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EXTENSION OF SATO'S HYPERFUNCTIONS

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Dedicated to the memory of Susanne Dierolf

Abstract: We characterize possible bounds for representing functions of arbitrary hyperfunctions. Specifically, there are always representing functions decreasing rapidly outside each strip near $\mathbb{R}$. Also, exponential decrease of any type on any strip $\mathbb{R} \times \pm i[c, C], 0 < c < C < \infty$, can be achieved. This will be used in [10] to define an asymptotic Fourier and Laplace transformation on the space of hyperfunctions.

Keywords: hyperfunctions, Fourier hyperfunctions, representing functions, vector valued hyperfunctions.

1. Introduction

Fourier hyperfunctions have been introduced by Kawai [3] to provide a general and natural frame for Fourier transformation. In fact, these Fourier techniques may also be applied to general hyperfunctions. This is due to the important result that any hyperfunction $[u] \in \mathcal{B}(\mathbb{R}) := H(\mathbb{C} \setminus \mathbb{R})/H(\mathbb{C})$ may be extended to a Fourier hyperfunction, that is, $[u]$ admits a representing function $h \in [u]$ which is of exponential type 0 on each strip outside the real axis, i.e.

$$\forall j \in \mathbb{N}\ \exists C_j > 0 : |h(z)| \leqslant C_j e^{|z|/j} \qquad \text{if } 1/j \leqslant |\operatorname{Im} z| \leqslant j.$$

In fact, the following stronger result of Kaneko and Komatsu holds:

Theorem. *Let $F \subset \mathbb{C}\setminus\mathbb{R}$ be closed and $k \in \mathbb{N}$. Then any hyperfunction $[u] \in \mathcal{B}(\mathbb{R})$ has a representing function $h \in [u]$ such that $|h(z)| \leqslant 1/k$ on F.*

The theorem was stated without proof by Sato [14]. Classical proofs are based on Hörmander's $\overline{\partial}-$ techniques (by an idea of B.A. Taylor, see [8, Lemma 1]) or on the flabbiness of the sheaf of hyperfunctions (see [2, Theorem 8.4.4] and also [7] and [9]) which is usually also proved by solving the $\overline{\partial}-$ equation.

In the present paper, we will first present a new short proof of this central result using the well known Köthe duality and the surjectivity criterion [11, Theorem

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26.1] for operators in Fréchet spaces, and we will then discuss the natural question which are the possible bounds for representing functions of general hyperfunctions.

A more refined version of the above argument will lead to the following characterization: let $\mathcal{V} := \{v_n \mid n \in \mathbb{N}\}$ be an increasing system of continuous (weight) functions $v_n : \mathbb{C} \setminus \mathbb{R} \to [0, \infty[$. Let $F_n, n \in \mathbb{N}$, be a closed exhaustion of $\mathbb{C} \setminus \mathbb{R}$ and let

$$\mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) := \{f \in H(\mathbb{C} \setminus \mathbb{R}) \mid \forall n \in \mathbb{N} : \sup_{z \in F_n} |f(z)| e^{v_n(z)} < \infty\}.$$

Using some mild assumptions on F_n and $\mathcal{V}$ we get (see Theorem 3.1)

Theorem. *The following are equivalent:*

(a) *Any* $[u] \in \mathcal{B}(\mathbb{R})$ *has a representing function* $h \in \mathcal{HV}(\mathbb{C} \setminus \mathbb{R})$.
(b) $\mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) \cap H(\mathbb{C}) \neq 0$
(c) $\mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) \cap H(\mathbb{C} \setminus \{0\}) \neq 0$

Though there is no representing function of Dirac's $\delta-$distribution decaying at infinity faster than $1/|z|$ we finally get the following surprising improvement of Kaneko's and Komatsu's theorem (see Corollary 3.3):

Let $F \subset \mathbb{C} \setminus \mathbb{R}$ be closed and $k \in \mathbb{N}$. Then any hyperfunction $[u] \in \mathcal{B}(\mathbb{R})$ has a representing function $h \in [u]$ such that $|z^k h(z)| \leqslant 1/k$ on F and such that $|z^n h(z)|$ is bounded on F for any n.

For Fourier and Laplace transformation of hyperfunctions it is interesting to notice that we can always find a representing function in the space

$$\mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R}) := \{f \in H(\mathbb{C} \setminus \mathbb{R}) \mid \forall k \in \mathbb{N} : \sup_{1/k \leqslant |\operatorname{Im} z| \leqslant k} |f(z)| e^{k|\operatorname{Re} z|} < \infty\}.$$

(see Corollary 2.4).

Using the extension of $[u] \in \mathcal{B}(\mathbb{R})$ to $\overline{[u]} \in \mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R})/\mathcal{H}_{-\infty}(\mathbb{C})$ we can thus define an asymptotic Fourier transform on $\mathcal{B}(\mathbb{R})$. This will be discussed in detail in [10].

Representing functions with bounds as above also exist in the case of several variables. Also, the results are easily transferred to the Fréchet valued situation which is needed for applications e.g. to the abstract Cauchy problem.

2. The theorem of Kaneko and Komatsu

We will give a new short proof of the theorem of Kaneko and Komatsu in this section. Recall that $[u]$ is a hyperfunction on $\mathbb{R}$ if

$$[u] \in \mathcal{B}(\mathbb{R}) := H(\mathbb{C} \setminus \mathbb{R})/H(\mathbb{C})$$

and that $h \in H(\mathbb{C} \setminus \mathbb{R})$ is called a representing function of $[u]$ if $h \in [u]$.

Let F_n be a closed exhaustion of $\mathbb{C} \setminus \mathbb{R}$, that is,

$$F_n \text{ is closed}, \qquad F_n \subset \operatorname{int}(F_{n+1}) \quad \text{for any } n \qquad \text{and} \qquad \bigcup_{n \in \mathbb{N}} F_n = \mathbb{C} \setminus \mathbb{R}.$$

Let $\mathcal{V} := \{v_n \mid n \in \mathbb{N}\}$ be an increasing system of continuous (weight) functions $v_n : \mathbb{C} \setminus \mathbb{R} \to [0, \infty[$. We define the Fréchet space $\mathcal{HV}(\mathbb{C} \setminus \mathbb{R})$ by

$$\mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) := \{f \in H(\mathbb{C} \setminus \mathbb{R}) \mid \forall n \in \mathbb{N} : \|f\|_n := \sup_{z \in F_n} |f(z)| e^{v_n(z)} < \infty\}.$$

The main result of this section is the following slight improvement of the theorem of Kaneko and Komatsu mentioned in the introduction:

Theorem 2.1. *Let $F \subset \mathbb{C} \setminus \mathbb{R}$ be closed and let $k \in \mathbb{N}$. Then any $[u] \in \mathcal{B}(\mathbb{R})$ has a representing function h such that $|zh(z)| \leqslant 1/k$ on F.*

The proof of Theorem 2.1 is based on the Köthe - duality: For a compact $K \subset \mathbb{C}$ and $\nu \in H(K)'$ let $S(\nu) := u_\nu$ where

$$u_\nu(t) := \frac{i}{2\pi} \langle_\xi \nu, 1/(t - \xi)\rangle, t \notin K.$$

Theorem 2.2 ([4]). *Let $K \subset \mathbb{C}$ be compact. Then*

$$S : H(K)'_b \to H(\mathbb{C} \setminus K)/H(\mathbb{C})$$

is a topological isomorphism and

$$\nu(f) = \int_\gamma u_\nu(\xi) f(\xi) d\xi \text{ for any } f \in H(U) \tag{2.1}$$

if U is a neighborhood of K and γ is a path in U around K with clockwise orientation.

Proof of Theorem 2.1. Choose a closed exhaustion F_n of $\mathbb{C} \setminus \mathbb{R}$ such that $F_1 := F$ and set $\mathcal{V} := \{\ln_+(|z|)\}$.

(a) We first show that the mapping

$$T : \mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) \times H(\mathbb{C}) \to H(\mathbb{C} \setminus \mathbb{R}), (f, g) \to f + g,$$

is surjective. Since the transpose of T is

$${}^tT(\nu) = (\nu \mid_{\mathcal{HV}(\mathbb{C}\setminus\mathbb{R})}, \nu \mid_{H(\mathbb{C})}) \qquad \text{for } \nu \in H(\mathbb{C} \setminus \mathbb{R})'$$

we have to show the following (by the surjectivity criterion [11, 26.1]) :

$B \subset H(\mathbb{C}\setminus\mathbb{R})'$ is equicontinuous on $H(\mathbb{C}\setminus\mathbb{R})$ if B is equicontinuous on $\mathcal{HV}(\mathbb{C}\setminus\mathbb{R})$ and on $H(\mathbb{C})$.

Since B is equicontinuous on $H(\mathbb{C})$ there are $n \in \mathbb{N}$ and $C_j > 0$ such that B is equicontinuous on $H(K_n)$, $K_n := \{z \in \mathbb{C} \mid |z| \leqslant n\}$, and

$$|u_\nu(z)| \leqslant C_1 \sup_{\xi \in K_n} |1/(z - \xi)| \leqslant C_2 \qquad \text{if } \nu \in B \text{ and } n + 1 \leqslant |z| \leqslant n + 2. \tag{2.2}$$

Since B is equicontinuous on $\mathcal{HV}(\mathbb{C} \setminus \mathbb{R})$ there are $n_0 \in \mathbb{N}$ and $C_3 > 0$ such that we get for $x \in I_n := [-n - 2, n + 2]$ and $l \in \mathbb{N}_0$

$$|u_\nu^{(l)}(x)| = \frac{l!}{2\pi} |\langle_\xi \nu, (x - \xi)^{-l-1}\rangle| \leqslant C_3 l! \|(x - \ \cdot\ )^{-l-1}\|_{n_0} \leqslant C_4 l! \operatorname{dist}(x, F_{n_0})^{-l}$$

where the first equality is due to the fact that the difference quotients $[1/(x-\ \cdot\ )-1/(x+h-\ \cdot\ )]/h, h \in \mathbb{R}$, converge in $H(\mathbb{C}\setminus\mathbb{R})$. Hence the functions u_ν (being holomorphic near $\mathbb{R}$ by Theorem 2.2) can be holomorphically extended such that $\{u_\nu \mid \nu \in B\}$ is uniformly bounded on $\{z \in \mathbb{C} \mid |\operatorname{Re} z| < n+2, |\operatorname{Im} z| < \gamma/2\}$ for $\gamma :=$ dist(I_n, F_{n_0}). By Theorem 2.2 and (2.2), this shows a).

(b) The mapping T is surjective by a) and hence open. Thus there are a compact $K \subset \mathbb{C}\setminus\mathbb{R}$ and $C_5 > 0$ such that $[u] \in \mathcal{B}(\mathbb{R})$ (with representing function $v \in H(\mathbb{C}\setminus\mathbb{R})$) has a representing function $f \in \mathcal{HV}(\mathbb{C}\setminus\mathbb{R})$ such that

$$\sup_{z\in F} |zf(z)| \leqslant \|f\|_1 \leqslant C_5 \sup_{z\in K} |v(z)|.$$

Since $H(\mathbb{C})$ is dense in $H(\mathbb{C}\setminus\mathbb{R})$ by Runge's theorem, we may choose v such that $C_5 \sup_{z\in K} |v(z)| \leqslant 1/k$. The theorem is proved. ■

The bounds obtained in Theorem 2.1 will be essentially improved in Corollary 3.3 below.

As we mentioned already in the introduction, Theorem 2.1 directly implies the extension of Sato's hyperfunctions to Fourier hyperfunctions $\mathcal{Q}(\mathbb{R})$ and to modified Fourier hyperfunctions $\mathcal{R}(\mathbb{R})$. For the convenience of the reader we recall the respective definitions: Let $F_n := \{z \in \mathbb{C} \mid 1/n \leqslant |\operatorname{Im} z| \leqslant n\}$ and

$$\mathcal{O}_{exp}(\mathbb{C}\setminus\mathbb{R}) := \{f \in H(\mathbb{C}\setminus\mathbb{R}) \mid \forall n \in \mathbb{N} : \sup_{z\in F_n} |f(z)|e^{-|z|/n} < \infty\}$$

and

$$\mathcal{O}_{exp}(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \forall n \in \mathbb{N} : \sup_{|\operatorname{Im} z|\leqslant n} |f(z)|e^{-|z|/n} < \infty\}.$$

The Fourier hyperfunctions are defined by $\mathcal{Q}(\mathbb{R}) := \mathcal{O}_{exp}(\mathbb{C}\setminus\mathbb{R})/\mathcal{O}_{exp}(\mathbb{C})$. The modified Fourier hyperfunctions $\mathcal{R}(\mathbb{R})$ are defined similarly using the radial compactification of $\mathbb{C}$, especially $\widetilde{F_n} := \{z \in \mathbb{C} || \operatorname{Im} z| \geqslant \frac{1}{n}(1+|\operatorname{Re} z|)\}$ is used instead of F_n in the definition above (see [13]).

Corollary 2.3. *The canonical (restriction) mappings*

$$R : \mathcal{Q}(\mathbb{R}) \to \mathcal{B}(\mathbb{R}) \textit{ and } \mathcal{R}(\mathbb{R}) \to \mathcal{B}(\mathbb{R})$$

are surjective.

Proof. This follows from Theorem 2.1 with $F := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \geqslant 1/(1+|\operatorname{Re}(z)|)\}$. ■

The next example provides an elementary theory of Fourier and Laplace transformation (see [10]) : let

$$\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R}) := \{f \in H(\mathbb{C}\setminus\mathbb{R}) \mid \forall n \in \mathbb{N} : \sup_{1/n\leqslant|\operatorname{Im} z|\leqslant n} |f(z)|e^{n|\operatorname{Re} z|} < \infty\}$$

and

$$\mathcal{H}_{-\infty}(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \forall k \in \mathbb{N} : \sup_{|\operatorname{Im} z|\leqslant k} |f(z)|e^{k|\operatorname{Re} z|} < \infty\}.$$

Corollary 2.4. *The canonical (restriction) mapping*

$$R : \mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R})/\mathcal{H}_{-\infty}(\mathbb{C}) \to \mathcal{B}(\mathbb{R})$$

is surjective.

Proof. For $[u] \in \mathcal{B}(\mathbb{R})$ and $f(z) := e^{z^2}$ let h be a representing function for $f[u] := [fu]$ by Theorem 2.1 with $F := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \geqslant 1/(1 + |\operatorname{Re}(z)|)\}$. Then h/f is a representing function for $[u]$ as desired. ■

In [2, 8.4.5] representing functions h were obtained such that, for fixed $k \in \mathbb{N}$, $|h(z)|e^{k|z|}$ is bounded on each strip F_n.

$\mathcal{H}_{-\infty}(\mathbb{C})$ is the space of test functions for the Fourier ultra-hyperfunctions (see [12]).

The decay of representing functions in Corollary 2.4 can easily be improved. By essentially the same proof we obtain:

Remark 2.5. Let $F_n := \{z \in \mathbb{C} \mid 1/n \leqslant |\operatorname{Im} z| \leqslant n\}$ and $\mathcal{V} := \{e^{n|\operatorname{Re} z|^k} \mid n \in \mathbb{N}\}$ for fixed $k \in \mathbb{N}$. Then the canonical (restriction) mapping

$$R : \mathcal{HV}(\mathbb{C} \setminus \mathbb{R})/\mathcal{HV}(\mathbb{C}) \to \mathcal{B}(\mathbb{R})$$

is surjective.

Proof. Apply the proof of Corollary 2.4 with $f(z) := e^{z^l}$ for $l > k$ even. ■

3. Bounds for representing functions

In view of the different bounds obtained so far for representing functions of hyperfunctions it is an interesting question how far we can push the bounds for representing functions. From now on we will generally assume that $\mathcal{HV}(\mathbb{C} \setminus \mathbb{R})$ is invariant under real shifts and finite derivatives. To have this guaranteed (by application of Cauchy's formula) we will use the following assumptions:

$$\forall n \in \mathbb{N}, x \in \mathbb{R}\ \exists k > n, \gamma > 0 : \qquad F_n - B_\gamma(x) \subset F_k. \tag{3.1}$$

$$\forall n \in \mathbb{N}, x \in \mathbb{R}\ \exists k > n, \gamma, C > 0 \forall z \in F_n : \qquad v_n(z) \leqslant C + \inf_{|x-\xi| \leqslant \gamma} v_k(z - \xi). \tag{3.2}$$

(3.1) is clearly satisfied by the standard choices $F_n := \{z \in \mathbb{C} \mid 1/n \leqslant |\operatorname{Im} z| \leqslant n\}$ or $\widetilde{F}_n := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \geqslant (1 + |\operatorname{Re} z|)/n\}$ used for the examples above.

We now have the following characterization:

Theorem 3.1. *Let F_n and $\mathcal{V}$ satisfy* (3.1) *and* (3.2). *The following are equivalent:*

(a) *Any $[u] \in \mathcal{B}(\mathbb{R})$ has a representing function $h \in \mathcal{HV}(\mathbb{C} \setminus \mathbb{R})$.*
(b) $\mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) \cap H(\mathbb{C}) \neq 0$
(c) $\mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) \cap H(\mathbb{C} \setminus \{0\}) \neq 0$

Proof. "$(a) \Rightarrow (b)$" If $\mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) \cap H(\mathbb{C}) = 0$, then the mapping

$$T : \mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) \times H(\mathbb{C}) \to H(\mathbb{C} \setminus \mathbb{R}), (f, g) \to f + g,$$

is injective. Since T is surjective by assumption, T is a topological isomorphism, hence $H(\mathbb{C})$ is a closed subspace of $H(\mathbb{C} \setminus \mathbb{R})$, a contradiction, since $H(\mathbb{C})$ is dense in $H(\mathbb{C} \setminus \mathbb{R})$ by Runge's theorem.

"$(b) \Rightarrow (c)$" This is evident.

"$(c) \Rightarrow (a)$" We modify the proof of Theorem 2.1: Let $B \subset H(\mathbb{C} \setminus \mathbb{R})'$ be equicontinuous on $H(\mathbb{C})$ and on $\mathcal{HV}(\mathbb{C} \setminus \mathbb{R})$. Then there are $n \in \mathbb{N}$ and $C_j > 0$ such that

$$|\nu(h)| \leqslant C_0 \sup_{|z| \leqslant n} |h(z)| \qquad \text{if } \nu \in B \text{ and } h \in H(\mathbb{C}) \tag{3.3}$$

and thus

$$|u_\nu(z)| \leqslant C_1 \qquad \text{if } \nu \in B \text{ and } n + 1 \leqslant |z| \leqslant n + 2 \tag{3.4}$$

and such that

$$|\nu(h)| \leqslant C_2 \|h\|_n \qquad \text{if } \nu \in B \text{ and } h \in \mathcal{HV}(\mathbb{C} \setminus \mathbb{R}). \tag{3.5}$$

Choose $g \in \mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) \cap H(\mathbb{C} \setminus \{0\})$ by assumption. Then $h(z) := g(z)z^{-d}$ also satisfies the assumptions for g since for any j

$$|h(z)| \leqslant |g(z)|(\operatorname{dist}(0, F_j))^{-d} \qquad \text{for } z \in F_j.$$

We may thus assume that the singularity of g at 0 is not removable. By Theorem 2.2, g then is a representing function for $0 \neq \mu \in H(\{0\})'$, that is,

$$g = u_\mu + h \qquad \text{for some } h \in H(\mathbb{C}).$$

By (3.1) and (3.2) we know that $g^{(l)}(\,\cdot\, - x) \in \mathcal{HV}(\mathbb{C} \setminus \mathbb{R})$ for $x \in \mathbb{R}$, hence we get for $x \in I_n := [-n-2, n+2]$ by (3.3) and (3.5)

$$\begin{aligned} |(\frac{d}{dx})^l u_{\nu*\mu}(x)| &= \frac{l!}{2\pi} |\langle_\xi (\nu * \mu), (x - \xi)^{-l-1}\rangle| = \frac{l!}{2\pi} |\langle_\xi \nu, \langle_\eta \mu, (\xi - x - \eta)^{-l-1}\rangle\rangle| \\ &= |\langle_\xi \nu, u_\mu^{(l)}(\xi - x)\rangle| \leqslant |\langle_\xi \nu, g^{(l)}(\xi - x)\rangle| + |\langle_\xi \nu, h^{(l)}(\xi - x)\rangle| \\ &\leqslant C_2 \|g^{(l)}(\,\cdot\, - x)\|_n + C_3 l! \leqslant C_4 l! \gamma^{-l}(\|g\|_k + 1) \end{aligned} \tag{3.6}$$

where k and γ are chosen uniformly for $x \in I_n$ by compactness and by (3.1) and (3.2). Using also (3.4), the functions $u_{\nu*\mu}$ can thus be holomorphically extended such that $\{u_{\nu*\mu} \mid \nu \in B\}$ is uniformly bounded on $\{z \in \mathbb{C} \mid |\operatorname{Re} z| \leqslant n+2, |\operatorname{Im} z| \leqslant \gamma/2 \text{ or } n+1 \leqslant |z| \leqslant n+2\}$. Therefore, $\{\nu * \mu \mid \nu \in B\}$ is equicontinuous on $H(\mathbb{C} \setminus \mathbb{R})$ by Theorem 2.2. Since the convolution operator $\check{\mu}*$ is surjective on $H(\mathbb{C} \setminus \mathbb{R})$ for any $0 \neq \mu \in H(\{0\})'$ by [6] this implies that B is equicontinuous on $H(\mathbb{C} \setminus \mathbb{R})$. The theorem is proved. ■

Of course, for given $k \in \mathbb{N}$ the representing function h in Theorem 3.1 can be chosen such that $\|h\|_k \leqslant 1/k$ (by part b) of the proof of Theorem 2.1).

The decay at infinity of the representing function in Theorem 2.1 seems to be optimal since there is no representing function for Dirac's $\delta-$distribu- tion decaying on $\mathbb{C}$ at infinity faster than the standard representing function $-1/(2\pi i z)$. However, on the closed exhaustion $F_n := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \geqslant 1/n\}$ of $\mathbb{C} \setminus \mathbb{R}$ much better bounds for representing functions can be obtained. The relevant result is the following

Theorem 3.2. *Let $F_n := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \geqslant 1/n\}$. Let $\mathcal{V}$ satisfy (3.2) and*

$$\exists 0 < \delta < 1 \ \forall n \in \mathbb{N}: \qquad v_n(z) = O(|z|^\delta) \quad \text{on } F_n. \tag{3.7}$$

Then any $[u] \in \mathcal{B}(\mathbb{R})$ has a representing function $h \in \mathcal{HV}(\mathbb{C} \setminus \mathbb{R})$ if for any $k \in \mathbb{N}$ there is $0 \neq g \in H(\mathbb{C} \setminus \{0\})$ such that $\|g\|_k < \infty$ and such that $|g(z)| \leqslant C_j \exp(C_j |z|^\delta)$ on F_j for any j.

Proof. We modify the estimate for $|\langle_\xi \nu, g^{(l)}(\xi - x)\rangle|$ in (3.6): For $\delta < \alpha < 1$ let $f_m(z) := e^{-(-iz)^\alpha/m}$ for $\operatorname{Im}(z) > 0$ and $f_m(z) := e^{-(iz)^\alpha/m}$ for $\operatorname{Im}(z) < 0$ where $(\ )^\alpha$ is chosen positive on $]0, \infty[$. Then there is $\varepsilon > 0$ such that

$$|f_m(z)| \leqslant \exp(-\varepsilon |z|^\alpha/m) \leqslant 1 \qquad \text{on } \mathbb{C} \setminus \mathbb{R}. \tag{3.8}$$

This implies that

$$h_m := f_m g^{(l)}(\,\cdot\, - x) \in \mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) \qquad \text{for } l \in \mathbb{N}_0 \tag{3.9}$$

since also $|g^{(l)}(\,\cdot\, - x)| \leqslant \widetilde{C}_j e^{\widetilde{C}_j |z|^\delta}$ on any F_j by Cauchy's estimate and therefore

$$|f_m(z) g^{(l)}(z - x)| \leqslant \widetilde{C}_j \exp(-\varepsilon |z|^\alpha/m + \widetilde{C}_j |z|^\delta) \leqslant D_j e^{-v_j(z)} \qquad \text{on } F_j$$

by (3.7). Since $f_m \to 1$ uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$ we have $\lim_m h_m = g^{(l)}(\,\cdot\, - x)$ in $H(\mathbb{C} \setminus \mathbb{R})$. We thus get for $x \in I_n := [-n-2, n+2]$ by (3.9), (3.5) and (3.8)

$$\begin{aligned} |\langle_\xi \nu, g^{(l)}(\xi - x)\rangle| &= \lim_m |\langle_\xi \nu, h_m\rangle| \leqslant C_1 \sup_m \|h_m\|_n \\ &\leqslant C_1 \|g^{(l)}(\,\cdot\, - x)\|_n \leqslant C_2 l! \gamma^{-l} \|g\|_k \end{aligned}$$

where k and γ are chosen uniformly for $x \in I_n$ by compactness and by (3.2). The conclusion now follows as in the proof of "$(c) \Rightarrow (b)$" of Theorem 3.1. ■

Corollary 3.3. *Let $F \subset \mathbb{C} \setminus \mathbb{R}$ be closed and let $k \in \mathbb{N}$. Then any $[u] \in \mathcal{B}(\mathbb{R})$ has a representing function h such that $|z^k h(z)| \leqslant 1/k$ on F and such that $|z^n h(z)|$ is bounded on F for any n.*

Proof. Choose a closed exhaustion F_n of $\mathbb{C} \setminus \mathbb{R}$ such that $F_1 := F$. We may apply the proof of Theorem 3.2 for $g(z) := z^{-k}$ since the special form of F_n assumed in 3.2 is not needed in the present case. In fact it is clear that $h_m := f_m g^{(l)}(\,\cdot\, - x) \in \mathcal{HV}(\mathbb{C} \setminus \mathbb{R})$ for $l \in \mathbb{N}_0$ and that $\|g^{(l)}(\,\cdot\, - x)\|_n \leqslant C_1 C_2^l l!$. ■

Andreas Schmidt [15] has studied rapidly decreasing hyperfunctions $[u]$ (i.e. $|z^n u(z)| \leqslant C_n$ if $1/n \leqslant |\operatorname{Im} z| \leqslant n$ for any n) in connection with asymptotic expansions at ∞ and quantum field theory.

The assumption (3.7) is not very restrictive for our problem since there are no representing functions h decreasing exponentially on $F_n := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \geqslant 1/n\}$. More precisely we have

Remark 3.4. Let $F_n := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \geqslant \frac{1}{n}\}$ and $\mathcal{V} := \{n \ln_+(|z|) | n \in \mathbb{N}\}$. If $0 \neq h \in \mathcal{HV}(\mathbb{C}_+)$ then $|h(z)|e^{\varepsilon|\operatorname{Re} z|}$ is unbounded on any strip $F_\varepsilon(y_0) := \{z \in \mathbb{C} \mid |\operatorname{Im} z - y_0| < \varepsilon\}$ for any $y_0 > \varepsilon > 0$.

Proof. If $|h(z)|e^{\varepsilon|\operatorname{Re} z|}$ is bounded on $F_\varepsilon(y_0)$ for some $y_0 > \varepsilon > 0$, then $H(t) := h(t+iy_0)$ is holomorphic for $|\operatorname{Im}(t)| < \varepsilon$ and $|H(t)|e^{\varepsilon|\operatorname{Re} t|}$ is bounded for $|\operatorname{Im}(t)| < \varepsilon$, hence H is in the Gelfand/Shilov space S_1^1 (see [1]) and therefore also $\widehat{H} \in S_1^1$ (see [1]). Especially, $\widehat{H}$ is holomorphic on a strip near $\mathbb{R}$. Since $h \in \mathcal{HV}(\mathbb{C}_+)$ we get by Cauchy's theorem for any $n \in \mathbb{N}_0$ and any $y \geqslant 0$

$$|\widehat{H}^{(n)}(0)| = |\int_{\mathbb{R}} (x+iy)^n H(x+iy)dx| \leqslant C_1 \int_{\mathbb{R}} |x + i(y_0+y)|^{-2} dx \to 0$$

for $y \to \infty$. Hence $\widehat{H}^{(n)}(0) = 0$ for $n \in \mathbb{N}_0$ and thus $H = 0$ and $h = 0$. ■

It is not known if any $[u] \in \mathcal{B}(\mathbb{R})$ has a representing function h such that for some $\delta > 0$, $\sup_{|\operatorname{Im} z| \geqslant 1/n} |h(z)|e^{|z|^\delta} < \infty$ for any $n \in \mathbb{N}$.

4. Hyperfunctions supported in $[0, \infty[$

The results obtained sofar can also be proved for hyperfunctions $[u]$ with support in $[0, \infty[$, i.e. for

$$[u] \in \mathcal{B}([0,\infty[) := H(\mathbb{C} \setminus [0,\infty[)/H(\mathbb{C}).$$

We will only state the part of the results here which is needed in [10]. The formulation of the remaining statements is left to the reader. Let

$$G_k := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \leqslant k\} \setminus \{z \in \mathbb{C} \mid \operatorname{Re} z > -1/k \text{ and } |\operatorname{Im} z| < 1/k\}$$

and let $\mathcal{V} := \{v_n \mid n \in \mathbb{N}\}$ be an increasing system of continuous (weight) functions $v_n : \mathbb{C} \setminus [0,\infty[\to [0,\infty[$. We define the spaces $\mathcal{HV}(\mathbb{C} \setminus [0,\infty[)$ similarly as before by

$$\mathcal{HV}(\mathbb{C} \setminus [0,\infty[) := \{f \in H(\mathbb{C} \setminus [0,\infty[) \mid \forall n \in \mathbb{N} : \sup_{z \in G_n} |f(z)|e^{v_n(z)} < \infty\}.$$

Notice that representing functions in $\mathcal{HV}(\mathbb{C} \setminus [0,\infty[)$ also satisfy bounds near $-\infty$, hence the following result is stronger than Theorem 2.1 in that respect.

Theorem 4.1. *Let $F \subset \mathbb{C} \setminus [0,\infty[$ be closed and let $k \in \mathbb{N}$. Then any $[u] \in \mathcal{B}([0,\infty[)$ has a representing function h such that $|zh(z)| \leqslant 1/k$ on F.*

Proof. We repeat the proof of Theorem 2.1 with $\mathbb{C}\setminus[0,\infty[$ instead of $\mathbb{C}\setminus\mathbb{R}$ and $x\in[0,n+2]$ instead of $x\in I_n$. ■

The following example will be used for an elementary theory of an asymptotic Laplace transformation in [10]: let

$$\mathcal{H}_{-\infty}(\mathbb{C}\setminus[0,\infty[) := \{f\in H(\mathbb{C}\setminus[0,\infty[) \mid \forall k\in\mathbb{N}: \sup_{z\in G_k}|f(z)|e^{k|\operatorname{Re} z|}<\infty\}.$$

As before we get

Corollary 4.2. *The canonical (restriction) mapping*

$$R:\mathcal{H}_{-\infty}(\mathbb{C}\setminus[0,\infty[)/\mathcal{H}_{-\infty}(\mathbb{C})\to\mathcal{B}([0,\infty[)$$

is surjective.

5. Vector valued hyperfunctions

Using the $\pi-$tensorproduct it is no major effort to transfer our results to hyperfunctions with values in Fréchet spaces. This is interesting for application of the Laplace transform to the abstract Cauchy problem for closed operators in Fréchet spaces (see [10]). Similarly, results for hyperfunctions in several variables can be obtained from the single variable case considered sofar. Both extensions of the preceding results are shortly discussed in this section.

Let X be a Fréchet space with system $(\|\ \ \|_n)_{n\in\mathbb{N}}$ defining the topology. The space of X-valued holomorphic functions on an open set $U\subset\mathbb{C}$ is denoted by $H(U,X)$. The space of X-valued hyperfunctions on $\mathbb{R}$ (supported in $[0,\infty[$, respectively) is by definition

$$\mathcal{B}(\mathbb{R},X) := H(\mathbb{C}\setminus\mathbb{R},X)/H(\mathbb{C},X)$$

$$(\text{and } \mathcal{B}([0,\infty[,X) := H(\mathbb{C}\setminus[0,\infty[,X)/H(\mathbb{C},X), \text{ respectively}).$$

For a closed exhaustion F_k of $\mathbb{C}\setminus\mathbb{R}$ and a system $\mathcal{V}$ of weight functions we define $\mathcal{HV}(\mathbb{C}\setminus\mathbb{R},X)$ by

$$\mathcal{HV}(\mathbb{C}\setminus\mathbb{R},X) := \{f\in H(\mathbb{C}\setminus\mathbb{R},X) \mid \forall n\in\mathbb{N}: \sup_{z\in F_n}\|f(z)\|_n e^{v_n(z)}<\infty\}.$$

The spaces $\mathcal{HV}(\mathbb{C},X)$ (and $\mathcal{HV}(\mathbb{C}\setminus[0,\infty[,X)$) are defined similarly (see also Section 4).

Theorem 5.1. *Assume that*

$$\forall n\in\mathbb{N}\ \exists m\in\mathbb{N}: \qquad v_n(z)+2\ln(1+|z|)\leqslant C+v_m(z) \quad \text{on } F_n. \tag{5.1}$$

If the canonical mappings

$$R:\mathcal{HV}(\mathbb{C}\setminus\mathbb{R})\to\mathcal{B}(\mathbb{R}) \quad (\text{and } R:\mathcal{HV}(\mathbb{C}\setminus[0,\infty[)\to\mathcal{B}([0,\infty[), \text{ resp.})$$

are surjective, then the canonical mappings

$$R : \mathcal{HV}(\mathbb{C} \setminus \mathbb{R}, X) \to \mathcal{B}(\mathbb{R}, X) \quad (\textit{and } R : \mathcal{HV}(\mathbb{C} \setminus [0,\infty[, X) \to \mathcal{B}([0,\infty[, X))$$

are surjective.

Proof. By assumption the mappings

$$T : H(\mathbb{C}) \times \mathcal{HV}(\mathbb{C} \setminus \mathbb{R}) \to H(\mathbb{C} \setminus \mathbb{R}), \qquad (f, g) \to f + g$$

and

$$T : H(\mathbb{C}) \times \mathcal{HV}(\mathbb{C} \setminus [0,\infty[) \to H(\mathbb{C} \setminus [0,\infty[), \qquad (f, g) \to f + g$$

are surjective. Since X is a Fréchet space, the mappings

$$T \otimes id : H(\mathbb{C})\widehat{\otimes}_\pi X \times \mathcal{HV}(\mathbb{C} \setminus \mathbb{R})\widehat{\otimes}_\pi X \to H(\mathbb{C} \setminus \mathbb{R})\widehat{\otimes}_\pi X$$

and

$$T : H(\mathbb{C})\widehat{\otimes}_\pi X \times \mathcal{HV}(\mathbb{C} \setminus [0,\infty[)\widehat{\otimes}_\pi X \to H(\mathbb{C} \setminus [0,\infty[)\widehat{\otimes}_\pi X$$

are also surjective by [5, §41.5(7)]. $H(U)$ is nuclear for open $U \subset \mathbb{C}$. By the mean value property of holomorphic functions and (5.1), the spaces $\mathcal{HV}(\mathbb{C} \setminus \mathbb{R})$ and $\mathcal{HV}(\mathbb{C} \setminus [0,\infty[)$ could be defined using L_1-norms instead of $\sup -$norms. Hence these spaces are also nuclear. This shows the claim since $H(U, X) = H(U)\widehat{\otimes}_\pi X, \mathcal{HV}(\mathbb{C} \setminus \mathbb{R}, X) = \mathcal{HV}(\mathbb{C} \setminus \mathbb{R})\widehat{\otimes}_\pi X$ and $\mathcal{HV}(\mathbb{C} \setminus [0,\infty[, X) = \mathcal{HV}(\mathbb{C} \setminus [0,\infty[)\widehat{\otimes}_\pi X$ by nuclearity. ■

The following Corollary will be applied in [10]:

Corollary 5.2. *Let X be a Fréchet space. The canonical mappings*

$$R : \mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R}, X) \to \mathcal{B}(\mathbb{R}, X) \qquad \textit{and} \qquad R : \mathcal{H}_{-\infty}(\mathbb{C}\setminus[0,\infty[, X) \to \mathcal{B}([0,\infty[, X)$$

are surjective.

We finally discuss representing functions for hyperfunctions on $\mathbb{R}^k$ and hyperfunctions supported in the cone $[0,\infty[^k$, respectively. These can be treated similar to the vector valued case. We shortly recall the respective definitions: Let $F \subset \mathbb{C}$ be closed and let

$$\mathbb{C}^k \sharp F^k := (\mathbb{C} \setminus F)^k$$

and

$$\mathbb{C}^k \sharp_j F^k := (\mathbb{C} \setminus F) \times \cdots \times (\mathbb{C} \setminus F) \times \mathbb{C} \times (\mathbb{C} \setminus F) \times \cdots \times (\mathbb{C} \setminus F)$$

where the factor $\mathbb{C}$ is put in the j^{th} place. The spaces of hyperfunctions on $\mathbb{R}^k$ (and of hyperfunctions supported in $[0,\infty[^k$, respectively) are defined by

$$\mathcal{B}(\mathbb{R}^k) := H(\mathbb{C}^k \sharp \mathbb{R}^k) / \sum_{j=1}^{k} H(\mathbb{C}^k \sharp_j \mathbb{R}^k)$$

and

$$\mathcal{B}([0,\infty[^k) := H(\mathbb{C}^k\sharp[0,\infty[^k)/\sum_{j=1}^{k} H(\mathbb{C}^k\sharp_j[0,\infty[^k).$$

Let $\mathcal{V}_j := \{v_{j,n} \mid n \in \mathbb{N}\}, j = 1,\dots,k$, be systems of weight functions on $\mathbb{C}\setminus\mathbb{R}$ as before and set $\mathcal{V} := \{v_n(z) := \sum_{j=1}^{k} v_{j,n}(z_j) \mid n \in \mathbb{N}\}$. For $j = 1,\dots,k$ let $F_{j,n}, n \in \mathbb{N}$, be closed exhaustions of $\mathbb{C}\setminus\mathbb{R}$ and set $F_n := F_{1,n}\times\cdots\times F_{k,n}$. Then

$$\mathcal{HV}(\mathbb{C}^k\sharp\mathbb{R}^k) := \{f \in H(\mathbb{C}^k\sharp\mathbb{R}^k) \mid \forall n \in \mathbb{N}: \sup_{z\in F_n} |f(z)|e^{v_n(z)} < \infty\}.$$

Theorem 5.3. *Let $\mathcal{V}_j$ satisfy (5.1) for $j \leqslant k$. If the canonical mappings*

$$R_j : \mathcal{HV}_j(\mathbb{C}\setminus\mathbb{R}) \to \mathcal{B}(\mathbb{R}) \quad \textit{(and } R_j : \mathcal{HV}_j(\mathbb{C}\setminus[0,\infty[) \to \mathcal{B}([0,\infty[), \textit{ resp.)}$$

are surjective for $j = 1,\dots,k$, then the canonical mappings

$$R : \mathcal{HV}(\mathbb{C}^k\sharp\mathbb{R}^k) \to \mathcal{B}(\mathbb{R}^k) \quad \textit{(and } R : \mathcal{HV}(\mathbb{C}^k\sharp[0,\infty[^k) \to \mathcal{B}([0,\infty[^k))$$

are also surjective.

Proof. The proof is given only for $\mathcal{HV}(\mathbb{C}^k\sharp\mathbb{R}^k)$. The other case is treated similarly. The mappings

$$T_j : \mathcal{HV}_j(\mathbb{C}\setminus\mathbb{R}) \times H(\mathbb{C}) \to H(C\setminus\mathbb{R}), \qquad (f,g) \to f+g,$$

are surjective by assumption. Since X is a Fréchet space, the mapping

$$T := \widehat{\otimes}_{j=1}^{k} T_j : \widehat{\otimes}_{j=1}^{k}\big(\mathcal{HV}_j(\mathbb{C}) \times H(\mathbb{C}\setminus\mathbb{R})\big) \to \widehat{\otimes}_{j=1}^{k} H(C\setminus\mathbb{R}) = H(\mathbb{C}^k\sharp\mathbb{R}^k)$$

is surjective by [5, §41.5(7)]. (5.1) implies that the spaces $\mathcal{HV}_j(\mathbb{C}\setminus\mathbb{R})$ are nuclear and hence

$$\mathcal{HV}(\mathbb{C}^k\sharp\mathbb{R}^k) = \mathcal{HV}_1(\mathbb{C}\setminus\mathbb{R})\widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi \mathcal{HV}_k(\mathbb{C}\setminus\mathbb{R}).$$

Since also $H(\mathbb{C}\setminus\mathbb{R})$ is nuclear, similar equations hold for the tensor products of some factors $H(\mathbb{C})$ and some factors $\mathcal{HV}_j(\mathbb{C}\setminus\mathbb{R})$. Since these tensor products are all contained in $\sum_{j=1}^{k} H(\mathbb{C}^k\sharp_j\mathbb{R}^k)$ the theorem is proved. ■

Corollary 5.4. *The canonical mappings*

$$R : \mathcal{H}_{-\infty}(\mathbb{C}^k\sharp\mathbb{R}^k) \to \mathcal{B}(\mathbb{R}^k) \textit{ and } R : \mathcal{H}_{-\infty}(\mathbb{C}^k\sharp[0,\infty[^k) \to \mathcal{B}([0,\infty[^k)$$

are surjective.

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