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THE MATHEMATICAL WORK OF SUSANNE DIEROLF

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Dedicated to the memory of Susanne Dierolf

1. Introduction

In April 2009, Susanne Dierolf died at the age of 66 years. For almost 40 years she contributed an overwhelming amount of most original mathematical ideas mainly to the theory of topological vector spaces. The aim of this article, thus, cannot be to collect each single result due to Susanne Dierolf, and we will not even mention all of her publications, which, nevertheless, are all included in the bibliography. What we try to do is to present a selection of results and methods which we consider as her most important contributions to functional analysis and which are, according to our personal belief, typical for her highly original, creative, and accurate way of doing mathematics. Among these highlights are the definite solutions to four open problems posed by her "mathematical hero" A. Grothendieck and, though not quite explicit, the solution of a longstanding conjecture of V.A. Raĭkov in category theory.

Most of these and many more answers to questions from further mathematicians are "negative" and occasionally Susanne had been called "Mrs. Counterexample". However, as we will try to describe in this article, what one could call counterexample is very often an extremely clever existence theorem, which, by clarifying the reason for the failure of a too general conjecture, often triggered further research ending in "positive" applicable theorems.

The main area of Susanne Dierolf's mathematical work was the theory of topological or locally convex vector spaces (tvs and lcs, for short) with an emphasis on Fréchet spaces and their close relatives. We will thus concentrate on these topics neglecting her contributions to other branches of mathematics like general topology and, in particular, her work on topological groups culminating in the joint book [24] with W. Roelcke. This does of course not mean that we consider the latter less important but only that we do not feel competent to judge them properly.

2. General topological vector spaces and locally convex spaces

2.1. Incomplete quotients

Susanne Dierolf entered the scene of functional analysis in the seventies with a spectacular and curios result about incomplete quotients of complete topological vector spaces. It is of course a classical result that quotients of completely metrizable topological vector spaces (or even groups or even uniform spaces) are complete (for instance, the composition of the quotient map with the embedding into the completion is almost open hence surjective by the Schauder lemma), but beyond this situation not much can be said in general. G. Köthe constructed a closed subspace of a complete LB-space with incomplete corresponding quotient as well as a Montel space with a quotient which is not Montel, see [Köt69, §31]. What Susanne did in [4, 3] was to show that these examples are not at all rare pathological phenomena:

Theorem 1. *Every tvs (lcs) is the quotient of a complete tvs (lcs) all whose bounded sets are finite dimensional (and which is therefore a semi-Montel space).*

Let us sketch her proof in the separated and locally convex case. One of the ingredients is that every separated lcs $(X,\mathcal{T})$ is a *closed* (!) subspace of another one having very good properties: Forming the completion of $(X,p)/\operatorname{kern}(p)$ for every continuous seminorm p one gets a product Y of Banach spaces containing $(X,\mathcal{T})$ as a subspace. This is of course a standard procedure, but clearly X is closed in Y only if $(X,\mathcal{T})$ is complete. Here is the amazing trick to make X a *closed* subspace of a space Z which shares many of the properties of Y:

Denoting by Y' and Y^* the continuous and algebraic duals and $X^\perp = \{f \in Y^* : f|_X = 0\}$, Susanne defined

$$Z = \prod_{f\in Y'\cap X^\perp} \ker(f) \times \left(\left(\prod_{f\in X^\perp\setminus Y'} \ker(f)\right) + \left(\bigoplus_{f\in X^\perp\setminus Y'} Y\right)\right)$$

endowed with the relative topology of the huge product $\prod_{f\in X^\perp} Y$. Now the first factor of Z is easily seen to share most good properties of Y^I since $\ker(f)$ is then complemented in Y and one only has to deal with the second factor which is "very dense" in Y^J. Using her results on associated locally convex topologics – the main subject of the article [3] containing the present theorem – Susanne proved that Z is a barrelled space such that there is no strictly finer locally convex topology which coincides with the topology of Z on all absolutely convex compact sets – this property is called $\mathcal{C}_a$. Almost the same construction (modifying only the space Y) shows that one can assume that Z carries its weak topology $\sigma(Z,Z')$ if $\mathcal{T} = \sigma(X,X')$. The diagonal map $X \to Z$, $x \mapsto (x)_{f\in X^\perp}$ identifies X with a subspace of Z which can be shown to be closed. Now we are prepared for the proof of the theorem if $(X,\mathcal{T})$ is a Mackey space (i.e. there is no strictly finer locally convex topology on X having the same dual, which, by Mackey's theorem, means that $\mathcal{T}$ is the Mackey topology $\tau(X, X')$ of uniform convergence on all absolutely convex weak*-compact sets of X'): Applying the above construction to $(X', \sigma(X', X))$ one gets a barrelled space Z with $\mathcal{C}_a$ carrying its weak topology and containing $(X', \sigma(X', X))$ as a closed subspace. Then $E = (Z', \tau(Z', Z))$ is the desired complete space with finite dimensional bounded sets having X as a quotient: completeness follows from Grothendieck's construction of the completion using that Z has property $\mathcal{C}_a$, the bounded sets are finite dimensional since Z carries its weak topology and a direct calculation shows $E/(X')^{\perp} \simeq X$, since X is Mackey. If X is any separated locally convex space the first part of the proof shows in particular that there is a Mackey superspace X_1 of X, and if X_1 is a quotient of E_1 as required it is easy to obtain X as a quotient of $E = q^{-1}(X)$ where q is the quotient map.

Let us make some remarks on this proof which, for a functional analyst who learned his matters from, say, Rudin's textbook [Rud91] or, if he is specialized in the theory of locally convex spaces, Köthe's books [Köt69, Köt79] or even the one of Bonet and Perez-Carreras [PCB87], is certainly extremely astonishing. Such ideas did not come from nothing, one might feel an influence of Köthe's "konvergenzfreie Räume", and in any case constructions on a similar level of abstraction were discussed in the functional analysis group at Munich led by W. Roelcke, the advisor of Susanne's PhD thesis, further members were P. Dierolf, V. Eberhardt, P. Lurje, U. Schwanengel, and others.

Let us also remark that the second factor in the construction of Z is of the form $\prod_{i\in I} E + \bigoplus_{i\in I} F$ with a subspace E of a locally convex space F. This construction anticipated, in a certain sense, a type of locally convex spaces which turned out to be a rich source of examples and counterexamples.

2.2. Three-space problems

Having a nice class of topological or locally convex spaces one is quickly led to ask for stability properties of that class with respect to operations like forming closed subspaces, separated quotients, products, sums, direct (= inductive) limits, inverse (=projective) limits, or arbitrary initial or final linear or locally convex topologies. Typically it is not too hard to prove the stability properties which are actually satisfied but it is already less simple to find concrete examples showing that a stability property fails. Let us only mention one such non-trivial result published jointly with P. Lurje in the "comptes rendus" [6] (this, by the way, is the only article published as *Suzanne* Dierolf): There is a bornological and barrelled space containing a countable codimensional non-bornological subspace.

Here we want to describe another stability property, namely the three-space property. A property of objects in a fixed category like tvs or lcs is a three-space property if an object X has the property whenever there is a subobject L of X such that L and X/L have it. In other words, if $0 \to E \to F \to G \to 0$ is an exact sequence (F is then called an extension of E and G) the property inherits from E and G to F (a three-space property is thus stable by extensions). Put in this form, the problem has some relevance for splitting theories which ask for a right inverse of $F \to G$ in the exact sequence above (modulo an obvious equivalence relation, there is thus only the trivial extension $E \times G$): this can be much easier if one knows a priori that F has certain properties. Classical examples of three-space properties in the categories of tvs or lcs are metrizability and completeness. In her dissertation [2] Susanne had a list of 24 properties and she decided for almost each of them whether it is a three-space property or not. Many of these results were published jointly with Roelcke in [23]. We will give lists of such properties below but let us first mention a typical result showing how Susanne solved many such three-space problems:

Theorem 2. *Let $\mathcal{C}$ be a class of tvs (lcs) such that*

(a) *$\mathcal{C}$ is stable w.r.t. separated quotients,*
(b) *$X \in \mathcal{C}$ whenever X contains a onedimensional separated subspace L with $X/L \in \mathcal{C}$,*
(c) *there are $X \in \mathcal{C}$ and $a \in \tilde{X} \setminus X$ (where $\tilde{X}$ is the completion) such that $X + [a] \notin \mathcal{C}$,*
(d) *there is $Z \in \mathcal{C}$ with $Z' \neq Z^*$.*

Then $\mathcal{C}$ is not three-space-stable.

Before indicating the main idea of the proof let us mention that the theorem yields that, for instance, the classes of bornological and ultrabornological lcs are not three-space stable ((d) is trivial with any infinite dimensional Banach space and (c) can be seen with $X = \{f : \mathbb{R} \to \mathbb{R} : \{t \in \mathbb{R} : f(t) \neq 0\}$ countable$\}$ endowed with the topology of pointwise convergence and a the constant function 1).

Sketch of proof. Since $Z' \neq Z^*$ there is a dense hyperplane Y of $Z \times \mathbb{K}$ not containing $b = (0, 1) \in Z \times \mathbb{K}$. The trick is now the construction of a strictly coarser topology on $X \times Y$ making X a topological subspace such that the corresponding quotient is Y: Consider the subspace $X \times Y + [(a, b)]$ of the completion $\tilde{X} \times \tilde{Y}$ and factor out the additional line $[(a, b)]$. Then $X \times Y$ (with this coarser topology) does not belong to $\mathcal{C}$ since $(X + Y)/Y$ is topologically isomorphic to $(X + [a])/[a]$ which does not belong to $\mathcal{C}$ because of (b) and (c). ■

The most striking result in Susanne's dissertation is perhaps the fact that the class of DF-spaces is not three-space stable – since the DF-property was conceived by Grothendieck [Gro54] as the dual property of metrizability this could have been hoped. Let us mention that it seems to be unknown whether the classes of complete LB- or even LS-spaces are three-space stable.

The dissertation also contains several important positive three-space theorems, e.g. the classes of Schwartz or nuclear spaces are three-space stable. Now for the promised lists. Except for semimetrizability, which in [23] is attributed to M. I. Graev [Gra50], and completeness, which was probably first proved by N. Ya. Vilenkin [Vil48], to our best knowledge all results either stem from Susanne's dissertation [2], the joint paper with Roelcke [23], or other collaborations of Susanne.

The following are three-space properties in the category of locally convex spaces:

- semimetrizability
- seminormability
- barrelledness
- completeness
- countable barrelledness
- Schwartz
- nuclear
- Mackey
- reflexive Fréchet
- Fréchet-Montel
- quasinormable Fréchet
- countable neighbourhood condition

The following fail to be three-space properties:

- bornological
- ultrabornological
- quasibarrelled
- DF-space
- having a fundamental sequence of bounded sets
- distinguishedness
- semireflexive
- semi-Montel
- quasicompleteness
- sequential completeness
- local (= Mackey) completeness
- quasinormability
- distinguished Fréchet
- dual Fréchet space

The only question Susanne was not able to solve in her dissertation asked whether local convexity is a three-space property in the category of topological vector spaces. What she managed to do was an astonishing reduction of the problem: it suffices to investigate the inheritance of local convexity from X/L to X for onedimensional subspaces. The question was eventually solved in the negative by Kalton [Kal78] and, independently, Ribe [Rib79].

2.3. LB-spaces are not three-space stable

In early 2009 a PhD student at Trier, D. Sieg, gave a series of seminar talks on the homological foundation of the splitting theory for PLB- and PLS-spaces (countable projective limits of LB- or LS-spaces). This would be evident if PLB or PLS would be a three-space property which, apparently, was not known. Only very recently Susanne's son Bernhard Dierolf found some notes of Susanne pointing in this direction, most probably it was the last mathematical question she was working on. Susanne proved that LB is not a three-space property:

Though it is not known whether the completion of every LB-space is again LB, there are many examples of incomplete LB-spaces L with LB-completion $(E,\mathcal{T})$ – in the last published paper [71] of Susanne it is shown that this is always so in the class of so-called Moscatelli-LB-spaces. Moreover one can have $\operatorname{card}(E/L) = \operatorname{card}(\mathbb{R})$ and there is thus a separated LB-topology $\mathcal{R}$ on E/L (algebraically, E/L is isomorphic to each LB-space with separable steps). Now consider the initial topology $\mathcal{S}$ on E with respect to the identity $E \to (E,\mathcal{T})$ and the quotient map $E \to (E/L,\mathcal{R})$. Since $\mathcal{T}/L$ is the trivial (coarsest) topology due to the fact that L is dense, one gets $\mathcal{S}/L = \mathcal{R}$, and since L is the kernel of the quotient map, one has $\mathcal{S} \cap L = \mathcal{T} \cap L$. Moreover, $\mathcal{S}$ is strictly finer than $\mathcal{T}$ – and thus not an LB-topology because of the open mapping theorem for LB- or LF-spaces – but $\mathcal{S}/L$ and $\mathcal{S} \cap L$ are LB-topologies.

Arguing with the open mapping theorem for webbed spaces one sees that $(E,\mathcal{S})$ is not a PLB-space either and one can have $\mathcal{T}/L$ even a Hilbert space topology.

The crux in the counterexample was of course the incompleteness of the LB-space L and thus, the three-space properties for complete LB-spaces, LS-spaces, and PLS-spaces remain open. Susanne noticed however that LS is indeed a three-space property within the class of lcs having a fundamental sequence of bounded sets. Although she gave a different proof this can be seen by duality: If $0 \to X \to Y \to Z \to 0$ is exact with X and Z LS and Y having a fundamental sequence of bounded sets then, according to the above list, Y is barrelled, complete, and Schwartz and hence a reflexive DF-space. The dual sequence $0 \to Z'_\beta \to Y'_\beta \to X'_\beta \to 0$ is then topologically exact, too, whence Y'_β in Fréchet-Schwartz and $Y = Y''$ is LS.

3. Fréchet and DF-spaces

Although Susanne liked very much to consider mathematical problems just as brain-twisters where each question has equal right to be posed as a riddle she knew of course that there are specific properties in the huge zoo of locally convex conditions which are much more important for applications than others and she spent a lot of energy to clarify the relevant topics in the theory of locally convex spaces, specifically, the structure and duality theory of Fréchet spaces.

3.1. The lifting of bounded sets

Whenever one uses duality theory, e.g. to get hold of compactness arguments via the Banach-Alaoğlu theorem, one is quickly led to lifting problems for bounded sets. Indeed, if $q : X \to Y$ is a quotient map between locally convex spaces then Y' can be considered as the subspace $(\ker q)^\circ$ of X' and using the theorem of bipolars one gets that the strong topologies $\beta(Y',Y)$ and $\beta(X',X)$ coincide on $(\ker q)^\circ$ if and only if q lifts bounded sets with closure, i.e., every bounded set of Y is contained in $\overline{q(B)}$ for some bounded set in X.

Also many three-space properties become much easier if one knows that the involved quotient map lifts bounded sets (with or without closure). For example, it is shown in [23] that (ultra-)bornological is a three-space property under the additional assumption of lifting of bounded sets (or only local null sequences). Another example is in [40] where it is shown that distinguishedness is three-space stable in the class of Fréchet spaces under the assumption of lifting with closure.

The following theorem of Susanne and José Bonet [45] was a great surprise and served as a corner stone in Meise's and Vogt's theorem that the dual of a short exact sequence $0 \to E \to F \to G \to 0$ of Fréchet spaces is again topologically exact if and only if the quotient map $F \to G$ lifts bounded sets [MV97, proposition 26.12].

Theorem 3. *A quotient map between Fréchet spaces lifts bounded sets with closure if and only if it lifts without closure.*

The proof of Susanne and Bonet was a quite delicate Mittag-Leffler construction. A variant of the argument which makes it easier to get a starting point for the proof is in [Wen03]: It is quite obvious that $q : X \to Y$ lifts bounded sets if and only if the induced map $\tilde{q} : \ell_I^\infty(X) \to \ell_I^\infty(Y)$ between the bounded I-families in X and Y is surjective for every index set I and, in view of the Schauder lemma, this is the case if $\tilde{q}$ is almost open, i.e. *closures* of images of 0-neighbourhoods are again 0-neighbourhoods.

It is remarkable that the theorem does not hold for each bounded set individually. The example is of the following form which Susanne and Bonet called *Moscatelli-type* because a similar construction allowed V. B. Moscatelli to construct so-called twisted Fréchet spaces, i.e., strict projective limits of Banach spaces which are not isomorphic to a countable product of Banach spaces. Take two Banach spaces X, Y such that $X \hookrightarrow Y$ and set $G = X^{\mathbb{N}} + c_0(Y)$. The corresponding intersection $X^{\mathbb{N}} \cap c_0(Y)$ is called a Fréchet space of Moscatelli type. For this type of spaces (and more general variants where c_0 is replaced by other normal (= solid) Banach sequence spaces and sequences of Banach spaces $X_n \hookrightarrow Y_n$ are considered) Susanne and Bonet characterized almost all locally convex properties Fréchet spaces may have in terms of X and Y, see [33, 35]. If X is not closed in Y one obtains bounded sets in G which are lifted with closure (by the quotient map $+ : X^{\mathbb{N}} \times c_0(Y) \to G$) but not without closure.

3.2. Distinguishedness

The duality theory for Fréchet spaces is faced with the annoying fact that the strong dual of a Fréchet space may fail to be barrelled and bornological and that, therefore, basic principles of functional analysis like the Banach-Steinhaus theorem are not directly applicable. The first example is due to Köthe and Grothendieck but later on more and more such Fréchet spaces showed up, for instance the space $C(\mathbb{R}) \cap L^1(\mathbb{R})$ of Lebesgue-integrable continuous functions on the real line [Tas89].

Grothendieck called a Fréchet space distinguished if its strong dual is barrelled and he proved in his fundamental article [Gro54] that this is the same as being ultrabornological, hence an LB-space. There he asked whether the bidual of a distinguished Fréchet space is again distinguished (question non-résolue no. 5). Note that this is a question about the third dual of a Fréchet space and very subtle differences between the various topologies one has there. In a joint work with Bonet and C. Fernández [41] Susanne managed to show that $E = X^{\mathbb{N}} \cap c_0(Y)$ is always distinguished wehreas its bidual is distinguished if and only if X is closed in Y. Thus, $X = \ell_1$ and $Y = \ell_2$ yield:

Theorem 4. *There are distinguished Fréchet spaces with non-distinguished bidual.*

3.3. DF-spaces

Grothendieck conceived the class of DF-spaces as a model for duals of Fréchet (or metrizable) locally convex spaces, in particular, strong duals of Fréchet spaces are DF and strong duals of DF-spaces are Fréchet. Therefore, the strong bidual of a DF-space E, i.e., the strong dual of $(E', \beta(E', E))$, is a DF-space. However, there is a different locally convex topology on the bidual which Grothendieck called *natural*, namely the locally convex topology having $\{U^{\circ\circ}, U \in \mathscr{U}_0(E)\}$ as a basis of the 0-neighbourhood filter. Grothendieck asked [Gro54, page 78] whether E'' endowed with this topology is again DF. This was confirmed by Susanne in [22]:

Theorem 5. *For every DF-space the natural bidual is again DF.*

Another question from [Gro54] asked whether the space of $L_b(E, F)$ of continuous linear operators from a Fréchet space E to a DF-space F (endowed with the topology of uniform convergence on the bounded sets of E) is always DF. It is clear that $L_b(E, F)$ has a fundamental sequence of bounded sets and thus, the question is whether $L_b(E, F)$ is countably quasibarrelled (i.e., if U is the intersection of countably many absolutely convex 0-neighbourhoods and absorbs all bounded sets, then U should be itself a 0-neighbourhood).

As Grothendieck remarked in [Gro55, chap. I, §1.1] this question is intimately connected with the even more famous "problème des topologies" whether each bounded set in the completed tensor product $X \tilde{\otimes}_\pi Y$ of two Fréchet spaces is contained in a bounded set of canonical form.

In her Habilitationsschrift "On spaces of continuous linear mappings between locally convex spaces" [27] Susanne made among many other things a very serious attempt to Grothendieck's questions and naturally related problems. She did not solve them as stated but she obtained besides several positive results very clever examples showing that $L_b(E, F)$ need not be barrelled (quasibarrelled, bornological, ultrabornological) even if so are E'_β and F. The "problème des topologies" was solved in the negative afterwards by Taskinen [Tas86] using also ideas from [27]. Susanne then noticed that Taskinen's example also led to an example where $L_b(E, F)$ is not DF.

Let us explain some ingredients of Susanne's approach to Grothendieck's question for the case of even the Banach space $E = \ell_1$ and a DF-space F. Of course, one could try to *characterize* in terms of F when $L_b(E, F)$ has a certain property and then check whether it is always satisfied. This turns out to be difficult but Susanne proved an important necessary condition for the barrelledness of $L_b(\ell_1, F)$ [27, proposition 4.5]:

Theorem 6. *Let F be a locally complete DF-space and $(B_n)_{n\in\mathbb{N}}$ an increasing sequence of closed absolutely convex bounded sets such that*

(a) *the sequence $(nB_n)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets,*
(b) $\forall U \in \mathscr{U}_0(F)\ \exists B \subseteq F$ *bounded* $\forall n \in \mathbb{N}$ *one has* $U \cap B \not\subseteq B_n$.

Then $L_b(\ell_1, F)$ is not quasibarrelled.

This theorem is of the kind of results which are harder to find than to prove: For $B_n^* = \{T \in L(\ell_1, F) : T(B_{\ell_1}) \subseteq B_n\}$ one checks quite easily that $W = \frac{1}{2}\overline{\bigcup_{n\in\mathbb{N}} B_n^*}$ is a bornivorous barrel and assuming $L_b(\ell_1, F)$ quasibarrelled one gets $U \in \mathscr{U}_0(F)$ such that W contains $U^* = \{T \in L(\ell_1, F) : T(B_{\ell_1}) \subseteq U\}$. Using (b) one gets a bounded sequence $x_n \in U \setminus B_n$, and since B_n is closed there are $V_n \in \mathscr{U}_0(F)$ with $x_n \notin B_n + V_n$. Now $V = \bigcap_{n\in\mathbb{N}} \frac{1}{2}B_n + V_n$ is bornivorous in F and hence $V \in \mathscr{U}_0(F)$ so that

$$U^* \subseteq \overline{\bigcap_{n\in\mathbb{N}} B_n^*} \subseteq \frac{1}{2}\bigcup_{n\in\mathbb{N}} B_n^* + V^*.$$

Defining $T : \ell_1 \to F$ by $(\lambda_n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} \lambda_n x_n$ one gets $T \in U^*$ and hence $T \in \frac{1}{2}B_n^* + V^*$ for some $n \in \mathbb{N}$. For $\lambda = (\delta_{n,k})_{k\in\mathbb{N}}$ one has $x_n = T(\lambda)$, and this yields the contradiction $x_n \in B_n + V_n$.

Having the theorem it "remains" to find a DF-space F with these properties. Susanne gave two different classes of such spaces F which are even complete LB-spaces, namely one of Moscatelli type like $F = \bigoplus_{n\in\mathbb{N}} X + \ell_p(Y)$ for two Banach spaces $Y \hookrightarrow X$ such that Y is not closed in X, and also a certain Köthe LB-space F (a co-echelon space of order p). In the latter case, Susanne showed that also $L_b(\ell_q, F)$ is not quasibarrelled for all $1 \leqslant q < \infty$. One can thus have even a Hilbert space E and an inductive limit F of Hilbert spaces such that $L_b(E, F)$ is not quasibarrelled.

Theorem 6 above and many other results in Susanne's Habilitationsschrift had been very influential in further research on Fréchet spaces and their relatives, in particular in the work of Bierstedt and Bonet about the (dual) density condition which led to many important characterizations in Fréchet space theory, a survey of all this is [BB03].

4. Inductive limits

Dieudonné and Schwartz [DS49] introduced LF-spaces as countable *strict* inductive limits of Fréchet spaces in order to have a good general setting for spaces $\mathcal{D}(\Omega)$ of test functions for distribution theory. In [Gro54] Grothendieck answered all questions posed by Dieudonné and Schwartz and in [Gro55] he used the term LF-space (more precisely, "espace $(\mathscr{L}\mathscr{F})$") for general countable inductive limits of Fréchet spaces and gave several examples of non-strict LF-spaces which appear naturally in analysis like the space of all bounded operators (i.e., mapping some

0-neighbourhood into a bounded set) between Fréchet spaces. As Grothendieck noted, this LF-space is, in general, not complete, its completeness is characterized in [56]. Also in his "thèse" [Gro55] he proved the fundamental theorems called A and B about LF-spaces (the factorization theorems saying that every operator with closed graph from a Fréchet space into an LF-space factorizes over some step, and the open mapping and closed graph theorems for LF-spaces) which despite their simplicity had always been among Susanne's favorites.

4.1. Duality of inductive limits

Conceptually, inductive and projective (or direct and inverse) limits are dual concepts and a good understanding thus requires to know the precise relations between an inductive limit $E = \operatorname{ind} E_n$ and its strong dual. In [Gro54] Grothendieck thus asked whether the bidual of a strict LF-space is the inductive limit of the biduals and whether this holds if the steps are distinguished Fréchet spaces (Grothendieck wrote: *"J'avais annoncé oralement ce dernier résultat, mais ne retrouvant pas la démonstration, il y avait peut-être erreur"*). In a joint work with Bonet [31], Susanne solved these questions in the negative with, again, very simple LF-spaces of the form

$$E = \bigoplus_{n\in\mathbb{N}} X + Y^{\mathbb{N}}$$

where Y is a closed subspace of a Fréchet space X. Algebraically, the bidual is then $\bigoplus_{n\in\mathbb{N}} X'' + (Y'')^{\mathbb{N}}$, and Bonet and Susanne could show that one can find X and Y such that the strong topology of E'' differs from the LF-topology even if X and Y are distinguished.

Related problems were also studied jointly with Bonet and the second author of the present note in [67]. There, a slightly more general class of LF-spaces was investigated and the characterization for the special class suggested a *positive* answer to Grothendieck's question which could indeed be confirmed:

Theorem 7. *Let $E = \operatorname{ind} E_n$ be an inductive limit of quasinormable Fréchet spaces such that $c_0(E) = \bigcup_{n\in\mathbb{N}} c_0(E_n)$ (this retractivity condition is, in particular, satisfied by strict LF-spaces). Then E''_β is again an LF-space.*

4.2. Metrizable LF-spaces and another question of Grothendieck

That LB- and LF-spaces can be quite pathological was quickly realized after their creation when Köthe constructed an incomplete LB-space [Köt69, §31.6]. On the other hand, proper LB-spaces $X = \operatorname{ind} X_n$ (with $X \neq X_n$ for all $n \in \mathbb{N}$) are never metrizable: Otherwise, since metrizable spaces have the countable boundedness property, there would be $\varepsilon_n > 0$ such that $B = \Gamma\Big(\bigcup_{n\in\mathbb{N}} \varepsilon_n B_n\Big)$ is a bounded zero-neighbourhood in X and therefore, $B \subseteq \bar{B}_n$ for some $n \in \mathbb{N}$. But then, $id : X_n \to X$ is almost open and hence open and surjective by the Schauder lemma. As Susanne showed, the situation for LF-spaces is even worse [22]:

Theorem 8. *There are inductive limits of nuclear Fréchet spaces which are incomplete, metrizable, and carry their weak topology.*

The construction is similar as above.: For $X = \omega$ and $Y = s$ the LF-space $E = \bigoplus_{n\in\mathbb{N}} X + s^{\mathbb{N}}$ is a dense topological subspace of $X^{\mathbb{N}} \cong \omega$ (which of course needs a proof).

This simple class of examples enabled Susanne to answer yet another question of Grothendieck [Gro54, question non résolue 6]. Given a closed subspace F of a locally convex space E one has two "strong topologies" on $E'/F^\circ \cong F'$, namely $\beta(E', E)/F^\circ$ and $\beta(F', F)$, and Grothendieck proved that they coincide if F is quasibarrelled and $(F', \beta(F', F))$ bornological. In general these topologies are different and he asked whether, at least, they always have the same bounded sets.

It is now no surprise that Susanne's solution in general is negative, but the construction is very involved, the simple class of LF-spaces above being only the starting point. The main idea is that for $z \in \tilde{E}\setminus E$ one obtains a strictly coarser locally convex topology on $E = (E+[z])/[z]$ which is again barrelled and metrizable, but a very fine analysis is needed to investigate the corresponding strong topologies (metrizability of the coarser topology is essential to see that the topologies have different bounded sets).

4.3. Completeness of LB- and LF-spaces

We already mentioned Köthe's incomplete LB-space, and since then one of the main problem in the theory of locally convex spaces had been to characterize completeness of inductive limits. In particular, one would like to know the completeness of locally complete LB-spaces (every closed, absolutely convex, and bounded set is the unit ball of a Banach space or, equivalently, every bounded set is contained and bounded in some step, i.e., the inductive limit is regular). It seems that Klaus-Dieter Bierstedt, who died only four weeks after Susanne, attributed this problem to Grothendieck. However, we could not find this question in Grothendieck's work. What he did ask, and this is the last open problem from his thesis [Gro55], is whether every quasicomplete LF-space (i.e., all closed bounded sets are complete) is necessarily complete. Anyway, both questions are obviously related and the best way to answer them would be a proof that locally complete LF-spaces are complete (Susanne awarded a laurel leaf from her garden for the proof of this result for inductive limits of Fréchet Montel spaces).

Although Susanne could not solve these problems, she always came back to them and their relatives. A very serious attempt was made in a joint work with Paweł Domański [50, 47] on an apparently quite different subject.

A celebrated theorem of Davis, Figiel, Johnson, and Pełczyński [DFJP74] says that every weakly compact operator between Banach spaces factorizes through a reflexive Banach space. In the realm of locally convex spaces there are two natural candidates for the generalization of (weakly) compact operators: Either mapping a 0-neighbourhood into a relatively (weakly) compact set or mapping all bounded sets into relatively (weakly) compact ones. The former operators are still called (weakly) compact (and questions reduce quite easily to the Banach case) and for the latter Susanne and Domański coined the term Montel (reflexive) operator. The natural question is then whether every Montel operator between Fréchet spaces factorizes through a Fréchet Montel space and, by duality, whether every Montel operator between LB-spaces factorizes through a Montel LB-space. Although the question in general remained open, Susanne and Domański obtained a very surprising "meta-result":

Theorem 9. *Consider the following conditions.*

(a) *Every locally complete LB-space is complete.*
(b) *Every Montel operator between LB-spaces factorizes through a Montel LB-space.*
(c) *For every complete LB-space F the space $C(\beta\mathbb{N}, F)$ is bornological.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) holds.

The main point in the proof of this result is the investigation of spaces $C(\Omega, F)$ of continuous F-valued functions on a compact set Ω and an LB-space F and its subspace $C_f(\Omega, F)$ of *factorizable* continuous functions $f : \Omega \to F$, i.e., $f(\Omega)$ is contained in the unit ball of a Banach space $E \hookrightarrow F$ such that this inclusion factorizes through a Montel LB-space. Susanne and Domański proved that $C_f(\Omega, E)$ is always a locally complete LB-space which is dense in $C(\Omega, E)$.

Along their investigations they discovered a surprising new phenomenon: If F is not locally complete there are local Cauchy sequences without limits. Adding limits (which exist in the completion of F) one gets a new space $F^{(1)}$ which has more local Cauchy sequences and thus still may fail to be locally complete. Defining $F^{(\alpha+1)} = (F^{(\alpha)})^{(1)}$ for successor ordinals and $F^{(\beta)} = \bigcup_{\alpha<\beta} F^{(\alpha)}$ for limit ordinals one eventually ends up with a locally complete space but it is shown in [47]:

Theorem 10. *There is an LB-space F such that for each countable ordinal α the LB-space $F^{(\alpha)}$ is not locally complete.*

Let us briefly indicate the difficulty when trying to prove that $C(\Omega, F)$ is bornological for a (complete) LB-space $F = \text{ind } F_n$: The typical 0-neighbourhoods in $C(\Omega, F)$ are of the form

$$U^* = \{f \in C(\Omega, F) : f(\Omega) \subseteq U\}$$

for 0-neighbourhoods U in F and a similar description holds for the bounded sets. Moreover, if B_n are the unit balls of F_n, a typical 0-neighbourhood in F is of the form $\sum_{n\in\mathbb{N}} \varepsilon_n B_n = \bigcup_{N\in\mathbb{N}} \sum_{n=1}^{N} \varepsilon_n B_n$. Therefore, $C(\Omega, F)$ is bornological if and only if for every sequence $(\varepsilon_n)_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ there is $(\delta_n)_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ such that every continuous $f : \Omega \to F$ with values in $\sum_{n\in\mathbb{N}} \delta_n B_n$ can be written as $f = \sum_{n=1}^{N} f_n$ with continuous functions $f_n : \Omega \to \varepsilon_n B_n$.

It seems that there is no general device for such a decomposition of continuous functions. It was thus a surprise that nevertheless something can be said at least in simple cases [50]:

Theorem 11. *For every Montel LB-space F the space $c_0(F)$ of null sequences in F is bornological.*

This is indeed the special case $\Omega = \alpha\mathbb{N}$, the Alexandrov compactification of the natural numbers, and by an easy induction argument one sees that $\delta_n = \varepsilon_n/2^{n+1}$ are feasible once one has proved that every null sequence $x \in (K_1 + K_2)^{\mathbb{N}}$ can be written as the sum of two null sequences $y \in (2K_1)^{\mathbb{N}}$ and $z \in (2K_2)^{\mathbb{N}}$ where K_1 and K_2 are absolutely convex compact sets of a normed space. For this one has to show for the unit ball B

$$\forall \alpha > 0\, \exists \beta > 0 \quad (K_1 + K_2) \cap \beta B \subseteq (2K_1 \cap \alpha B) + (2K_2 \cap \alpha B),$$

and assuming the contrary one gets $\alpha > 0$ and $x_n \in (K_1 + K_2) \cap \frac{1}{n}B$ which are not in the right hand side of the above inclusion. Writing $x_n = a_n + b_n$ and using the compactness of K_1 on gets $a \in K_1$ and a subsequence $a_{n_k} \to a$ Then $b_{n_k} = x_{n_k} - a_{n_k} \to -a \in \bar{K}_2 = K_2$, and thus the contradiction $x_{n_k} = (a_{n_k} - a) + (a + b_{n_k}) \in 2K_1 \cap \alpha B + 2K \cap \alpha B$ for k large enough.

Readers interested in this topic are asked to prove that there is $M > 0$ such that for all absolutely convex compact sets K_1, K_2 of a normed space and all compact spaces Ω one has

$$C(\Omega, K_1 + K_2) \subseteq MC(\Omega, K_1) + MC(\Omega, K_2).$$

Some more results related to the bornologicity of $C(\Omega, F)$ are contained in [FW08].

5. Topological algebras

Mainly in connection with the research of three of her PhD students, Susanne contributed several ideas to the theory of topological algebras and published five papers [60, 62, 63, 65, 68] about this subject.

There are many different possibilities to define categories of topological algebras in order to balance good structural properties and, of course, applicability, namely by requiring that

– multiplication is separately continuous (semitopological algebras),
– multiplication is jointly continuous (topological algebras),
– the topology is defined by a family of submultiplicative seminorms (locally m-convex or *lmc* algebras).

The literature is not consistent concerning the first two notions and it even happened that authors did not realize that the notion of semitopological algebras clashes with a number of usual constructions, for instance, if A is only semitopological then $C(\Omega, A)$, the space of continuous A-valued functions on a topological space Ω, need not be an algebra (e.g., for $\Omega = A \times A$ and the two projections the product is not continuous).

5.1. Inductive limits of topological algebras

Another pitfall occurs when dealing with inductive limits of, say, an increasing sequence of topological algebras. In one of the standard references of the theory it is claimed that the locally convex inductive limit of locally convex topological algebras is always a topological algebra and that only local m-convexity is in question. Since the locally convex limit only respects linear structures and multiplication is bilinear this is not very plausible and indeed completely wrong [60]:

Theorem 12. *There is a strict inductive limit of commutative lmc Fréchet algebras which is not a topological algebra.*

The strategy to prove this result is again to characterize for Fréchet algebras $Y \hookrightarrow X$ when the "Moscatelli type" algebra $\bigoplus_{n\in\mathbb{N}} X + \ell_\infty(Y)$ is locally m-convex and then to find suitable "entries" X and Y. The characterization of m-convexity is quite complicated but as simple entries one can take Y as the algebra of entire functions and X as the algebra of continuous functions on $\mathbb{C}$.

In view of this example one could say that locally convex inductive limits are not suitable for the category of locally convex topological algebras and one should instead consider the inductive limit in that category (i.e., the finest locally convex topology making all inclusions continuous and the union a topological algebra). But then another problem appears which contrasts the classical result of Dieudonné and Schwartz [DS49] that strict LF-spaces are complete:

Theorem 13. *There is a strict inductive limit in the category of locally convex topological algebras of commutative lmc Fréchet algebras which is not complete.*

Concerning *lmc* algebras, [60] contains a very simple proof that countable locally convex inductive limits of Banach algebras are *lmc* as well as some generalizations to *lmc* algebras satisfying the countable neighbourhood condition.

5.2. The three-space problem for lmc algebras

As in the case of locally convex spaces, one can ask for three-space properties in categories of topological algebras, and the first natural question is whether the class of topological algebras is three-space stable in the class of algebras with a locally convex topology, i.e., whether the continuity of the product inherits from an ideal and the corresponding quotient to the algebra.

This fails drastically. By inventing the concept of semidirect products of topological algebras, Susanne and her PhD students Khin Aye Aye and K.-H. Schröder showed in [65]:

Theorem 14. *There is a commutative algebra A endowed with a Banach space topology containing a closed ideal I such that I and A/I are Banach algebras but A is not a topological algebra.*

In view of this and the fact mentioned in 2.2 that local convexity is not a three space property, the following nice result of Susanne and her student Th. Heintz came as a surprise [68]:

Theorem 15. *Let A be a locally convex topological algebra and I a two-sided ideal such that I and A/I are locally m-convex. Then A is locally m-convex.*

The proof of this result is completely elementary. But in order to feel the difference between "elementary" and "easy" we invite the reader to try to prove the theorem himself before consulting [68].

5.3. Characters and spectra

In [63] Susanne and her coauthors investigated whether every character (linear multiplicative scalar functional) on an algebra $C(X, A)$ of algebra valued continuous functions is of the canonical form $\chi = \phi \circ \delta_x$ for some $x \in X$ and a character ϕ on A, i.e., $\chi(f) = \phi(f(x))$. As mentioned at the beginning of this section, for $C(X, A)$ being an algebra one has to assume that A is a topological algebra. In order to have such a description one needs of course that every character on $C(X)$ is an evaluation which is the case for so-called real compact spaces X (for example, σ-compact spaces satisfy this condition). A typical result is the following.

Theorem 16. *If X is real compact and A is a metrizable topological algebra, then every character on $C(X, A)$ is of the canonical form.*

It is of course not difficult to guess the canonical representation: After reducing the result to the case where A has a unit e, one can embed $C(X)$ into $C(X, A)$ by $I(f) = f \otimes e$ and one gets $x \in X$ with $\chi \circ I = \delta_x$. On the other hand one can embed A into $C(X, A)$ by $J(a) = 1 \otimes a$ and gets a character $\phi = \chi \circ J$ on A. The point of the theorem is that one does not assume a priori that χ should be continuous and it is therefore not evident how to extend the identity $\chi = \phi \circ \delta_x$ from $C(X) \otimes A$ to the whole algebra $C(X, A)$.

A similar description of all characters is obtained in [63] for the algebra of A-valued holomorphic functions $H(\Omega, A)$ on an open set $\Omega \subseteq \mathbb{C}$ where now A can be any locally complete and locally convex topological algebra.

Further investigations on the algebra $C(X, A)$ are contained in [62]. Denoting by $\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible}\}$ the spectrum of an element a of a unital algebra, Susanne and Khin Aye Aye investigated under which circumstances one has

$$\sigma_{C(X,A)}(f) = \bigcup_{x \in X} \sigma_A(f(x)).$$

6. Functorial and categorical aspects

Susanne's highly accurate way of doing mathematics was always guided by the aim to have most transparent proofs where, whenever possible, she preferred structural considerations (using, e.g., universal properties) to technical arguments. It is thus not incidental that many articles of Susanne contain abstract functorial parts either to set the scene for more concrete investigations or, more importantly, to clarify results which without a general viewpoint appear technically difficult or even cumbersome. Susanne did not even scruple to include "general abstract nonsense" in her teaching.

In this part of the article we present three of Susanne's results which are explicitely related to functors and categories: a pretty unifying result about various completeness properties of spaces of operators from her Habilitationsschrift, some remarks on associated locally convex topologies, as well as a counterexample to a longstanding conjecture of Raĭkov in category theory.

6.1. A completeness theorem

Different completeness properties are omnipresent in analysis – the ingenious art to hide burdensome approximations in their limits. Of course, almost all natural locally convex spaces appearing in applications are complete (in the sense that every Cauchy filter or net has a limit); if this is not so one should doubt whether the model is appropriate. However, natural constructions may lead to spaces having only weaker completeness properties, e.g., an infinite dimensional Banach space is never complete for its weak topology but in the reflexive case it is at least quasicomplete (in the sense that every bounded Cauchy net has a limit).

Generalising and even simplifying a result of Grothendieck Susanne investigated in [43, chapter 1] completeness properties of spaces of operators $L_{\mathfrak{M}}(E,F)$ endowed with the topology of uniform convergence on all sets of a given class $\mathfrak{M}$ of bounded subsets of the lcs E. In order to have a single result for all possible completeness properties, Susanne considered a special class of functors $\mathscr{A}$ from the category of locally convex spaces to the category of sets satisfying

(a) $\mathscr{A}(E) \subseteq \mathscr{P}(E)$ (the power set of E) for each lcs E and
(b) $\mathscr{A}(T)(A) = T(A)$ for all $A \in \mathscr{A}(E)$

In other words, $\mathscr{A}$ assigns to every lcs E a class $\mathscr{A}(E)$ of subsets of E such that $T(A) \in \mathscr{A}(F)$ for every continuous linear $T : E \to F$ and $A \in \mathscr{A}(E)$. A lcs E is then called $\mathscr{A}$-complete if $\bar{A}^E$ is complete for every $A \in \mathscr{A}(E)$, i.e., every Cauchy filter with basis in some $A \in \mathscr{A}(E)$ has a limit in E.

Theorem 17. *Let $\mathscr{A}$ be as above, E and F two lcs with $L(E,F) = L(E,F_\sigma)$ (where F_σ is the space F in its weak topology, this condition is satisfied if, e.g., E is barrelled or, more generally, a Mackey space), and $\mathfrak{M}$ a family of bounded subsets of E with $\bigcup \mathfrak{M} = E$. Then $L_{\mathfrak{M}}(E,F)$ is $\mathscr{A}$-complete whenever $E'_{\mathfrak{M}} = L_{\mathfrak{M}}(E,\mathbb{K})$ and F are $\mathscr{A}$-complete.*

The proof of this result is not too hard: Given a Cauchy net $(T_i)_{i\in I}$ in $A \in \mathscr{A}(L_{\mathfrak{M}}(E,F))$ one first finds a limit T in $F^E = \prod_{x\in E} F$ which is again $\mathscr{A}$-complete, and then one has to show $T \in L(E,F_\sigma) = L(E,F)$ and that $T_i \to T$ holds in fact in $L_{\mathfrak{M}}(E,F)$.

It is remarkable that this single theorem with its lucid proof contains many results (some of which would be even harder to prove directly). Here is a list of examples with the corresponding completeness properties:

$\mathscr{A}(E) = \mathscr{P}(E)$	complete
$\mathscr{A}(E) = \{\text{bounded sets}\}$	quasicomplete
$\mathscr{A}(E) = \{\text{Cauchy sequences}\}$	sequentially complete
$\mathscr{A}(E) = \{\Gamma(x_n : n \in \mathbb{N}) : x_n \to 0\}$	Mackey complete
$\mathscr{A}(E) = \{\Gamma(K) : K \text{ compact}\}$	convex compactness property
$\mathscr{A}(E) = \{\text{precompact sets}\}$	p-complete.

6.2. Associated topologies

The functors $\mathscr{A}$ from above appeared in fact already in Susanne's papers [3] and [8] (the latter is a joint work with her husband Peter Dierolf).

A locally convex space $(X, \mathscr{T})$ is said to have property $(\mathscr{A})$ if every locally convex topology $\mathscr{S}$ on E is coarser than $\mathscr{T}$ whenever $\mathscr{S} \cap M$ is coarser than $\mathscr{T} \cap M$ for all sets $M \in \mathscr{A}(X)$.

This property $(\mathscr{A})$ is stable under forming final locally convex topologies, so it fits into the following situation: Given some property E such that $\{0\}$ has E and which is stable under forming final locally convex topologies, Susanne defined in [3] for a lcs $(X, \mathscr{T})$ the associated topology $\mathscr{T}^E$ on X as the coarsest locally convex topology with property E finer than $\mathscr{T}$.

Examples are properties like barrelled, quasibarrelled, ultrabornological, or bornological. In [8] and [3] Susanne investigated whether the passage $\mathscr{T} \mapsto \mathscr{T}^E$ commutes with other processes, like forming products, direct sums, or completions.

We remark that Susanne also considered in [8] initial associated topologies. As applications of these theories Susanne got Theorem 1 above and

Theorem 18. *Every locally convex space appears as a closed subspace of a barrelled and compactly determined locally convex space, more precisely a barrelled locally convex space with* $(\mathscr{A})$*, where*

$$\mathscr{A}(X) = \{B \subset X : B = \overline{\Gamma(B)} \text{ compact}\}.$$

6.3. Raĭkov's conjecture

V.A. Raĭkov conjectured that every semiabelian category is quasiabelian. We will describe here Susanne's and José Bonet's example from [69] disproving this longstanding conjecture.

An additive category with kernel and cokernels is called semiabelian if for all morphisms f the canonical morphism $\bar{f} : \operatorname{coim}(f) \to \operatorname{im}(f)$ is a monomorphism and an epimorphism. A semiabelian category is called quasiabelian if in any pullback square

$$\begin{array}{ccc} P & \xrightarrow{p_f} & Y \\ {\scriptstyle p_g}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

with a cokernel f the morphism p_f is also a cokernel, and a dual condition for push-out squares holds.

These conditions are essential for numerous homological constructions in the category, for instance to have a good splitting theory in terms of the derived functor Ext^1. An important example of a non-abelian but quasiabelian category in functional analysis is the category LCS of all locally convex spaces.

The category BOR of bornological locally convex spaces is semiabelian: The cokernel of a morphism (continuous linear map) $f : E \to F$ is the canonical map $F \to F/f(E)$ where $F/f(E)$ is equipped with the (again bornological) quotient topology (consequently, a morphism is a cokernel exactly if it is open), whereas the kernel is the embedding of $f^{-1}(0) \to E$ where the subspace has to be endowed with the associated bornological topology (since subspaces of bornological spaces need not be bornological).

The canonical morphism is then (as in linear algebra) the bijection

$$\bar{f} : E/f^{-1}(0) \to f(E), \qquad x + f^{-1}(0) \mapsto f(x),$$

hence as well a mono- as an epimorphism.

Susanne and Bonet showed that the condition on pull-backs above is violated in BOR, thus disproving Raĭkov's conjecture. Given two morphisms $f : X \to Z$, $g : Y \to Z$ the pullback in BOR is

$$P = \{(x, y) \in X \times Y : f(x) = g(y)\}$$

equipped with the associated bornological topology of the product, p_f is the projection on Y, and p_g the projection on X. Therefore, Susanne and Bonet had to find continuous linear maps $f : X \to Z$, $g : Y \to Z$ between bornological locally convex spaces such that f is open but

$$p_f : P \to Y, \quad (x, y) \mapsto y, \quad \text{is not open.}$$

Here again, Susanne's favorite LB-spaces (of Moscatelli type) are used: One takes two Banach spaces E and F, where F is continuously included in E such that $F \cap \overline{B_F}^E$ is not absorbed by B_F (i.e. $\overline{B_F}^E$ induces a strictly weaker norm q on F), and one defines the LB-space

$$Z = \bigoplus_{n\in\mathbb{N}} E + c_0(F)$$

which is an incomplete quotient of the LB-space $X = \bigoplus\limits_{n\in\mathbb{N}} E \times c_0(F)$, the quotient map f (in fact, the sum) is then a cokernel.

The space Y is just the normed space (F, q) and $g : Y \to Z$ is defined as $g(y) = (\frac{1}{n}y)_{n\in\mathbb{N}}$. Thus, g has in fact values in $c_0(F)$ but it is continuous only as a Z-valued map. Having found these spaces it is not obvious but possible to prove that the projection p_f is indeed not open.

We remark that another example of a semiabelian and not quasiabelian category was given (later) by Rump [Rum08], see also the remarks in [SW10].

7. Susanne as a teacher

Susanne felt in classrooms like a fish in water and all students – the good as well as the weaker ones – benefited enormously from her unsurpassably precise way of presenting mathematics. Whenever one could be tempted to explain something by hand waving she increased instead the accuracy. Students loved her lectures and also in the case of optional courses she always had a full classroom which she entered quite a while prior to starting in order to talk to her students not only about mathematics in which case she answered most patiently even to not so clever questions, but also about all sorts of sorrow students may have had.

Consequently, Susanne had much more diploma and PhD students than usual and once she put a student under her wings she gave him or her all necessary support. Some students had weekly appointments with her, either in her office, her kitchen, or her garden during which Susanne destilled new theorems out of coffee and cigarettes writing them on all sorts of paper available – typically she used old envelopes and back sides of all the important correspondence a university professor receives. There is thus some truth in saying that Susanne wrote much more than only one PhD thesis.

But also in cases where such an unusual kind of help was not necessary Susanne supported her students in any way she could, sometimes even investing her private money. In the early nineties she managed to smuggle a PhD student into an Oberwolfach conference describing him as the nurse of her twin children.

Let us finally give the list of Susanne's PhD-students.

- Yolanda Melendez, 1991: *Límites inductivos de tipo Moscatelli en espacios localmente convexos.*
- Leonhard Frerick, 1994: *On vector valued sequence spaces.*
- Stephan Müller, 1994: *Azyklische und schwach azyklische induktive Limiten.*
- Khin Aye Aye, 1995: *Projective limits of Moscatelli type.*
- Jochen Wengenroth, 1995: *Retractive (LF)-spaces.*
- Helmut Hüser, 1995: *Lokalkonvexe Topologien auf Räumen n-linearer Abbildungen und n-homogener Polynome.*
- Norbert Berscheid, 1997: *Baire Properties of Locally Convex Spaces.*
- Karl-Heinz Schröder, 1998: *Locally convex algebras.*
- Thomas Heintz, 2002: *Locally convex and m-convex algebras.*
- Philipp Kuß, 2008: *Vollständigkeit und vollständige Hüllen induktiver Limiten vom Moscatellischen Typ.*

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