tau(\zeta_R)| = |\operatorname{Im}\sqrt{-R^3}| = R^{3/2}.$$

If we assume that V is hyperbolic for ω then there exists $C > 0$ such that

$$R^{3/2} = |\operatorname{Im}\tau(\zeta_R)| \leqslant C(|\operatorname{Im}\zeta_R| + \omega_1(\tau(\zeta_R)) + \omega_2(\zeta_R) + 1) = C(R^{\frac{3}{2}\alpha_1} + R^{\alpha_2} + 1),$$

which gives a contradiction for large R since $0 \leqslant \alpha_1, \alpha_2 < 1$ and proves our claim.

Definition 9. *Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be of degree $m \geqslant 1$ and let P_m be its principal part.*

(a) *P is said to be hyperbolic with respect to $N \in \mathbb{R}^n \setminus \{0\}$ if $P_m(N) \neq 0$ and if there exists $\tau_0 \in \mathbb{R}$ such that*

$$P(\xi + i\tau N) \neq 0 \text{ if } \xi \in \mathbb{R}^n \qquad \text{and} \qquad \tau < \tau_0.$$

(b) *P is said to be σ_α-hyperbolic with respect to $N \in \mathbb{R}^n \setminus \{0\}$ for $0 \leqslant \alpha < 1$ if $P_m(N) \neq 0$ and if the differential operator $P(D) := P(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n})$ admits a fundamental solution $E \in \mathcal{D}'_{\sigma_\alpha}(\mathbb{R}^n)$ that has its support in the closed half space $\{x \in \mathbb{R}^n : \langle x, N\rangle \geqslant 0\}$.*

Note that by well-known results σ_0-hyperbolicity is equivalent to hyperbolicity.

Proposition 10. *Let $P \in \mathbb{C}[\tau, \zeta_1, \dots, \zeta_n]$ be homogeneous of degree $m \geqslant 1$ with $P(1, 0, \dots, 0) \neq 0$. Then for*

$$V := \{(\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^n : \ P(\tau, \zeta) = 0\}$$

the following are equivalent:

(a) *V satisfies $\mathrm{PL}(\omega)$ for some/all weight functions ω.*
(b) *For each $\xi \in \mathbb{R}^n$ the polynomial $\tau \mapsto P(\tau, \xi)$ has only real roots.*
(c) *There exists $c > 0$ such that $|\operatorname{Im}\tau| \leqslant c|\operatorname{Im}\zeta|$ for all $(\tau, \zeta) \in V$.*
(d) *P is hyperbolic with respect to $N = (1, 0, \dots, 0)$.*

Proof. *(a)* $\Rightarrow$ *(b)*: Assume that V satisfies $\mathrm{PL}(\omega)$ for $\omega(\tau, \zeta) = \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$. If we assume that *(b)* does not hold, then there exists $(\tau^0, \xi^0) \in V \cap (\mathbb{C} \times \mathbb{R}^n)$ with $\operatorname{Im}\tau^0 \neq 0$. Take $\zeta_R = R\xi^0$. By homogeneity $\theta_R = R(\tau^0, \xi^0) = (R\tau^0, \zeta_R) \in V$. Then fix μ with $\max\{\alpha_1, \alpha_2\} < \mu < 1$, let W_R denote the connected component of $V \cap (\mathbb{C} \times B(\zeta_R, R^\mu))$, and define $u : \ V \to \mathbb{R}$ by

$$u(\tau, \zeta) = \begin{cases} \max\left\{\dfrac{R^\mu}{2} + R^\mu H\left(\dfrac{\zeta - \zeta_R}{R^\mu}\right), |\operatorname{Im}\zeta|\right\} & (\tau, \zeta) \in W_R, \ |\zeta - \zeta_R| \leqslant R^\mu \\ |\operatorname{Im}\zeta| & \text{otherwise,} \end{cases} \tag{3}$$

where $H(\theta) := (|\operatorname{Im}\theta|^2 - |\operatorname{Re}\theta|^2)/2$ is a harmonic function on $\mathbb{C}^n$ whose properties are described in Lemma 2.9 of [8]. Next we claim that there exist $\delta > 0$ and $R_0 > 1$ such that for each $(\tau, \zeta) \in W_R$ we have

$$|\operatorname{Im}\tau| \geqslant \delta R \geqslant \frac{R^\mu}{2} \qquad \text{for } R \geqslant R_0. \tag{4}$$

To prove this claim, note first that there is a homogeneous algebraic variety B in $\mathbb{C}^n$ such that the map $\pi : (\tau, \zeta) \mapsto \zeta$ on V is unbranched over $\mathbb{C}^n \setminus B$. Since $\mathbb{R}^n \setminus B$ is open and dense in $\mathbb{R}^n$, we may assume that we have chosen (τ^0, ξ^0) in such a way that it is a regular point of V and that there is a holomorphic map $\varphi : B(\xi^0, \epsilon) \to \mathbb{C}$ such that $\{(\varphi(\zeta), \zeta) : \zeta \in B(\xi^0, \epsilon)\}$ parametrizes a neighborhood of (τ^0, ξ^0). Moreoever, we may choose $\epsilon > 0$ so small that $|\operatorname{Im}\varphi(\zeta)| \geqslant |\operatorname{Im}\tau^0|/2$ for $\zeta \in B(\xi^0, \epsilon)$.

Now note that for $(\tau, \zeta) \in W_R$ we have $\zeta = \zeta_R + h$, $|h| < R^\mu$ and

$$0 = P(\tau, \zeta_R + h) = R^m P\Big(\frac{\tau}{R}, \xi^0 + \frac{h}{R}\Big).$$

Because of $\mu < 1$ there exists $R_0 > 1$ such that $R^\mu/R < \epsilon$ for $R \geqslant R_0$ and hence $\xi^0 + h/R \in B(\xi^0, \epsilon)$. This implies $\tau/R = \varphi(\xi^0 + h/R)$ and consequently $|\operatorname{Im}\tau/R| \geqslant |\operatorname{Im}\tau^0|/2$. Thus we proved the estimate (4) with $\delta := |\operatorname{Im}\tau^0|/2$.

Therefore, u satisfies (α) and (β) of $\mathrm{PL}(\omega)$ and hence from (γ) at θ_R:

$$\begin{aligned}\frac{R^\mu}{2} \leqslant u(\theta_R) &\leqslant A(\omega_1(R\tau^0) + \omega_2(R\xi^0) + 1) \\ &= A(R^{\alpha_1}|\tau^0|^{\alpha_1} + R^{\alpha_2}|\xi^0|^{\alpha_2} + 1)\end{aligned}$$

which gives a contradiction for large R since $\mu > \max\{\alpha_1, \alpha_2\}$.

$(b) \Rightarrow (c)$: Apply the classical Phragmén-Lindelöf theorem for $\mathbb{C}^n$ to

$$u(\zeta) := \max\{|\operatorname{Im}\tau| : (\tau, \zeta) \in V\}.$$

$(c) \Rightarrow (a)$: Obvious.

$(d) \Leftrightarrow (b)$: This holds by Hörmander [6], Theorem 5.5.3. ■

The following corollary is an immediate consequence of Proposition 10 and Hörmander [6], Corollary 5.5.1.

Corollary 11. *Let $P \in \mathbb{C}[\tau, \zeta_1, \ldots, \zeta_n]$ be homogeneous of degree $m \geqslant 1$ and assume that $P(1, 0, \ldots, 0) \in \mathbb{R} \setminus \{0\}$. If $V(P)$ satisfies* $\mathrm{PL}(\omega)$ *for some weight function ω, then P has real coefficients.*

Proposition 12. *Let $P \in \mathbb{C}[\tau, \zeta_1, \ldots, \zeta_n]$ be of degree $m \geqslant 1$, denote by P_m its principal part and assume that $P_m(1, 0, \ldots, 0) \neq 0$. If $V(P)$ satisfies* $\mathrm{PL}(\omega)$ *for some weight function ω then also $V(P_m)$ satisfies* $\mathrm{PL}(\omega)$.

Proof. To argue by contradiction, we assume that P satisfies PL(ω) for the weight function $\omega(\tau,\zeta)=\sigma_{\alpha_1}(|\tau|)+\sigma_{\alpha_2}(|\zeta|)$ and that $V(P_m)$ does not satisfy PL(ω). Then Proposition 10 implies the existence of $\xi\in\mathbb{R}^m$ such that not all the zeros of the polynomial $\tau\mapsto P_m(\tau,\xi)$ are real. As we indicated in the proof of Proposition 10 we can therefore assume the existence of a regular point (τ_0,ξ_0) in $V(P_m)$ with $\xi_0\in\mathbb{R}^m$ and $\tau_0\notin\mathbb{R}$. Next let $\Gamma(\xi_0,\delta):=\bigcup_{t>0}t(\xi_0+B(0,\delta))$. Since P_m is homogeneous and satisfies $P_m(1,0,\ldots,0)\neq 0$ by hypothesis, we can choose $\delta>0$ and a holomorphic function $\varphi:\Gamma(\xi_0,\delta)\to\mathbb{C}$ such that $\{(\varphi(\zeta),\zeta):\ \zeta\in\Gamma(\xi_0,\delta)\}$ is the connected component of $V(P_m)\cap\{(\tau,\zeta):\zeta\in\Gamma(\xi_0,\delta)\}$ which contains (τ_0,ξ_0). Since $\operatorname{Im}\tau_0\neq 0$, we can choose δ so small that

$$|\operatorname{Im}\varphi(\xi_0+h)|\geqslant|\operatorname{Im}\varphi(\xi_0)|/2=|\operatorname{Im}\tau_0|/2,\qquad h\in B(0,\delta).$$

Then fix $\mu<1$ satisfying $\mu>\max\{\alpha_1,\alpha_2\}$. We claim that there exists $R_1>1$, and $0<\epsilon<\frac{|\operatorname{Im}\tau_0|}{4}$ such that for $R\geqslant R_1$ and $h\in\mathbb{C}^n$ with $|h|<R^\mu$ the polynomials

$$q_{R,h}:\tau\mapsto P_m(\tau,R\xi_0+h)\qquad\text{and}\qquad p_{R,h}:\tau\mapsto P(\tau,R\xi_0+h)$$

have the same number of zeros in the disk $B^1(\varphi(R\xi_0+h),\epsilon R)$.

To prove this claim, using the Theorem of Rouché, we note first that because of $\mu<1$ there exists $R_0>1$ such that $R\xi_0+h\in\Gamma(\xi_0,\delta)$ for $R\geqslant R_0$ and each $h\in\mathbb{C}^n$ satisfying $|h|<R^\mu$. Next note that $P_m(1,0,\ldots,0)\neq 0$ implies the existence of $C>0$ such that

$$|\tau|\leqslant C|\zeta|\qquad\text{for }\ (\tau,\zeta)\in V(P_m).\tag{5}$$

From this estimate it follows that we can choose R_0 even so large that for $C_1:=2R(C+1)|\xi_0|$ we have

$$|(\varphi(R\xi_0+h),R\xi_0+h)|=R|(\varphi(\xi_0+\frac{h}{R}),\xi_0+\frac{h}{R})|\leqslant R(C+1)|\xi_0+\frac{h}{R}|\leqslant C_1R.\tag{6}$$

Next let $P(\tau,\zeta)=\sum_{j=0}^m P_j(\tau,\zeta)$, where P_j is either homogeneous of degree j or $P_j\equiv 0$. Hence there exists $M\geqslant 1$ such that for (τ,ζ) with $|(\tau,\zeta)|\geqslant R_0$ we have

$$|P(\tau,\zeta)-P_m(\tau,\zeta)|\leqslant\sum_{j=0}^{m-1}|P_j(\tau,\zeta)|\leqslant M|(\tau,\zeta)|^{m-1}.$$

Now fix $0<\epsilon<1$ and $\lambda\in\mathbb{C}$ with $|\lambda|=\epsilon R$ and note that this estimate together with (6) implies

$$|p_{R,h}(\varphi(R\xi_0+h)+\lambda)-q_{R,h}(\varphi(R\xi_0+h)+\lambda)|\leqslant M(C_1+\epsilon)^{m-1}R^{m-1}.\tag{7}$$

To derive an estimate for $q_{R,h}(\varphi(R\xi_0+h)+\lambda)$ from below, note that for $(\tau,\zeta)\in V(P_m)$ we have the expansion

$$P_m(\tau+\lambda,\zeta)=\sum_{j=1}^m\frac{1}{j!}\frac{\partial^j P_m}{\partial\tau^j}(\tau,\zeta)\lambda^j.$$

Because of $m \geqslant 1$, there exists k with $1 \leqslant k \leqslant m$ such that

$$\frac{\partial^j P_m}{\partial \tau^j}(\tau_0, \xi_0) = 0, \quad 0 \leqslant j < k \qquad \text{and} \qquad \frac{\partial^k P_m}{\partial \tau^k}(\tau_0, \xi_0) \neq 0.$$

Since $\frac{\partial^j P_m}{\partial \tau^j}$ is homogeneous of degree $m - j$, this implies that for $\sigma > 0$, small enough, there exists $\delta_1 = \delta_1(\sigma)$ such that

$$\left| \frac{\partial^j P_m}{\partial \tau^j}(\tau, \zeta) \right| \leqslant \sigma |(\tau, \zeta)|^{m-j}, \qquad (\tau, \zeta) \in \Gamma((\tau_0, \xi_0), \delta_1),\ 0 \leqslant j < k.$$

Next let $\eta := \left| \frac{\partial^k P_m}{\partial \tau^k}(\tau_0, \xi_0) \right| / 2$. Then we can choose $\delta_2 > 0$ such that

$$\left| \frac{\partial^k P_m}{\partial \tau^k}(\tau, \zeta) \right| \geqslant \eta |\zeta|^{m-k}, \qquad (\tau, \zeta) \in \Gamma((\tau_0, \xi_0), \delta_2).$$

Then there exists $D \geqslant 1$, $C_2 > 0$, and $R_1 \geqslant R_0$ such that for $R \geqslant R_1$:

$$\begin{aligned} |q_{R,h}(\varphi(R\xi_0 + h) + \lambda)| &= |P_m(\varphi(R\xi_0 + h) + \lambda, R\xi_0 + h)| \\ &\geqslant \frac{1}{k!} \left| \frac{\partial^k P_m}{\partial \tau^k}(\varphi(R\xi_0 + h) + \lambda, R\xi_0 + h)\lambda^k \right| \\ &\quad - \sum_{j=1, j\neq k}^{m} \frac{1}{j!} \left| \frac{\partial^j P_k}{\partial \tau^j}(\varphi(R\xi_0 + h) + \lambda, R\xi_0 + h) \right| |\lambda|^j \\ &\geqslant \frac{1}{k!} \eta |R\xi_0 + h|^{m-k} \epsilon^k R^k \\ &\quad - \sum_{j=1}^{k-1} \frac{1}{j!} \sigma (C_1 R)^{m-j} \epsilon^j R^j - D R^m \epsilon^{k+1} \\ &\geqslant R^m \left(\frac{\eta}{2k!} |\xi_0|^{m-k} \epsilon^k - \sigma C_2 - D\epsilon^{k+1} \right). \end{aligned}$$

Now we choose $0 < \epsilon < \min\left(\frac{\eta}{8Dk!} |\xi_0|^{m-k}, \frac{|\operatorname{Im} \tau_0|}{4} \right)$ and $\sigma < \frac{\eta}{8C_2 k!} |\xi_0|^{m-k} \epsilon^k$. Then the estimate above implies that, if we choose R_1 large enough, we get that for $R \geqslant R_1$

$$|q_{R,h}(\varphi(R\xi_0 + h) + \lambda)| \geqslant \frac{\eta}{4k!} |\xi_0|^{m-k} \epsilon^k R^m > M(C_1 + 1)^{m-1} R^{m-1}.$$

From this estimate and (7) it follows that we can apply the Theorem of Rouché, to see that our claim is true.

Next choose $\delta_3 > 0$ so small that for $\zeta \in \Gamma(\xi_0, \delta_3)$ we have $(\varphi(\zeta), \zeta) \in \Gamma((\tau_0, \xi_0), \min(\delta_1(\sigma), \delta_2))$. After enlarging R_1 if necessary, we now get that for $R \geqslant R_1$ and $h \in \mathbb{C}^n$ with $|h| < R^\mu$ each point $(\tau(R\xi_0 + h), R\xi_0 + h)$ in $V(P)$ which is close to $V(P_m) \cap \{(\varphi(\zeta), \zeta) : \zeta \in \Gamma(\xi_0, \delta_3)\}$ satisfies $|\tau(R\xi_0 + h) - \varphi(R\xi_0 + h)| < \epsilon R$, provided that we have chosen $\epsilon > 0$ so small that each zero (τ, ζ) of P_m which satisfies $\zeta \in \Gamma(\xi_0, \delta_3)$ and $\tau \neq \varphi(\zeta)$ already satisfies $|\tau - \varphi(\zeta)| \geqslant 8\epsilon|\zeta|$. Then our choice of ϵ implies for $R \geqslant R_1$ and $|h| < R^\mu$:

$$|\operatorname{Im} \tau(R\xi_0 + h)| \geqslant |\operatorname{Im} \varphi(R\xi_0 + h)| - |\operatorname{Im}(\tau(R\xi_0 + h) - \varphi(R\xi_0 + h))|$$
$$\geqslant \frac{|\operatorname{Im} \tau_0|}{2} R - \epsilon R \geqslant \frac{|\operatorname{Im} \tau_0|}{4} R.$$

Using this estimate we can now argue as in the proof of Proposition 10 to see that $V(P)$ does not satisfy $\mathrm{PL}(\omega)$. ■

Remark 13. From now on we will concentrate on quadratic polynomials $P \in \mathbb{C}[\tau, \zeta_1, \ldots, \zeta_n]$. If the τ-variable is non-characteristic for such a polynomial, i.e., if $P_2(1, 0, \ldots, 0) \neq 0$ holds for the principal part P_2 of P, then there exist complex linear forms l and L in ζ, a quadratic form Q in ζ, and complex numbers a, C such that, up to a complex constant factor, P has the following form

$$P(\tau, \zeta) = \tau^2 + 2\tau l(\zeta) + Q(\zeta) + a\tau + L(\zeta) + C. \tag{8}$$

If $V(P)$ satisfies $\mathrm{PL}(\omega)$ then $V(P_2)$ satisfies $\mathrm{PL}(\omega)$ by Proposition 12, where

$$P_2(\tau, \zeta) = \tau^2 + 2\tau l(\zeta) + Q(\zeta)$$

has real coefficients by Proposition 10 and Corollary 11. To characterize those P for which $V(P)$ satisfies $\mathrm{PL}(\omega)$ we can therefore restrict our attention to polynomials P of the form (8) for which l is a real linear form and Q is a real quadratic form.

Note also that by Proposition 10 for each $\xi \in \mathbb{R}$ the polynomial $p_\xi : \tau \mapsto P_2(\tau, \xi)$ has only real zeros. It is easy to check that p_ξ has the zeros $\tau_\pm = -l(\xi) \pm (l(\xi)^2 - Q(\xi))^{1/2}$. Since l and Q are real for real ξ, this implies that the quadratic form $Q_l : \zeta \mapsto l(\zeta)^2 - Q(\zeta)$ is positive semidefinite.

Lemma 14. *Let $P \in \mathbb{C}[\tau, \zeta_1, \ldots, \zeta_n]$ be of the form*

$$P(\tau, \zeta) = \tau^2 + 2\tau l(\zeta) + Q(\zeta) + 2a\tau + L(\zeta) + C,$$

where l is a real linear form, Q a real quadratic form, L a complex linear form, and a, C are complex numbers. Then there exist a real linear form λ, $0 \leqslant m \leqslant n$, a complex linear form $\Lambda_0(z) = \sum_{j=m+1}^n l_j z_j$, $C_0 \in \mathbb{C}$ and, if $m \neq 0$, a real quadratic form $D(z) = \sum_{j=1}^m d_j z_j^2$ with $d_j \neq 0$ for $1 \leqslant j \leqslant m$, such that for

$$P_0(\tau, z) := (\tau + \lambda(z))^2 + D(z) + \Lambda_0(z) + C_0$$

the following holds: $V(P)$ satisfies $\mathrm{PL}(\omega)$ for a weight function ω if and only if $V(P_0)$ satisfies $\mathrm{PL}(\omega)$.

Proof. If we let $Q_l(\zeta) := Q(\zeta) - l(\zeta)^2$, $L_l(\zeta) := L(\zeta) - 2al(\zeta)$ and $C_1 := C - a^2$ then we have

$$P(\tau, \zeta) = (\tau + l(\zeta) + a)^2 + Q_l(\zeta) + L_l(\zeta) + C_1.$$

If the quadratic form $Q_l \equiv 0$ then we let $D := 0$, $m := 0$ and $A := \mathrm{id}_{\mathbb{C}^n}$. Otherwise we can choose $A \in GL(\mathbb{R}^n)$ and D as in the statement such that

$$Q_l(\zeta) = D(A\zeta).$$

Note that m is the number of non-zero eigenvalues of the real symmetric matrix which defines Q_l. Next define $\lambda(z) := l(A^{-1}z)$, $\Lambda_1(z) := L_l(A^{-1}z)$ and

$$P_1(\tau, z) := (\tau + \lambda(z) + a)^2 + D(z) + \Lambda_1(z) + C_1.$$

Then we have

$$P(\tau, \zeta) = (\tau + \lambda(A\zeta) + a)^2 + D(A\zeta) + \Lambda_1(A\zeta) + C_1 = P_1(\tau, A\zeta).$$

As in the proof of Meise, Taylor, and Vogt [9], Lemma 3, we can find $b = (b_1, \dots, b_n)$ with $b_j = 0$ for $m + 1 \leqslant j \leqslant n$, such that

$$D(z + b) + \Lambda_1(z + b) + C_1 = D(z) + \Lambda_0(z) + C_0,$$

where Λ_0 is defined as in the assertion of the Lemma. Hence we have

$$\begin{aligned} P(\tau, \zeta + A^{-1}b) &= (\tau + \lambda(A\zeta + b) + a)^2 + D(A\zeta + b) + \Lambda_1(A\zeta + b) + C_1 \\ &= (\tau + \lambda(A\zeta) + d)^2 + D(A\zeta) + \Lambda_0(A\zeta) + C_0. \end{aligned}$$

If we now define P_0 as in the statement of the Lemma, then we have

$$P(\tau - d, \zeta + A^{-1}(b)) = P_0(\tau, A\zeta).$$

Next we note that $\mathrm{PL}(\omega)$ is invariant under real linear changes of variables and also under complex shifts in the variables. Hence the result follows from the last equality. ∎

Proposition 15. *Assume that $P_0 \in \mathbb{C}[\tau, \zeta_1, \dots, \zeta_n]$ is given by*

$$P_0(\tau, \zeta) = (\tau + \lambda(\zeta))^2 + D(\zeta) + \Lambda_0(\zeta) + C_0,$$

where λ is a real linear form, $D(\zeta) = \sum_{j=1}^m d_j \zeta_j^2$ for $d_j \in \mathbb{R} \setminus \{0\}$ for $1 \leqslant j \leqslant m$ and $1 \leqslant m \leqslant n$, $\Lambda_0(\zeta) = \sum_{j=m+1}^n l_j \zeta_j$ is a complex linear form, and $C_0 \in \mathbb{C}$.

(a) *If $\Lambda_0 \equiv 0$ then the following assertions are equivalent:*
   (1) *$V(P_0)$ satisfies $\mathrm{PL}(\omega)$ for some/each weight function ω.*
   (2) *$V(P_0)$ is hyperbolic for some/each weight function ω.*
   (3) *D is negative semidefinite.*

(b) *If $\Lambda_0 \not\equiv 0$ then $V(P_0)$ satisfies $\mathrm{PL}(\omega)$ only if D is negative semidefinite and if one of the following conditions is satisfied:*
   (1) *$\alpha_2 \geqslant 1/2$ if there exists $\xi \in \{0\} \times \mathbb{R}^{n-m}$ such that $\Lambda_0(\xi) \neq 0$ and $\lambda(\xi) = 0$.*

(2) $\max\{\alpha_1, \alpha_2\} \geqslant 1/2$ *if for each* $\xi \in \{0\} \times \mathbb{R}^{n-m}$ *we have* $\lambda(\xi) \neq 0$ *whenever* $\Lambda_0(\xi) \neq 0$.

Proof. (a) (1) $\Rightarrow$ (3): If $V(P_0)$ satisfies PL(ω) for some weight function ω then Proposition 12 shows that also $V(P_2)$ satisfies PL(ω), where P_2 is the principal part of P_0. Since $P_2(\tau, \zeta) = (\tau + \lambda(\zeta))^2 + D(\zeta)$ in this case, it follows from Proposition 10 and Remark 13 that the quadratic form D is negative semidefinite. This is equivalent to $d_j < 0$ for $1 \leqslant j \leqslant m$, because $d_j \in \mathbb{R} \setminus \{0\}$.

(3) $\Rightarrow$ (2): Since $\Lambda_0 \equiv 0$, we have $(\tau, \zeta) \in V(P_0)$ if and only if $\tau = -\lambda(\zeta) \pm \sqrt{-D(\zeta) - C_0}$. This implies the existence of $C > 0$ such that

$$|\operatorname{Im} \tau| \leqslant |\tau| \leqslant C(|\zeta| + 1), \ \ (\tau, \zeta) \in V(P_0).$$

Next we define $v : \mathbb{C}^n \to \mathbb{R}$ by

$$v(\zeta) := \max\{|\operatorname{Im} \tau| : (\tau, \zeta) \in V(P_0)\}.$$

By Hörmander [7], Lemma 4.4, v is in PSH($\mathbb{C}^n$). Since λ is a real linear form and since D is negative semidefinite by hypothesis, there exists $C_1 > 0$ such that

$$v(\xi) \leqslant C_1, \qquad \xi \in \mathbb{R}^n.$$

Therefore, the function $\varphi : \mathbb{C}^n \to \mathbb{R}$, $\varphi(\zeta) := \frac{1}{C}(v(\zeta) - C_1)$ satisfies the hypotheses of the classical Phragmén-Lindelöf Theorem for $\mathbb{C}^n$. Hence φ satisfies

$$\varphi(\zeta) \leqslant |\operatorname{Im} \zeta|, \qquad \zeta \in \mathbb{C}^n.$$

By the definition of v, this implies

$$|\operatorname{Im} \tau| \leqslant C|\operatorname{Im} \zeta| + C_1, \qquad (\tau, \zeta) \in V(P_0).$$

Hence (2) holds.

(2) $\Rightarrow$ (1): This implication holds by Remark 7.

(b) The arguments in part (a) show that the negative semidefinitness of D is necessary also in this case. To show that also the other condition is necessary, note first that after a real linear change of variables in $(x_{m+1}, \ldots, x_n)$, we may assume that ξ in (1) is the canonical basis vector e_{m+1}, so that we have $\Lambda_0(e_{m+1}) = l_{m+1} \neq 0$. For $R > 1$ we then let $\zeta_R := \rho e_{m+1}$ where $\rho = R$ or $\rho = -R$ is chosen so that $\operatorname{Im}(-l_{m+1}\rho - C_0)^{1/2} \neq 0$. We also let $\theta_R := (-\lambda(\zeta_R) + (-l_{m+1}\rho - C_0)^{1/2}, \zeta_R)$ and we fix $0 < \mu < 1/2$. Then let $V := V(P_0)$ and denote W_R the connected component of $V \cap (\mathbb{C} \times B(\zeta_R, R^\mu))$ which contains θ_R and define $u : V \to \mathbb{R}$ by formula (3). If $h \in \mathbb{C}^n$ satisfies $|h| < R^\mu$ and $\tau(\zeta_R + h)$ satisfies $P(\tau(\zeta_R + h), \zeta_R + h) = 0$, then it follows easily that there exist $\delta > 0$ and $R_0 > 1$ such that for $R \geqslant R_0$ we have

$$|\operatorname{Im} \tau(\zeta_R + h) + \operatorname{Im} \lambda(\zeta_R + h)| \geqslant \delta R^{1/2}.$$

Since $\operatorname{Im} \lambda(\zeta_R + h) = \operatorname{Im} \lambda(h)$ and since $|h| < R^\mu$, this estimate implies

$$|\operatorname{Im} \tau(\zeta_R + h)| \geqslant \frac{R^\mu}{2}, \qquad R \geqslant R_0. \tag{9}$$

Since u is plurisubharmonic on V and since the arguments that we used in the proof of Proposition 10 show that u satisfies the conditions (α) and (β) of $\mathrm{PL}(\omega)$, u also satisfies the condition (γ) of $\mathrm{PL}(\omega)$. Hence there exist $A, A' > 0$ such that

$$\begin{aligned}\frac{R^\mu}{2} &\leqslant u(\theta_R) \leqslant A(|\operatorname{Im}\zeta_R| + \omega_1(\tau(\zeta_R)) + \omega_2(\zeta_R) + 1)\\ &\leqslant A'\left(|\operatorname{Im}\zeta_R| + \left|-\lambda(\zeta_R) + \sqrt{-\Lambda_0(\zeta_R) - C_0}\right|^{\alpha_1} + R^{\alpha_2} + 1\right), \quad R \geqslant R_0.\end{aligned} \tag{10}$$

If we are in case (1) then $\lambda(\zeta_R) = 0$. Hence the inequality (10) implies the existence of $A_1 > 0$ such that

$$\frac{R^\mu}{2} \leqslant A_1\left(R^{\alpha_1/2} + R^{\alpha_2}\right), \qquad R \geqslant R_0.$$

Since $0 < \mu < 1/2$ was chosen arbitrarily, it now follows that we must have $\max\{\alpha_1/2, \alpha_2\} \geqslant 1/2$. Since $0 \leqslant \alpha_1 < 1$, this implies $\alpha_2 \geqslant 1/2$.

If we are in case (2), then the inequality (10) implies that for each $0 < \mu < 1/2$ there exist $A_2 > 0$ and $R_0 > 1$ such that

$$\frac{R^\mu}{2} \leqslant A_2(R^{\alpha_1} + R^{\alpha_2}), \qquad R \geqslant R_0$$

and hence $\max\{\alpha_1, \alpha_2\} \geqslant 1/2$. ■

Remark 16. Note that Proposition 15 is still valid if $m = 0$ and hence $D \equiv 0$.

To show that the necessary conditions in Proposition 15 are in fact sufficient, we need the following lemma.

Lemma 17. *The following inequality holds for each* $(z_1, z_2) \in \mathbb{C}^2$*:*

$$|\operatorname{Im}\sqrt{z_1 + z_2}| \leqslant |\operatorname{Im}\sqrt{z_1}| + |\operatorname{Im}\sqrt{z_2}|. \tag{11}$$

Proof. Note first that the inequality (11) is a consequence of the following one:

$$|\operatorname{Im}\sqrt{a^2 + b^2}| \leqslant \sqrt{|\operatorname{Im} a|^2 + |\operatorname{Im} b|^2}, \qquad (a, b) \in \mathbb{C}^2. \tag{12}$$

To prove (12), define

$$\varphi:\ \mathbb{C}^2 \to \mathbb{C}, \qquad \varphi(a, b) := |\operatorname{Im}\sqrt{a^2 + b^2}|.$$

It is easy to check that φ is plurisubharmonic on $\mathbb{C}^2$ and satisfies $\varphi(a, b) = 0$ for each $(a, b) \in \mathbb{R}^2$.

Moreover,

$$\varphi(a, b) \leqslant \sqrt{|a|^2 + |b|^2} = |(a, b)|.$$

Hence the classical Theorem of Phragmén-Lindelöf for $\mathbb{C}^2$ implies that

$$\varphi(a, b) \leqslant |\operatorname{Im}(a, b)| = \sqrt{|\operatorname{Im} a|^2 + |\operatorname{Im} b|^2},$$

which is the estimate (12). ■

Proposition 18. *Let P_0 be as in Proposition 15, assume that the conditions in Proposition 15 (b) (1) are fulfilled and let $\omega(\tau,\zeta) := \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$. Then the following conditions are equivalent:*

(1) *$V(P_0)$ satisfies $\mathrm{PL}(\omega)$.*
(2) *$\alpha_2 \geqslant 1/2$.*
(3) *P_0 is σ_{α_2}-hyperbolic with respect to $N = (1,0,\ldots,0)$ for $\sigma_{\alpha_2}(\tau,\zeta) = |(\tau,\zeta)|^{\alpha_2}$, $(\tau,\zeta) \in \mathbb{C}\times\mathbb{C}^n$.*
(4) *$V(P_0)$ is hyperbolic for ω.*

Proof. (1) ⇒ (2): This holds by Proposition 15 (b) (1).

(2) ⇒ (3): Since we can apply a real diagonal change of variables, it is no restriction to assume that

$$D(\zeta) = -\sum_{j=1}^{m} \zeta_j^2.$$

Then, for fixed $(\tau,\zeta) \in V(P_0)$, we have that

$$(\tau + \lambda(\zeta))^2 = \sum_{j=1}^{m} \zeta_j^2 - \Lambda_0(\zeta) - C_0.$$

Applying Lemma 17 to $z_1 = \sum_{j=1}^{m} \zeta_j^2$ and $z_2 = -\Lambda_0(\zeta) - C_0$:

$$\begin{aligned} |\operatorname{Im}(\tau + \lambda(\zeta))| &= \left|\operatorname{Im}\left(\sum_{j=1}^{m} \zeta_j^2 - \Lambda_0(\zeta) - C_0\right)^{1/2}\right| \\ &\leqslant \left|\operatorname{Im}\Big(\sum_{j=1}^{m} \zeta_j^2\Big)^{1/2}\right| + |\operatorname{Im}(-\Lambda_0(\zeta) - C_0)^{1/2}|. \end{aligned} \tag{13}$$

Since

$$\left|\operatorname{Im}\Big(\sum_{j=1}^{m} \zeta_j^2\Big)^{1/2}\right| \leqslant |\operatorname{Im}\zeta|,\ \zeta \in \mathbb{C}^m, \tag{14}$$

since Λ_0 is a linear form and since $\alpha_2 \geqslant 1/2$ there exists $C_3 > 0$ such that we get from (13)

$$\begin{aligned} |\operatorname{Im}(\tau + \lambda(\zeta))| &\leqslant |\operatorname{Im}\zeta| + |\Lambda_0(\zeta) + C_0|^{1/2} \\ &\leqslant C_3(|\operatorname{Im}\zeta| + |\zeta|^{1/2} + 1) \\ &\leqslant C_3(|\operatorname{Im}\zeta| + \sigma_{\alpha_2}(|\zeta|) + 2). \end{aligned}$$

Because λ is a real linear form this estimate implies the existence of $C_4 > 0$ such that

$$|\operatorname{Im}\tau| \leqslant C_4(|\operatorname{Im}\zeta| + \sigma_{\alpha_2}(|\zeta|) + 1). \tag{15}$$

Since $N = (1,0,\ldots,0)$ is non-characteristic for P_0, this estimate implies by Meise, Taylor and Vogt [10], Propositions 2.7 and 2.9, that P_0 is σ_{α_2}-hyperbolic with respect to N.

(3) ⇒ (4): This holds by [1], Proposition 3.9.
(4) ⇒ (1): This holds by Remark 7. ∎

Remark 19. Note that in Proposition 18 the implications (2) ⇒ (3) ⇒ (4) ⇒ (1) hold, whenever D is negative semidefinite (if $D \equiv 0$ we have only case 1 in the implication (2) ⇒ (3)).

Lemma 20. *Let $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m \setminus \{0\}$, set*

$$\tilde{\lambda}_j = \begin{cases} \lambda_j, & \text{if } \lambda_j \neq 0 \\ 1, & \text{if } \lambda_j = 0 \end{cases} \tag{16}$$

for $1 \leqslant j \leqslant m$, $\Lambda := \max_{1 \leqslant j \leqslant m} \tilde{\lambda}_j^2$ and take $0 < \varepsilon < 1/(m\Lambda)$. There exists then a constant $c > 0$ such that each $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$ which satisfies

$$\left| \sum_{j=1}^m \zeta_j^2 \right| \leqslant \varepsilon \left| \sum_{j=1}^m \lambda_j \zeta_j \right|^2, \tag{17}$$

also satisfies

$$\left| \sum_{j=1}^m \lambda_j \zeta_j \right| \leqslant c \sum_{j=1}^m |\operatorname{Im} \tilde{\lambda}_j \zeta_j| = c \sum_{j=1}^m |\tilde{\lambda}_j| \cdot |\operatorname{Im} \zeta_j|. \tag{18}$$

Proof. Let us first remark that if $\operatorname{Im} \tilde{\lambda}_j \zeta_j = 0$ for all $j = 1, \dots, m$, i.e. $\zeta_1, \dots, \zeta_m \in \mathbb{R}$, then by Cauchy-Schwarz inequality and (17) we have that

$$\left| \sum_{j=1}^m \lambda_j \zeta_j \right|^2 \leqslant \left| \sum_{j=1}^m \lambda_j^2 \right| \cdot \left| \sum_{j=1}^m \zeta_j^2 \right| \leqslant m\Lambda\varepsilon \left| \sum_{j=1}^m \lambda_j \zeta_j \right|^2$$

which gives $\sum_{j=1}^m \lambda_j \zeta_j = 0$ since $\varepsilon m \Lambda < 1$ by assumption. In this case (18) holds trivially.

Let us then assume that $\sum_{j=1}^m |\operatorname{Im} \tilde{\lambda}_j \zeta_j| > 0$. By homogeneity it is then sufficient to prove that (17) implies the existence of $c > 0$ such that

$$\sum_{j=1}^m |\operatorname{Im} \tilde{\lambda}_j \zeta_j| = 1 \Longrightarrow \left| \sum_{j=1}^m \lambda_j \zeta_j \right| \leqslant c. \tag{19}$$

To argue by contradiction, assume that this does not hold. Then we can find a sequence $(\zeta^{(k)})_{k \in \mathbb{N}}$ in $\mathbb{C}^m$ such that the inequality (17) holds for each $\zeta^{(k)}$ while

$$\sum_{j=1}^m |\operatorname{Im} \tilde{\lambda}_j \zeta_j^{(k)}| = 1, \qquad \text{and} \qquad \lim_{k \to \infty} \left| \sum_{j=1}^m \lambda_j \zeta_j^{(k)} \right| = \infty. \tag{20}$$

Next we choose $a_j^{(k)}, b_j^{(k)} \in \mathbb{R}$ for $j \in \{1, \ldots, m\}$ and $k \in \mathbb{N}$ such that $\tilde{\lambda}_j \zeta_j^{(k)} = a_j^{(k)} + i b_j^{(k)}$. Then (20) implies

$$1 = \sum_{j=1}^{m} |b_j^{(k)}| = \sum_{j=1}^{m} |\operatorname{Im} \tilde{\lambda}_j \zeta_j^{(k)}| \geqslant \sum_{j=1}^{m} |\operatorname{Im} \lambda_j \zeta_j^{(k)}|$$

and hence $\lim_{k\to\infty} \sum_{j=1}^{m} |\operatorname{Re} \lambda_j \zeta_j^{(k)}| = \infty$. Consequently we get

$$\sum_{j=1}^{m} |b_j^{(k)}| = 1 \qquad \text{and} \qquad \lim_{k\to\infty} \sum_{j=1}^{m} |a_j^{(k)}| = \infty. \tag{21}$$

Now fix $\zeta = (\zeta_1, \ldots, \zeta_m)$ with $\tilde{\lambda}_j \zeta_j = a_j + i b_j$ and let $\lambda := \min_{1\leqslant j\leqslant m} \tilde{\lambda}_j^2$. Then the Cauchy-Schwarz inequality implies

$$\begin{aligned}
\left| \sum_{j=1}^{m} \zeta_j^2 \right| &= \left| \sum_{j=1}^{m} \frac{1}{\tilde{\lambda}_j^2} (a_j^2 + 2i a_j b_j - b_j^2) \right| \\
&\geqslant \frac{1}{\Lambda} \left| \sum_{j=1}^{m} a_j^2 \right| - \frac{2}{\lambda} \left| \sum_{j=1}^{m} a_j^2 \right|^{1/2} \left| \sum_{j=1}^{m} b_j^2 \right|^{1/2} - \frac{1}{\lambda} \left| \sum_{j=1}^{m} b_j^2 \right| \\
&\geqslant \frac{1}{\Lambda} \sum_{j=1}^{m} a_j^2 - \frac{2}{\lambda} \left(\sum_{j=1}^{m} a_j^2 \right)^{1/2} \left(\sum_{j=1}^{m} |b_j| \right) - \frac{1}{\lambda} \left(\sum_{j=1}^{m} |b_j| \right)^2 \\
&= \frac{1}{\Lambda} \sum_{j=1}^{m} a_j^2 - \frac{2}{\lambda} \left(\sum_{j=1}^{m} a_j^2 \right)^{1/2} - \frac{1}{\lambda}.
\end{aligned}$$

On the other side,

$$\left| \sum_{j=1}^{m} \lambda_j \zeta_j \right| \leqslant \sum_{j=1}^{m} |a_j| + \sum_{j=1}^{m} |b_j| \leqslant \sqrt{m} \left(\sum_{j=1}^{m} a_j^2 \right)^{1/2} + 1.$$

Since $\zeta^{(k)}$ satisfies the inequality (17), the two estimates above imply

$$\frac{1}{\Lambda} \left(\sum_{j=1}^{m} (a_j^{(k)})^2 \right) - \frac{2}{\lambda} \left(\sum_{j=1}^{m} (a_j^{(k)})^2 \right)^{1/2} - \frac{1}{\lambda} \leqslant \varepsilon \left(\left(m \sum_{j=1}^{m} (a_j^{(k)})^2 \right)^{1/2} + 1 \right)^2$$

and hence

$$(1 - \varepsilon m \Lambda) \sum_{j=1}^{m} (a_j^{(k)})^2 - \Lambda\left(\frac{2}{\lambda} + 2\varepsilon\sqrt{m}\right) \left(\sum_{j=1}^{m} (a_j^{(k)})^2 \right)^{1/2} - \Lambda\left(\frac{1}{\lambda} + \varepsilon\right) \leqslant 0.$$

This is a contradiction for large $k \in \mathbb{N}$, since $1 - \varepsilon m\Lambda > 0$ and

$$\lim_{k\to\infty} \sum_{j=1}^{m} (a_j^{(k)})^2 = \infty. \qquad \blacksquare$$

Proposition 21. *Let P_0 be as in Proposition 15, assume that the conditions in Proposition 15 (b) (2) are fulfilled, and let $\omega(\tau, \zeta) := \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$. Then the following assertions are equivalent:*

(1) $V(P_0)$ *satisfies* PL(ω).
(2) $\max\{\alpha_1, \alpha_2\} \geqslant 1/2$.
(3) $V(P_0)$ *is hyperbolic for* ω.

Proof. (1) ⇒ (2): This holds by Proposition 15 (b) (2).

(2) ⇒ (3): As in the proof of Proposition 18 it is no restriction to assume that $D(\zeta) = -\sum_{j=1}^m \zeta_j^2$. Since λ is a real linear form, there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\lambda(\zeta) = \sum_{j=1}^n \lambda_j \zeta_j$. Obviously, condition 15 (b) (2) implies that

$$\ker(\lambda|_{\{0\}\times\mathbb{R}^{n-m}}) \subset \ker(\Lambda_0|_{\{0\}\times\mathbb{R}^{n-m}}),$$

where we consider both restrictions as $\mathbb{R}$-linear maps into the $\mathbb{R}$-vector space $\mathbb{C}$. Since $\Lambda_0 \not\equiv 0$, it follows that $\lambda|_{\{0\}\times\mathbb{R}^{n-m}} \not\equiv 0$. As λ has real coefficients by hypothesis, $\lambda|_{\{0\}\times\mathbb{R}^{n-m}}$ is linear and $\lambda(\{0\} \times \mathbb{R}^{n-m}) = \mathbb{R} \subset \mathbb{C}$. Consequently, $\dim_{\mathbb{R}}(\ker(\lambda|_{\{0\}\times\mathbb{R}^{n-m}})) = n - m - 1$. This implies that the two kernels are in fact equal and that $\dim_{\mathbb{R}} \Lambda_0(\{0\} \times \mathbb{R}^{n-m}) = 1$. Because of the special form of Λ_0 this shows that we can find $\tilde{\mu} \in \mathbb{C} \setminus \{0\}$ such that $\Lambda_0|_{\{0\}\times\mathbb{R}^{n-m}} = \tilde{\mu}\lambda_0$, where $\lambda_0 = \lambda \circ \pi_{n-m}$ for $\pi_{n-m}(\zeta) := (0, \dots, 0, \zeta_{m+1}, \dots, \zeta_n)$. Hence we can choose $\mu \in \mathbb{C} \setminus \{0\}$ and $C_0 \in \mathbb{C}$ such that $(\tau, \zeta) \in V(P_0)$ if and only if

$$(\tau + \lambda(\zeta))^2 = (\tau + \sum_{j=1}^n \lambda_j \zeta_j)^2 = \sum_{j=1}^m \zeta_j^2 + \mu \sum_{j=m+1}^n \lambda_j \zeta_j + C_0 = Q(\zeta) + \mu\lambda_0(\zeta) + C_0.$$

Note that $\Lambda_0 \not\equiv 0$ implies $(\lambda_{m+1}, \dots, \lambda_n) \in \mathbb{R}^{n-m} \setminus \{0\}$ and that $Q(\zeta) = \sum_{j=1}^m \zeta_j^2$.

If $\alpha_2 \geqslant 1/2$ then Remark 19 shows that $V(P_0)$ is hyperbolic for ω. Hence it suffices to show that this also holds if $\alpha_1 \geqslant 1/2$. To do this consider $\tilde{\lambda}_j$ as in (16), set $\Lambda := \max_{1\leqslant j\leqslant m} \tilde{\lambda}_j^2$ and consider the following cases for $(\tau, \zeta) \in V(P_0)$:

Case (1): $\left|\sum_{j=1}^m \lambda_j \zeta_j\right| \leqslant 2\sqrt{m\Lambda}\, |Q(\zeta)|^{1/2}$.

Subcase (1.1): $|Q(\zeta)| \leqslant 2|\mu\lambda_0(\zeta) + C_0|$.

Then the hypothesis of the subcase gives

$$|\tau + \lambda(\zeta)| = |Q(\zeta) + \mu\lambda_0(\zeta) + C_0|^{1/2} \leqslant \sqrt{3}(|\mu\lambda_0(\zeta)|^{1/2} + |C_0|^{1/2}).$$

On the other side, the present hypotheses imply

$$\begin{aligned} |\tau + \lambda(\zeta)| &\geqslant |\lambda_0(\zeta)| - \left|\sum_{j=1}^m \lambda_j \zeta_j\right| - |\tau| \geqslant |\lambda_0(\zeta)| - 2\sqrt{m\Lambda}|Q(\zeta)|^{1/2} - |\tau| \\ &\geqslant |\lambda_0(\zeta)| - 2\sqrt{2m\Lambda}(|\mu\lambda_0(\zeta)|^{1/2} + |C_0|^{1/2}) - |\tau|. \end{aligned}$$

Therefore,

$$|\lambda_0(\zeta)| - (2\sqrt{2m\Lambda|\mu|} + \sqrt{3|\mu|})\,|\lambda_0(\zeta)|^{1/2} - 2\sqrt{2m\Lambda|C_0|} - \sqrt{3|C_0|} - |\tau| \leqslant 0$$

and hence

$$\begin{aligned}|\lambda_0(\zeta)|^{1/2} &\leqslant \frac{1}{2}\Big(2\sqrt{2m\Lambda|\mu|} + \sqrt{3|\mu|} \\ &\qquad + \sqrt{(2\sqrt{2m\Lambda|\mu|} + \sqrt{3|\mu|})^2 + 8\sqrt{2m\Lambda|C_0|} + 4\sqrt{3|C_0|} + 4|\tau|}\,\Big) \\ &\leqslant |\tau|^{1/2} + c\end{aligned}$$

for $c = 2\sqrt{2m\Lambda|\mu|} + \sqrt{3|\mu|} + \sqrt[4]{8m\Lambda|C_0|} + \sqrt[4]{3|C_0|}$.

It thus follows that there exist $A, A' > 0$ such that

$$\begin{aligned}|\operatorname{Im}\tau| &= \Big|-\sum_{j=1}^{n}\operatorname{Im}(\lambda_j\zeta_j) \pm \operatorname{Im}\sqrt{Q(\zeta)+\mu\lambda_0(\zeta)+C_0}\Big| \\ &\leqslant \Big|\sum_{j=1}^{n}\lambda_j\operatorname{Im}\zeta_j\Big| + \big|Q(\zeta)+\mu\lambda_0(\zeta)+C_0\big|^{1/2} \\ &\leqslant A\left(|\operatorname{Im}\zeta| + \sqrt{3|\mu|}\,|\lambda_0(\zeta)|^{1/2} + \sqrt{3|C_0|}\right) \\ &\leqslant A'(|\operatorname{Im}\zeta| + |\tau|^{1/2} + 1) \\ &\leqslant A'(|\operatorname{Im}\zeta| + \sigma_{\alpha_1}(|\tau|) + 2)\end{aligned}$$

since $\lambda_j \in \mathbb{R}$ and $\alpha_1 \geqslant 1/2$.

Subcase (1.2): $|Q(\zeta)| \geqslant 2\,|\mu\lambda_0(\zeta) + C_0|$.

In this case we can write

$$\mu\lambda_0(\zeta) + C_0 = \alpha Q(\zeta)$$

for some $\alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1/2$. Then

$$\begin{aligned}\tau &= -\sum_{j=1}^{m}\lambda_j\zeta_j - \lambda_0(\zeta) \pm (Q(\zeta)+\mu\lambda_0(\zeta)+C_0)^{1/2} \\ &= -\sum_{j=1}^{m}\lambda_j\zeta_j - \frac{\alpha}{\mu}Q(\zeta) + \frac{C_0}{\mu} \pm Q(\zeta)^{1/2}\sqrt{1+\alpha}\end{aligned}$$

and hence by the hypothesis in case (1):

$$\left|\frac{\alpha}{\mu}Q(\zeta) \mp Q(\zeta)^{1/2}\sqrt{1+\alpha} + \tau\right| = \left|-\sum_{j=1}^{m}\lambda_j\zeta_j + \frac{C_0}{\mu}\right| \leqslant 2\sqrt{m\Lambda}\,|Q(\zeta)|^{1/2} + \left|\frac{C_0}{\mu}\right|.$$

On the other side,

$$\left|\frac{\alpha}{\mu}Q(\zeta) \mp Q(\zeta)^{1/2}\ \sqrt{1+\alpha} + \tau\right| \geqslant \left|\frac{\alpha}{\mu}Q(\zeta)\right| - |Q(\zeta)|^{1/2}\ \sqrt{|1+\alpha|} - |\tau|.$$

Therefore,

$$\left|\frac{\alpha}{\mu}Q(\zeta)\right| - (2\sqrt{m\Lambda} + \sqrt{|1+\alpha|})\,|Q(\zeta)|^{1/2} - \left|\frac{C_0}{\mu}\right| - |\tau| \leqslant 0,$$

which implies

$$|Q(\zeta)|^{1/2} \leqslant \Bigg(2\sqrt{m\Lambda} + \sqrt{|1+\alpha|} + \sqrt{(2\sqrt{m\Lambda} + \sqrt{|1+\alpha|})^2 + 4\left|\frac{\alpha C_0}{\mu^2}\right| + 4\left|\frac{\alpha}{\mu}\right|\cdot|\tau|}\Bigg)\cdot\frac{|\mu|}{2|\alpha|},$$

i.e.

$$\left|\alpha Q(\zeta)^{1/2}\right| \leqslant c'(|\tau|^{1/2}+1)$$

for some $c' > 0$. From these estimates and the identity

$$\sqrt{1+\alpha} - \sqrt{1} = \int_0^\alpha \frac{1}{2\sqrt{1+z}}\,dz$$

we now get the existence of $A, A', A'' > 0$ such that the following estimates hold:

$$\begin{aligned}
|\operatorname{Im}\tau| &= \left|-\sum_{j=1}^{n}\operatorname{Im}(\lambda_j\zeta_j) \pm \operatorname{Im}(Q(\zeta)+\mu\lambda_0(\zeta)+C_0)^{1/2}\right| \\
&\leqslant A|\operatorname{Im}\zeta| + \left|\operatorname{Im}\left(Q(\zeta)^{1/2}\sqrt{1+\alpha}\right)\right| \\
&\leqslant A|\operatorname{Im}\zeta| + \left|\operatorname{Im}Q(\zeta)^{1/2}\right|\cdot|\operatorname{Re}\sqrt{1+\alpha}| + \left|\operatorname{Re}Q(\zeta)^{1/2}\right|\cdot|\operatorname{Im}(\sqrt{1+\alpha}-\sqrt{1})| \\
&\leqslant A|\operatorname{Im}\zeta| + \sqrt{|1+\alpha|}|\operatorname{Im}\zeta| + \left|Q(\zeta)^{1/2}\right|\sup_{|z|\leqslant 1/2}\left|\frac{1}{2\sqrt{1+z}}\right|\cdot|\alpha| \\
&\leqslant A'|\operatorname{Im}\zeta| + \left|\alpha Q(\zeta)^{1/2}\right|\sup_{|z|\leqslant 1/2}\frac{1}{2\sqrt{1-|z|}} \\
&\leqslant A'|\operatorname{Im}\zeta| + \frac{\sqrt{2}}{2}c'(|\tau|^{1/2}+1) \\
&\leqslant A''(|\operatorname{Im}\zeta| + \sigma_{\alpha_1}(|\tau|)+1).
\end{aligned}$$

Case (2): $|Q(\zeta)|^{1/2} \leqslant \frac{1}{2\sqrt{m\Lambda}}\left|\sum_{j=1}^m \lambda_j\zeta_j\right|$.

Subcase (2.1): $|Q(\zeta)| \leqslant 2\,|\mu\lambda_0(\zeta)+C_0|$.

Subcase (2.1)(a): $|\lambda_0(\zeta)| \geqslant 2\left|\sum_{j=1}^m \lambda_j\zeta_j\right|$.

Then

$$\begin{aligned}
|\tau| &\geqslant |\lambda_0(\zeta)| - \Big|\sum_{j=1}^m \lambda_j\zeta_j\Big| - |Q(\zeta)+\mu\lambda_0(\zeta)+C_0|^{1/2} \\
&\geqslant |\lambda_0(\zeta)| - \frac{1}{2}\,|\lambda_0(\zeta)| - \sqrt{3|\mu|}\,|\lambda_0(\zeta)|^{1/2} - \sqrt{3|C_0|} \\
&\geqslant \delta\,|\lambda_0(\zeta)| - \delta'
\end{aligned}$$

for some $\delta, \delta' > 0$. Therefore,

$$\begin{aligned}
|\operatorname{Im}\tau| &\leqslant \Big|\sum_{j=1}^n \lambda_j \operatorname{Im}\zeta_j\Big| + |Q(\zeta)+\mu\lambda_0(\zeta)+C_0|^{1/2} \\
&\leqslant A\left(|\operatorname{Im}\zeta| + |\lambda_0(\zeta)|^{1/2} + 1\right) \\
&\leqslant A'(|\operatorname{Im}\zeta| + |\tau|^{1/2} + 1) \\
&\leqslant A'(|\operatorname{Im}\zeta| + \sigma_{\alpha_1}(|\tau|) + 2).
\end{aligned}$$

Subcase (2.1)(b): $|\lambda_0(\zeta)| \leqslant 2\left|\sum_{j=1}^m \lambda_j\zeta_j\right|$.

Then

$$\begin{aligned}
|\operatorname{Im}\tau| &\leqslant \Big|\sum_{j=1}^n \lambda_j \operatorname{Im}\zeta_j\Big| + |Q(\zeta)+\mu\lambda_0(\zeta)+C_0|^{1/2} \\
&\leqslant A\left(|\operatorname{Im}\zeta| + |\lambda_0(\zeta)|^{1/2} + 1\right) \leqslant A\left(|\operatorname{Im}\zeta| + |\lambda_0(\zeta)| + 2\right) \\
&\leqslant A\left(|\operatorname{Im}\zeta| + 2\Big|\sum_{j=1}^m \lambda_j\zeta_j\Big| + 2\right) \leqslant A'\left(|\operatorname{Im}\zeta| + \sum_{j=1}^m |\operatorname{Im}\tilde{\lambda}_j\zeta_j| + 1\right) \\
&\leqslant A''(|\operatorname{Im}\zeta| + 1)
\end{aligned}$$

for some $A, A', A'' > 0$, because of Lemma 20.

Subcase (2.2): $|Q(\zeta)| \geqslant 2\,|\mu\lambda_0(\zeta)+C_0|$.

In this case we can write

$$\mu\lambda_0(\zeta) + C_0 = \alpha Q(\zeta)$$

for some $\alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1/2$. Then

$$\begin{aligned}
|\operatorname{Im}\tau| &\leqslant \left|\sum_{j=1}^{n} \lambda_j \operatorname{Im}\zeta_j\right| + |Q(\zeta) + \mu\lambda_0(\zeta) + C_0|^{1/2} \\
&\leqslant A|\operatorname{Im}\zeta| + |Q(\zeta)|^{1/2}\sqrt{|1+\alpha|} \\
&\leqslant A|\operatorname{Im}\zeta| + \frac{1}{2\sqrt{m\Lambda}}\left|\sum_{j=1}^{m} \lambda_j\zeta_j\right|\sqrt{|1+\alpha|} \\
&\leqslant A'\left(|\operatorname{Im}\zeta| + \sum_{j=1}^{m} |\operatorname{Im}\tilde{\lambda}_j\zeta_j|\right) \\
&\leqslant A''|\operatorname{Im}\zeta|
\end{aligned}$$

for some $A, A', A'' > 0$, because of Lemma 20.

(3) $\Rightarrow$ (1): This holds by Remark 7. ■

Remark 22. Note that Proposition 21 holds also if $D \equiv 0$ (i.e. $Q \equiv 0$ in the proof).

Proof of the Main Theorem 1. This proof follows from the Propositions 15, 18 and 21 and from the Remarks 16, 19 and 22. ■

Let us now consider an example of an algebraic variety defined by a polynomial of order $m \geqslant 2$:

Example 23. Let $(a_1, \dots, a_n) \in \mathbb{C}^n \setminus \{0\}$ and

$$V = \{(\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^n : \ \tau^m + \sum_{j=1}^{n} a_j \zeta_j^m = 0\}.$$

We claim that V satisfies PL(ω) if and only if $m = 2$ and $a_j \leqslant 0$ for $1 \leqslant j \leqslant n$.

Indeed, if $m \geqslant 3$ then, taking $\zeta_R = (R, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\max\{\alpha_1, \alpha_2\} < \mu < 1$ and $h \in \mathbb{C}^n$ with $|h| \leqslant R^\mu$, we have that

$$\tau^m = -a_1(R + h_1)^m - \sum_{j=2}^{n} a_j h_j^m.$$

Hence we can choose an m-th root of $-a_1 R^m$ such that

$$|\operatorname{Im}\tau(\zeta_R + h)| \geqslant \delta R \geqslant \frac{R^\mu}{2}, \qquad R \gg 1.$$

For such a choice of the m-th root, taking $\theta_R = (\tau(\zeta_R), \zeta_R)$ and $u \in \mathrm{PSH}(V)$ as in (3), we have that u satisfies (α) and (β) of PL(ω) and hence, if V satisfies

PL(ω), from (γ):

$$\begin{aligned}\frac{R^\mu}{2} \leqslant u(\theta_R) &\leqslant A(\omega_1(\tau(\zeta_R)) + \omega_2(\zeta_R) + 1) \\ &\leqslant A'(R^{\alpha_1} + R^{\alpha_2} + 1)\end{aligned}$$

for some $A, A' > 0$, obtaining a contradiction for large R since $0 \leqslant \alpha_1, \alpha_2 < \mu$.

The case $m = 2$, i.e.

$$\tau^2 = \sum_{j=1}^{n} (-a_j)\zeta_j^2,$$

follows from Proposition 15.

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Addresses: Chiara Boiti: Dipartimento di Matematica, Università di Ferrara, Via Machiavelli 35, I-44121 Ferrara, Italy;
Reinhold Meise: Mathematisches Institut, Heinrich-Heine-Universität, Universitätsstraße 1, D-40225 Düsseldorf, Germany.

E-mail: chiara.boiti@unife.it, meise@math.uni-duesseldorf.de

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