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ON THE DIOPHANTINE EQUATION $x^2 + 5^a \cdot 11^b = y^n$

İsmail Naci Cangül, Musa Demirci, Gökhan Soydan, Nikos Tzanakis

Abstract: We give the complete solution (n, a, b, x, y) of the title equation when $\gcd(x, y) = 1$, except for the case when xab is odd. Our main result is Theorem 1.

Keywords: Exponential Diophantine equation, S-Integral points of an elliptic curve, Thue-Mahler equation, Lucas sequence, Linear form in logarithms of algebraic numbers.

1. Introduction

The literature on the exponential Diophantine equation

$$x^2 + C = y^n, \qquad x \geqslant 1, \quad y \geqslant 1, \quad n \geqslant 3 \tag{1.1}$$

goes back to 1850 when Lebesque [25] proved that the equation (1.1) has no solutions when $C = 1$. The title equation is actually a special case of the Diophantine equation $ay^2 + by + c = dx^n$, where a, b, c and d are integers, $a \neq 0$, $b^2 - 4ac \neq 0$, $d \neq 0$, which has only a finite number of solution in integers x and y when $n \geqslant 3$; see [23]. J.H.E. Cohn [20], solved (1.1) for most values of C in the range $1 \leqslant C \leqslant 100$. The equations $x^2 + 74 = y^5$ and $x^2 + 86 = y^5$ that are not solved in that paper, were later solved by Mignotte and de Weger in [33], and the remaining unsolved cases in Cohn's paper were solved by Bugeaud, Mignotte and Siksek in [14].

Upper bounds for the exponent n can be obtained as an application of the work of Bérczes, Brindza and Hajdu [9] and of Győry [22]. These results are based on the Theory of Linear Forms in Logarithms and the obtained upper bounds, though effective, are not explicit.

Recently, the case in which C is a power of a fixed prime gained the interest of several authors. In [1], Arif and Muriefah solve $x^2 + 2^k = y^n$ under certain assumptions. In [24], Le verifies a conjecture of J.H.E Cohn saying that

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$x^2 + 2^k = y^n$ has no solutions with even $k > 2$ and x odd, which was proposed in [19]. The equation $x^2+3^m = y^n$ is completely solved by Arif and Muriefah in [2] when m is odd and by Luca in [28] when m is even. Liqun solves the same equation independently in [26] for both odd and even m. All solutions of $x^2+5^m = y^n$ with m odd are given by Arif and Muriefah in [3] and with m even by Muriefah in [5]. Again, the same equation is independently solved by Liqun in [27]. In [4], Arif and Muriefah give the complete solution of $x^2+q^{2k+1} = y^n$ for q odd prime, $q \not\equiv 7 \pmod 8$ and $n \geqslant 5$ prime to $6h$, where h is the class-number of the number field $\mathbb{Q}(\sqrt{-q})$. Luca and Togbe solve $x^2 + 7^{2k} = y^n$ in [31] and Bérczes and Pink [10] solve (1.1) with $C = p^{2k}$, where $2 \leqslant p < 100$ is prime, $(x, y) = 1$ and $n \geqslant 3$.

More complicated cases, in which C is a product of at least two prime powers are considered in some recent papers. For example, the complete solution (n, x, y) with $n \geqslant 3$ and $\gcd(x, y) = 1$ of the equation (1.1), when C is one of 2^a3^b, 5^a13^b, $2^a5^b13^c$, 2^a11^b, $2^a3^b11^c$ is respectively given in [29], [7], [21], [16], [17]. In [32] the equation (1.1) with $C = 2^a5^b$ is solved when $n \in \{3, 4, 5, 6, 8\}$ and $\gcd(x, y) = 1$. In [35] all the *non-exceptional solutions* (in the terminology of that paper) of the equation (1.1) with $C = 2^a3^b5^c7^d$ are given (with $n \geqslant 3$). Note that finding all the *exceptional solutions* of this equation seems to be a very difficult task.

A survey of many relevant results can be found in [6].

In this paper, we study the equation

$$x^2 + 5^a \cdot 11^b = y^n\,, \qquad x \geqslant 1,\ y \geqslant 1,\ (x, y) = 1, \quad n \geqslant 3,\ a \geqslant 0,\ b \geqslant 0\,. \tag{1.2}$$

Our main result is the following.

Theorem 1. *When $n = 3$, the only solutions to the equation (1.2) are*

$$\begin{aligned}(a, b, x, y) =&(0, 1, 4, 3),\ (0, 1, 58, 15),\ (0, 2, 2, 5),\ (0, 3, 9324, 443),\ (1, 1, 3, 4),\\ &(1, 1, 419, 56),\ (2, 3, 968, 99),\ (3, 1, 37, 14),\ (5, 5, 36599, 1226)\,,\end{aligned} \tag{1.3}$$

and, consequently, $(a, b, x, y) = (1, 1, 3, 2)$ is the only solution when $n = 6$.

When $n = 4$, the equation (1.2) has no solutions.

When $n \geqslant 5, n \neq 6$, the equation (1.2) has no solutions (a, b, x, y) with ab odd and x even, or with at least one of a, b even.

Remark. For $n \geqslant 5, n \neq 6$, the above theorem lefts out the solutions (a, b, x, y) with xab odd. These are exactly the *exceptional solutions* of the equation 1.2 in the terminology of [35]; see also the remark at the end of this paper.

The proof of Theorem 1 is given in sections 2, 3 and 4, where the cases $n = 3$, $n = 4$ and $n \geqslant 5$ are respectively considered. Our numerous, crucial computations in Section 2 have been done mainly with the aid of Magma [13], [15]; to a less extent we have also been aided by the routines of Pari (http://pari.math.u-bordeaux.fr).

Note that since $n \geqslant 3$, it follows that n is either a multiple of 4 or a multiple of an odd prime p, therefore it suffices to study the equation (1.2) when $n = 3, 4$ or an odd prime $\geqslant 5$. Furthermore, note that if $b = 0$, then our equation reduces to the equation $x^2 + 5^a = y^n$, which is solved in [27]. Also, when $a = 0$, the equation (1.2) reduces to $x^2 + 11^b = y^n$ which is solved in [16].

2. Equation (1.2) with $n = 3$

This section is devoted to the proof of the following result.

Proposition 2. *The complete solution of the equation*

$$x^2 + 5^a 11^b = y^3\,, \qquad a \geqslant 0,\, b \geqslant 0,\, x > 0,\, y > 0,\, gcd(x, y) = 1 \tag{2.1}$$

is given in (1.3).

Writing in (2.1) $a = 6A + i$, $b = 6B + j$ with $0 \leqslant i, j \leqslant 5$ we see that

$$\left(\frac{y}{5^{2A}11^{2B}}, \frac{x}{5^{3A}11^{3B}}\right)$$

is an S-integral point (X, Y) on the elliptic curve

$$\mathcal{E}_{ij} : Y^2 = X^3 - 5^i 11^j\,,$$

where $S = \{5, 11\}$, with the numerator of X being prime to 55, in view of the restriction $\gcd(x, y) = 1$. A practical method for the explicit computation of all S-integral points on a Weierstrass elliptic curve has been developed by Pethő, Zimmer, Gebel and Herrmann in [34] and has been implemented in Magma. The relevant routine `SIntegralPoints` worked without problems for all (i, j) except for $(i, j) = (2, 5), (4, 4), (5, 4)$. Thus, in the non-exceptional cases (i, j), i.e. when $(a, b) \not\equiv (2, 5), (4, 4), (5, 4) \pmod 6$, all solutions to equation (2.1) turned out to be those appearing in (1.3). For the exceptional pairs $(i, j) = (2, 5), (4, 4), (5, 4)$ Magma returns no S-integral points *under the assumption that the rank of the corresponding curve $\mathcal{E}_{ij}$ is zero*, an assumption that the routine itself cannot certify. Again using Magma, we performed a 2-descent, followed by a 4-descent which proved that the rank is actually zero in the first two cases $(i, j) = (2, 5), (4, 4)$, allowing us to arrive safely to the following conclusion:

> When $(a, b) \not\equiv (5, 4) \pmod 6$, all solutions to equation (2.1) are those displayed in (1.3).

In the third exceptional case $(i, j) = (5, 4)$, the 4-descent reveals the non-torsion point

$$(X, Y) = \left(\frac{997597438498050698749}{101288668233063249}, \frac{31508127105495852851671290908932}{32236010714473507582283943}\right)$$

on the curve $\mathcal{E}_{54}$, which proves invalid the assumption under which Magma "claims" non-existence of S-integral points on $\mathcal{E}_{54}$. Thus, non-existence of integral solutions to (2.1) when $(a, b) \equiv (5, 4) \pmod 6$ cannot be considered as a fact that has been proved by Magma routines. Therefore we treat this equation separately, indicating thus an alternative method for resolving equations $x^2 + C = y^3$ when C has a prescribed ("small") set of distinct prime divisors. Moreover, the resolution of the Thue-Mahler equation (2.13) that we present in section 2 is interesting *per se*, as it deals with a totality of non-trivial computational problems that never before (to the best of our knowledge) have been encountered in the resolution of a Thue-Mahler equation; we acknowledge here the great usefulness of the relevant routines of MAGMA.

In conclusion, according to our discussion so far, for the proof of Proposition 2 it remains to show that the equation

$$x^2 + 5^a 11^b = y^3\,, \qquad (a,b) \equiv (5,4) \pmod 6,\ x \neq 0,\ \gcd(x, 5\cdot 11) = 1\,, \tag{2.2}$$

has no solutions. We write (2.2) as

$$y^3 - 5^2 \cdot 11 \cdot (5^c 11^d)^3 = x^2\,, \qquad cd \text{ odd},\ (y, 5\cdot 11) = 1. \tag{2.3}$$

and in what follows we will reduce its solution to a number of Thue or Thue-Mahler equations. A practical solution of Thue equations has been developed by Tzanakis and de Weger [36] which later was improved by Bilu and Hanrot [11] and implemented in PARI and MAGMA. We will make use of the relevant routines several times without special mentioning. Concerning the Thue-Mahler equations, no automatic resolution is available so far and we will follow the method of Tzanakis and de Weger [37].

Factorization of (2.3) in the field $\mathbb{Q}(\theta)$, where $\theta^3 = 5^2 \cdot 11$, gives

$$(y - 5^c 11^d \theta)(y^2 + 5^c 11^d y\theta + 5^{2c} 11^{2d} \theta^2) = x^2\,.$$

In the field $\mathbb{Q}(\theta)$ the ideal class-number is 3, an integral basis is given by $1, \theta, \theta^2/5$ and the fundamental unit is $\epsilon = 1 + 338\theta - 52\theta^2$ with norm $+1$. It is easily checked that the two factors in the left-hand side of the last equation above are relatively prime, hence we have an ideal equation $(y - 5^c 11^d \theta) = \mathfrak{a}^2$, where $\mathfrak{a}$ is an integral ideal. Since the class-number is relatively prime to the exponent of $\mathfrak{a}$, this ideal must be principal, generated by an integral element $u + v\theta + w\theta^2/5$. Then, passing to element equation, we get

$$y - 5^c 11^d \theta = \pm\epsilon^i (u + v\theta + w\theta^2/5)^2\,.$$

Taking norms we see that, necessarily, the plus sign must hold above. Also, comparing coefficients of θ in both sides we see very easily that w must be divisible by 5, hence, on replacing w by $5w$, we rewrite the last equation as follows:

$$y - 5^c 11^d \theta = \epsilon^i (u + v\theta + w\theta^2)^2\,. \tag{2.4}$$

We consider two cases, depending on the value of i.

Let $i = 0$. Equating coefficients of like powers of θ in both sides of (2.4) we obtain the following relations:

$$v^2 + 2uw = 0 \tag{2.5}$$

$$2uv + 275w^2 = -5^c \cdot 11^d \tag{2.6}$$

$$u^2 + 550vw = y \tag{2.7}$$

The above equations, along with the fact that $\gcd(y, 5 \cdot 11) = 1$, easily imply that $\gcd(u, w) = 1$ and w is odd, hence (2.5) implies that

$$u = 2sv_1^2\,, \quad w = -sv_2^2\,, \quad v = 2v_1v_2\,, \quad s \in \{-1, 1\}\,, \quad \gcd(2v_1, v_2) = 1$$

and substitution into (2.6) gives

$$-5^c \cdot 11^d = 275v_2^4 + 8sv_1^3v_2 = v_2((2sv_1)^3 + 275v_2^3)\,. \tag{2.8}$$

Since c is odd, it is $\geqslant 1$, therefore, from (2.8) one of v_1, v_2 is divisible by 5. If 5 divides v_1, then 25 divides the right-hand side, hence c must be $\geqslant 2$ and since it is odd, it must be at least 3. But then 5^3 divides $275v_2^4$, hence 5 divides v_2 which contradicts the fact that $\gcd(v_1, v_2) = 1$. Therefore, 5 divides v_2 and does not divide v_1.

If 11 also divides v_2, then, neither 5 nor 11 divides $(2sv_1)^3 + 275v_2^3$, hence, by (2.8) we have $v_2 = \pm 5^c \cdot 11^d$ and $(2sv_1)^3 + 275v_2^3 = \pm 1$. But since the only solutions of the Thue equation $X^3 + 275Y^3 = \pm 1$ is $(X, Y) = (\pm 1, 0)$, the previous equation is impossible. Therefore 11 divides v_1 and does not divide v_2. Now, if $d > 1$, then (2.8) implies that $275v_2^4$ is divisible by 11^2, hence 11 divides v_2, a contradiction. Therefore, $d = 1$ and by our discussion so far we conclude, in view of (2.8), that $v_2 = \pm 5^c$ and $(2sv_1)^3 + 275v_2^3 = \mp 11$, which is impossible since the Thue equation $X^3 + 275Y^3 = 11$ is impossible.

We conclude therefore that equation (2.4) is impossible when $i = 0$.

Next, let $i = 1$. Equating coefficients of like powers of θ in (2.4) gives

$$-52u^2 + 676vu + 2wu + v^2 + 92950w^2 - 28600wv = 0 \tag{2.9}$$

$$338u^2 + 2vu - 28600wu - 14300v^2 + 275w^2 + 185900wv = -5^c \cdot 11^d \tag{2.10}$$

$$u^2 - 28600vu + 185900wu + 92950v^2 - 3932500w^2 + 550wv = y \tag{2.11}$$

From the above equations it is easy to see that v is even w is odd and (since also $\gcd(y, 5 \cdot 11) = 1$) $\gcd(u, w) = 1$. On the other hand, equation (2.9) can be written as

$$2(52u - 2199w)(1099u - 46475w) = (v + 338u - 14300w)^2\,.$$

Since

$$\begin{vmatrix} 52 & -2199 \\ 1099 & -46475 \end{vmatrix} = 1$$

and $\gcd(u, w) = 1$, it follows that the two parenthesis in the left-hand side of the last equation are relatively prime, the first one being odd, because w is odd. It follows that

$$\begin{aligned} 52u - 2199w &= sX^2 \\ 1099u - 46475w &= 2sY^2 \\ v + 338u - 14300w &= \pm 2XY \end{aligned}$$

where X, Y are integers and $s \in \{-1, 1\}$. Solving the system in u, v, w we obtain expressions of u, v, w in terms of X, Y; then, substitution into (2.10) gives

$$150975X^4 \pm 185900X^3Y + 85800X^2Y^2 \pm 17592XY^3 + 1352Y^4 = 5^c \cdot 11^d\,.$$

Replacement of $-X$ by X shows that we may consider only the plus sign in the above equation. We have thus obtained a Thue-Mahler equation which we will solve in the next section.

2.1. The solution of the Thue-Mahler equation

In this section we prove that the Thue-Mahler equation

$$150975X^4 + 185900X^3Y + 85800X^2Y^2 + 17592XY^3 + 1352Y^4 = 5^c \cdot 11^d \quad (2.12)$$

has no solutions. We will follow closely the method of [37] which, to the best of our knowledge is the only systematic exposition found so far in the literature. For the convenience of the reader, we will use the same notation with [37] as far as possible. *The notation in this section is independent of the notation used in the others sections of the present paper.*

Putting $x = 2 \cdot 13^2 Y$, $y = X$ (obviously, $(x, y) = 1$), we transform equation (2.12) into

$$x^4 + 4398x^3y + 7250100x^2y^2 + 5309489900xy^3 + 1457454977550y^4 = 2 \cdot 13^6 5^c 11^d. \quad (2.13)$$

We work in the field $K = \mathbb{Q}(\theta)$, where θ is a root of the polynomial

$$g(t) = t^4 + 4398t^3 + 7250100t^2 + 5309489900t + 1457454977550 \in \mathbb{Q}[t].$$

The ideal-class number is 1 and an integral basis is 1, θ, $(4\theta + \theta^2)/169$, $(929950\theta + 173\theta^2 + \theta^3)/142805$. For shortness, we will use the notation $\gamma = [a, b, c, d]$, where $a, b, c, d \in \mathbb{Z}$, to mean that the algebraic integer $\gamma \in K$ has the coefficients a, b, c, d with respect to the above integral basis.

A pair of fundamental units is

$$\varepsilon_1 = [677070473, 1764897, 260182, 69044]$$
$$\varepsilon_2 = [7564704083, 22782192, 3852447, 1164105]\,.$$

The factorization of the rational primes 2, 5, 11 and 13 is as follows:

$$\begin{aligned}
2 &= \varepsilon_2^{-1}\pi_2^4 & \pi_2 &= [21436, 39, 3, 0]\\
5 &= \varepsilon_1^{-1}\varepsilon_2\pi_{51}\pi_{52}^3\,, & \pi_{51} &= [9690469, 26053, 3965, 1087]\\
& & \pi_{52} &= [653350925, 1762426, 269424, 74288]\\
11 &= \pi_{111}\pi_{112}^3\,, & \pi_{111} &= [1060859, 2835, 429, 117]\\
& & \pi_{112} &= [204919, 535, 79, 21]\\
13 &= -\pi_{131}\pi_{132}\,, & \pi_{131} &= [127759589, 344590, 52671, 14521]\\
& & \pi_{132} &= [16961503, 45062, 6773, 1833]\,,
\end{aligned}$$

where all the prime elements π_{ij} above, except for π_{132} are of degree 1, and π_{132} is of degree 3.

In the notation of relation (3) of [37],

$$f_0 \leftarrow 1, \quad c \leftarrow 2 \cdot 13^6, \quad p_1 \leftarrow 5, \quad z_1 \leftarrow c, \quad p_2 \leftarrow 11, \quad z_2 \leftarrow d.$$

Fix the prime $p \in \{5, 11\}$. For the elements of the ring $\mathbb{Z}_p$ of p-adic integers we will use the notation $0.d_0d_1d_2\ldots$, where $d_0, d_1, d_2, \ldots$ are integers between 0 and $p-1$, to mean the p-adic integer $d_0+d_1p+d_2p^2+\cdots$. In the sequel, all our computations with p-adic numbers have been done with the relevant MAGMA routines.

Over $\mathbb{Q}_p[t]$ we have the factorization of $g(t) = g_1(t)g_2(t)$ as in the following table.

Table 1. Factorization $g(t) = g_1(t)g_2(t)$ into irreducibles over $\mathbb{Q}_p$

p	$g_1(t)$	$g_2(t)$
5	$t-(0.20404\ldots)$	$t^3+(0.00011\ldots)t^2+(0.00422\ldots)t+(0.00444\ldots)$
11	$t-(0.25033\ldots)$	$t^3+(0.09363\ldots)t^2+(0.09900\ldots)t+(0.052(10)6\ldots)$

We denote by L_p the splitting field of $g(t)$ over $\mathbb{Q}_p$. This is obtained in two steps, as shown in the following table.

Table 2. $K_p = \mathbb{Q}_p(u)$, $g_{p1}(u) = 0$ and $L_p = K_p(v)$, $g_{p2}(v) = 0$

p	$g_{p1}(t)$	$g_{p2}(t)$
5	t^2+4t+2	$t^3+(0.03001\ldots)t^2+(0.00111\ldots)t+(0.04442\ldots)$
11	t^2+7t+2	$g_2(t)$

The roots of $g(t)$ are shown in the following table.

Table 3. The roots of $g(t)$ over $\mathbb{Q}_p$

p	$\theta^{(1)}$	$\theta^{(2)}, \theta^{(3)}, \theta^{(4)}$
5	$0.20404\ldots$	$(0.10001\ldots)v^2+(0.04220\ldots)v+(0.00011\ldots)$, $(0.30230\ldots)uv^2+(0.33140\ldots)v^2+(0.00424\ldots)uv$ $+(0.03143\ldots)v+(0.00344\ldots)u+(0.00132\ldots)$, $(0.24214\ldots)uv^2+(0.11303\ldots)v^2+(0.00120\ldots)uv$ $+(0.03032\ldots)v+(0.00203\ldots)u+(0.00444\ldots)$
11	$0.25033\ldots$	v, $(0.(10)4(10)71\ldots)uv^2+(0.26306\ldots)v^2+(0.(10)3(10)63\ldots)uv$ $+(0.72327\ldots)v+(0.026(10)3\ldots)u+(0.08801\ldots)$, $(0.16039\ldots)uv^2+(0.947(10)4\ldots)v^2+(0.17047\ldots)uv$ $+(0.38783\ldots)v+(0.09407\ldots)u+(0.05936\ldots)$

For $\gamma \in \mathbb{Q}_p$ we define, as usually, $\mathrm{ord}_p(\gamma) = m$ iff $\gamma = p^m \mu$, where μ is a p-adic unit. We extend the function ord_p to L_p by the formula

$$\mathrm{ord}_p(\gamma) = \frac{1}{6}\mathrm{ord}_p(\mathrm{N}_{L_p/\mathbb{Q}_p}(\gamma))$$

(see section 4 of [37]). By Statement (i) of the *First Corollary of Lemma 1*, p.231 of [37] we conclude that at most one among π_{51} and π_{52} divides $x - y\theta$. Moreover, if π_{52} divides $x - y\theta$, then Statement (ii) of the same *Corollary* asserts that at most π_{52}^2 divides $x - y\theta$. Similarly, at most one among π_{111} and π_{112} divides $x - y\theta$ and if this is the case with the second one, then at most its first power divides $x - y\theta$. These observations and standard arguments of Algebraic Number Theory lead to the following ideal equation (cf. relation (9) of [37])

$$\langle x - y\theta \rangle = \mathfrak{a}\mathfrak{b}\mathfrak{p}_1^{n_1}\mathfrak{p}_2^{n_2}\,,$$

where

$$\begin{gathered}
\mathfrak{a} \in \{\langle \pi_2\pi_{131}^6 \rangle, \langle \pi_2\pi_{131}^3\pi_{132} \rangle, \langle \pi_2\pi_{132}^2 \rangle\}, \\
\mathfrak{b} = \langle \pi_{52}^{j_1}\pi_{112}^{j_2} \rangle, \qquad (0 \leqslant j_1 \leqslant 2,\ 0 \leqslant j_2 \leqslant 1) \\
\mathfrak{p}_1 = \langle \pi_{51} \rangle, \qquad \mathfrak{p}_2 = \langle \pi_{52} \rangle\,.
\end{gathered}$$

$$\begin{aligned}
c &= n_1 + j_1 \qquad \text{with } (n_1 > 0 \text{ and } j_1 = 0) \text{ or } (n_1 = 0 \text{ and } j_1 \leqslant 2), \\
d &= n_2 + j_2 \qquad \text{with } (n_2 > 0 \text{ and } j_2 = 0) \text{ or } (n_2 = 0 \text{ and } j_2 \leqslant 1).
\end{aligned} \tag{2.14}$$

Following the strategy (and notation) of section 7 of [37] we obtain the following relation:

$$\begin{aligned}
\lambda &= \delta_2 \left(\frac{\varepsilon_1^{(i_0)}}{\varepsilon_1^{(j)}}\right)^{a_1} \left(\frac{\varepsilon_2^{(i_0)}}{\varepsilon_2^{(j)}}\right)^{a_2} \left(\frac{\pi_{51}^{(i_0)}}{\pi_{51}^{(j)}}\right)^{n_1} \left(\frac{\pi_{111}^{(i_0)}}{\pi_{112}^{(j)}}\right)^{n_2} \\
\lambda &= \delta_1 \left(\frac{\varepsilon_1^{(k)}}{\varepsilon_1^{(j)}}\right)^{a_1} \left(\frac{\varepsilon_2^{(k)}}{\varepsilon_2^{(j)}}\right)^{a_2} \left(\frac{\pi_{51}^{(k)}}{\pi_{51}^{(j)}}\right)^{n_1} \left(\frac{\pi_{111}^{(k)}}{\pi_{112}^{(j)}}\right)^{n_2} - 1\,,
\end{aligned} \tag{2.15}$$

where

$$\delta_1 = \frac{\theta^{(i_0)} - \theta^{(j)}}{\theta^{(i_0)} - \theta^{(k)}} \cdot \frac{\alpha^{(k)}}{\alpha^{(j)}}\,, \qquad \delta_2 = \frac{\theta^{(j)} - \theta^{(k)}}{\theta^{(k)} - \theta^{(i_0)}} \cdot \frac{\alpha^{(i_0)}}{\alpha^{(j)}}\,,$$

$$\alpha \in \{\pi_{131}^6, \pi_{131}^3\pi_{132}, \pi_{132}^2\} \cdot \pi_2\pi_{52}^{j_1}\pi_{112}^{j_2}$$

with j_1, j_2 as in (2.14).

We view (2.15) either as a relation in L_p, where $p \in \{5, 11\}$ as the case may be, and in this case $i_0 = 1, j = 2, k = 3$ (cf. table of roots of $g(t)$ over $\mathbb{Q}_p$), or as a relation in $\mathbb{C}$, in which case we number the real/complex roots of $g(t)$ as θ_1, θ_2, the real ones, and θ_3, θ_4 the pair of complex-conjugate roots, and we take $i_0 \in \{1, 2\}$ (one must consider both cases), $j = 3, k = 4$.

We set now

$$A = \max\{|a_1|, |a_2|\}, \qquad N = \max\{n_1, n_2\}, \qquad H = \max\{A, N\}.$$

We apply Yu's theorem (Theorem 1 in [38]) in its somewhat simplified version presented in Appendix A2 of [37]. Note that the least field generated over $\mathbb{Q}$ by the five algebraic numbers appearing in λ (cf. (2.15)) is of degree 24. Thus, in the notation of the above mentioned Appendix A2, $n_1 \leftarrow 24, n_2 \leftarrow 48, (q, u) \leftarrow (2, 2), f_2 \leftarrow 1$, and after the computation of the various parameters in Yu's theorem, we obtain the values c_{13}, c_{14} (p. 238 of [37]) for which

$$N \leqslant c_{13}(\log H + c_{14}),$$

namely, $c_{13} = 2.0564 \times 10^{39}, c_{14} = 7.7425$.

Next, we work with real/complex linear forms in logarithms of agebraic numbers. In the terminology of [37], p. 243, we encounter a "complex case", therefore, following that paper, we consider the linear form

$$\Lambda_0 = i^{-1}\mathrm{Log}(1+\lambda) = i^{-1}\mathrm{Log}\left(\delta_1 \left(\frac{\varepsilon_1^{(k)}}{\varepsilon_1^{(j)}}\right)^{a_1} \left(\frac{\varepsilon_2^{(k)}}{\varepsilon_2^{(j)}}\right)^{a_2} \left(\frac{\pi_{51}^{(k)}}{\pi_{51}^{(j)}}\right)^{n_1} \left(\frac{\pi_{111}^{(k)}}{\pi_{112}^{(j)}}\right)^{n_2}\right)$$

where Log denotes the principal branch of the complex logarithmic function. Since, for every $z \in \mathbb{C}$, $i^{-1}\mathrm{Log}(\overline{z}/z) = \mathrm{Arg}(\overline{z}/z)$, where Arg denotes the principal Argument, we have after expansion (remember that $i_0 = 1, 2$ and $j = 3, k = 4$),

$$\begin{aligned}\Lambda_0 = \mathrm{Arg}\frac{\theta^{(i_0)} - \theta^{(3)}}{\theta^{(i_0)} - \theta^{(4)}} \cdot \frac{\alpha^{(4)}}{\alpha^{(3)}} + a_1\mathrm{Log}(\delta_1 \left(\frac{\varepsilon_1^{(4)}}{\varepsilon_1^{(3)}}\right) + a_2\mathrm{Arg}\left(\frac{\varepsilon_2^{(4)}}{\varepsilon_2^{(3)}}\right) \\ + n_1\mathrm{Arg}\left(\frac{\pi_{51}^{(4)}}{\pi_{51}^{(3)}}\right) + n_2\mathrm{Arg}\left(\frac{\pi_{111}^{(k)}}{\pi_{112}^{(j)}}\right) + a_0(2\pi).\end{aligned} \tag{2.16}$$

According to relation (27), p. 245 of [37], we have

$$0 < |\Lambda_0| < 1.02c_{21}e^{-c_{15}A},$$

where c_{21} and c_{16} are explicit, and we need further a lower bound of the shape $|\Lambda_0| > \exp(-c_7(\log H + 2.5))$ (see p. 246 of [37]). Baker-Wüstholz's theorem [8] furnishes us $c_7 = 8.43 \times 10^{56}$. The constant c_{16} must be less than $3.809\ldots$ (for the choice of c_{16} see [37], bottom of p. 239). The constants c_7, c_{13}, c_{14} and c_{16} are the crucial ones in the computation of an upper bound for H. A number of other parameters must be obtained by elementary but quite cumbersome calculations. A very detailed exposition of how this list of parameters are calculated for the general Thue-Mahler equation is exposed in the first eleven sections of [37], culminating to an explicit upper bound for H. Fortunately, the computation of these constants, including that of c_7, c_{13}, c_{14} and c_{16}, can be rather easily implemented in (for example) Maple or Magma. For our equation the upper bound that we calculate is

$$H < K_0 = 5.792 \times 10^{58}. \tag{2.17}$$

Especially for N, a smaller upper bound is obtained by the *Corollary to Theorem 10*, p. 248 of [37], namely

$$N < N_0 = 2.942 \times 10^{41}\,. \tag{2.18}$$

Thus, the upper bound for H is, actually, the upper bound for A.

Reduction of the upper bound. In order to considerably reduce the upper bound (2.18) by the so called *p-adic reduction process*, we need the *p-adic logarithmic function* $\log_p z$, which is defined for every p-adic unit $z \in L_p$ and takes values in L_p; see the detailed exposition in section 12 of [37].

For $p \in \{5, 11\}$ we put (viewing λ in (2.15) as an element of L_p)

$$\Lambda = \log_p(1+\lambda) = \log_p \delta_1 + n_1 \log_p \frac{\pi_{51}^{(k)}}{\pi_{51}^{(j)}} + n_2 \log_p \frac{\pi_{111}^{(k)}}{\pi_{111}^{(j)}} + a_1 \log_p \frac{\varepsilon_1^{(k)}}{\varepsilon_1^{(j)}} + a_2 \log_p \frac{\varepsilon_2^{(k)}}{\varepsilon_2^{(j)}}\,,$$

where the indices can be chosen arbitrarily from the set $\{2,3,4\}$ $(k \neq j)$. Expressing Λ in terms of the basis $1, u, v, uv, v^2, uv^2$ of $L_p/\mathbb{Q}_p$, we can write

$$\Lambda = \Lambda_0 + \Lambda_1 u + \Lambda_2 v + \Lambda_3 uv + \Lambda_4 v^2 + \Lambda_5 uv^2\,,$$

where each Λ_i is a linear form

$$\Lambda_i = \alpha_{i0} + \alpha_{i1} n_1 + \alpha_{i2} n_2 + \alpha_{i3} a_1 + \alpha_{i4} a_2\,, \qquad \alpha_{ij} \in \mathbb{Z}_p\,, \quad (i = 0, \ldots, 5).$$

Following the discussion of section 14 of [37], for each i, we divide by the α_{ij} whose ord_p has a minimal value (actually, this is obtained for som $j > 0$), obtaining thus a linear form

$$\Lambda_i' = -\beta_0 - \beta_1 b_1 - \beta_2 + b_2 - \beta_3 b_3 + b_4\,,$$

where (b_1, b_2, b_3, b_4) is a permutation of (n_1, n_2, a_1, a_2). At this point we note that the p-adic numbers $\beta_0, \ldots, \beta_3$ are computed with a high p-adic precision m. We denote by $\beta^{(m)}$ the *rational integer* which approximates β with m p-adic digits; in other words, $\mathrm{ord}_p(\beta - \beta^{(m)}) \geqslant m$.

Following the p-adic reduction process described in section 15 of [37], we consider the lattice whose basis is formed by the columns of the matrix

$$\begin{pmatrix} W & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta_1^{(m)} & \beta_2^{(m)} & \beta_3^{(m)} & p^m \end{pmatrix},$$

where W is an integer somewhat larger than K_0/N_0; in this case we choose $W = 2 \cdot 10^{17}$. Then we obtain an *LLL-reduced basis* of the lattice. As explained in section 15 of [37], if m is sufficiently large, then it is highly probable that a certain condition stated in Proposition 15 of [37] (in which condition the reduced basis is, of course, involved) is fulfilled; and if the condition is fulfilled, then, according to that Proposition 15, $n_1, n_2 \leqslant m+1$. It turns out that, if $p = 5$, then $m = 306$

is sufficient for the condition of Proposition 15 to be fulfilled; and if $p = 11$, it suffices to have a precision of $m = 207$ 11-adic digits. Thus, in the first p-adic step we made a huge "jump", falling from (2.18) to $N \leqslant N_1 = 307$.

Now it is the turn of the *real reduction step.* We rewrite the linear form Λ_0 in (2.16) as

$$\Lambda_0 = \rho_0 + n_1\lambda_1 + n_2\lambda_2 + a_1\mu_1 + a_2\mu_2 + a_0\mu_3 \qquad (\mu_3 = 2\pi)$$

and for $C = 10^m$, with m a sufficiently large integer (having nothing to do with the m in the p-adic reduction process), we put

$$\phi_0 = [C\rho_0], \quad \phi_i = [C\lambda_i] \quad (i = 1, 2)\,, \qquad \psi_i = [C\mu_i] \quad (i = 1, 2, 3)\,,$$

where $[x] = \lfloor x \rfloor$ if $x \geqslant 0$ and $[x] = \lceil x \rceil$ if $xN0$. In practice, this means that we must compute our real numbers ρ, λ, μ with a precision of somewhat more than m decimal digits.

Following the discussion of section 16 of [37], we consider the lattice whose basis is formed by the columns of the matrix

$$\begin{pmatrix} W & 0 & 0 & 0 & 0 \\ 0 & W & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \phi_1 & \phi_2 & \psi_1 & \psi_2 & \psi_3 \end{pmatrix}.$$

Again, we compute an LLL reduced basis for the lattice and, according to Proposition 16 of [37], if a certain condition, in which the reduced basis is involved, is satisfied, then a considerably smaller upper bound for H is obtained. It is highly probable that this condition is satisfied if $C = 10^m$ is sufficiently large. As it turns out in our case, $m = 200$ is sufficient and the reduced upper bound implied by the above mentioned Proposition 16 is $H \leqslant K_1 = 546$, an enormous "jump" from (2.17)!

This strategy of a p-adic reduction process followed by a real reduction process is repeated, with K_1 in place of K_0 and N_1 in place of N_0, giving even smaller upper bounds, namely, $N \leqslant N_2 = 32$ and $H \leqslant K_2 = 74$. We repeat the process once more. The 5-adic reduction process gives $n_1 \leqslant 25$ and the 11-adic reduction process gives $n_2 \leqslant 18$. The real reduction process gives $H \leqslant 59$. Thus,

$$0 \leqslant n_1 \leqslant 25\,, \qquad 0 \leqslant n_2 \leqslant 18\,, \qquad A = \max\{|a_1|, |a_2|\} \leqslant 59\,. \tag{2.19}$$

The sieve after the reduction. The bounds (2.19) cannot be further improved, therefore we have to search whether there exist quadruples (a_1, a_2, n_1, n_2) in the range (2.19) and pairs (i_1, i_2) and (j_1, j_2) in the range

$$(i_1, i_2) \in \{(6,0), (3,1), (0,2)\}\,, \qquad 0 \leqslant j_1 \leqslant 2\,,\ 0 \leqslant j_2 \leqslant 1\,, \tag{2.20}$$

such that

$$h(i_1, i_2, j_1, j_2, a_1, a_2, n_1, n_2) := \pi_2 \pi_{131}^{i_1} \pi_{132}^{i_2} \pi_{52}^{j_1} \pi_{112}^{j_2} \varepsilon_1^{a_1} \varepsilon_2^{a_2} \pi_{51}^{n_1} \pi_{111}^{n_2} \tag{2.21}$$

is of the form $x - y\theta$, i.e., such that, after expanding the right-hand side in (2.21) and expressing it in terms of the basis $1, \theta, \theta^2, \theta^3$ of $K/\mathbb{Q}$, the coefficients of θ^2 and θ^3 are zero. Doing this check by "brute force" is very time consuming. Instead, we choose to do the following sieving process (see also section 18 of [37]).

Let q be a rational prime which splits into four distinct (first degree) prime divisors $\rho_1, \ldots, \rho_4$ of K. Then, for every algebraic integer $\gamma \in K$, there exist rational integers A_i, $i = 1, \ldots 4$, such that $\gamma \equiv A_i \pmod{\rho_i}$. As a consequence, every (rational) relation with algebraic integers of K implies congruences mod ρ_i, one for every $i = 1, \ldots, 4$. But since in these congruences the elements of K are replaced by *rational integers*, these are valid also as congruences in $\mathbb{Z}$ modulo q.

Take, for example, $q = 31$. Then, we have the ideal factorization $\langle q \rangle = \prod_{i=1}^{4} \langle \rho_i \rangle$, where

$$\theta \equiv 1 \pmod{\rho_1}, \quad \theta \equiv 17 \pmod{\rho_2}, \quad \theta \equiv 19 \pmod{\rho_3}, \quad \theta \equiv 29 \pmod{\rho_4}.$$

From a relation of the form $x - y\theta = h(\mathbf{i})$, where $\mathbf{i} = (i_1, i_2, j_1, j_2, a_1, a_2, n_1, n_2)$ is in the range (2.19) and (2.20), we obtain the four congruences

$$x - y \equiv H_1(\mathbf{i}), \quad x - 17y \equiv H_2(\mathbf{i}), \quad x - 19y \equiv H_3(\mathbf{i}), \quad x - 29y \equiv H_4(\mathbf{i}) \pmod{31} \tag{2.22}$$

where $H_1(\mathbf{i})$ is the rational integer resulting on replacing θ by 1 in $h(\mathbf{i})$, and similarly for the remaining $H_j(\mathbf{i})$'s. Then,

$$27H_1(\mathbf{i}) + 5H_2(\mathbf{i}) \equiv H_3(\mathbf{i}), \qquad 7H_1(\mathbf{i}) + 25H_2(\mathbf{i}) \equiv H_4(\mathbf{i}) \pmod{31}. \tag{2.23}$$

Note now that, for every algebraic integer $\gamma \in K$, the order of γ modulo 31 is a divisor of 30. The orders of $\varepsilon_1, \varepsilon_2, \pi_{51}, \pi_{111}$ modulo 31 are 30,15,15,30, respectively. Therefore, we check the congruences (2.23) for all $\mathbf{i}$'s with (i_1, i_2) and (j_1, j_2) as in (2.20) and $0 \leqslant a_1 \leqslant 29$, $0 \leqslant a_2 \leqslant 14$, $0 \leqslant n_1 \leqslant 14$, $0 \leqslant n_2 \leqslant 18$.

For example, when $(i_1, i_2, j_1, j_2) = (6, 0, 2, 1)$, there are 4275 quadruples (a_1, a_2, n_1, n_2) that satisfy the first congruence (2.23). We check which of them also satisfy the second congruence (2.22) and only 117 quadruples pass the test. Now, these 117 quadruples must be lifted to cover the range (2.19), resulting to 6532 quadruples. Thus, there are 6532 6-tuples $\mathbf{i} = (6, 0, 2, 1, a_1, a_2, n_1, n_2)$ with a_1, a_2, n_1, n_2 in the range indicated by (2.19), that satisfy both congruences (2.22).

Next, we work similarly with the prime $q = 79$ (which splits into four distinct prime divisors of K). The analogous to the congruences (2.22) are now

$$x - 6y \equiv H_1'(\mathbf{i}), \quad x - 14y \equiv H_2'(\mathbf{i}), \quad x - 41y \equiv H_3'(\mathbf{i}), \quad x - 44y \equiv H_4'(\mathbf{i}) \pmod{73},$$

implying the anlogous to (2.23) congruences

$$24H_1'(\mathbf{i}) - 23H_2'(\mathbf{i}) \equiv H_3'(\mathbf{i}), \qquad -22H_1'(\mathbf{i}) + 23H_2'(\mathbf{i}) \equiv H_4'(\mathbf{i}) \pmod{73}.$$

We check which of the 6532 6-tuples $\mathbf{i}$, obtained before, satisfy the last congruences and only three 6-tuples pass the test, which are tested by a final similar test with the prime 223 in place of 73; no one passes the test. This shows that no 6-tuple $(6, 0, 2, 1, a_1, a_2, n_1, n_2)$ is accepted.

A similar test is repeated for every (i_1, i_2) and (j_1, j_2) as in (2.20) and always we end up with no acceptable $(i_1, i_2, j_1, j_2, a_1, a_2, n_1, n_2)$.

Final conclusion of section 2.1: The Thue-Mahler equation (2.13) has no solutions and, consequently, neither the equation (2.12) has solutions. This completes the proof of Proposition 2.

3. Equation (1.2) with $n = 4$

In this section we prove the following result.

Proposition 3. *If $n = 4$, then the equation (1.2) has no solution.*

Proof. Since $n = 4$, equation (1.2) is written as

$$5^a \cdot 11^b = (y^2 + x)(y^2 - x)\,, \tag{3.1}$$

from which we obtain

$$\begin{aligned} y^2 + x &= 5^{a_1} 11^{b_1} \\ y^2 - x &= 5^{a_2} 11^{b_2} \end{aligned}$$

where $a_1, a_2, b_1, b_2 \geqslant 0$. From the equations above and the assumption $\gcd(x, y) = 1$ it follows that a_1, a_2 cannot both be positive, and similarly for b_1, b_2. Summing the two equations we obtain

$$Z^2 - Du^2 = 2 \cdot 5^{a_1} 11^{b_1} \tag{3.2}$$

where $D \in \{2, 10, 22, 110\}$, $Z = 2y$ and $u = 5^{a_2} 11^{b_2}$. We have $\gcd(Z, u) = 1$. Indeed, otherwise we would have $\gcd(2y^2, u) > 1$, hence $\gcd(5^{a_1} 11^{b_1} + 5^{a_2} 11^{b_2}, 5^{a_2} 11^{b_2}) > 1$ which contradicts our remark concerning the pairs a_1, a_2 and b_1, b_2 a few lines above.

We claim that $a_1 = 0$. Indeed, suppose that $a_1 \geqslant 1$. If $D = 2$ or 22, then by (3.2), $Z^2 \equiv 2u^2 \pmod 5$, implying $Z \equiv u \equiv 0 \pmod 5$, which contradicts $\gcd(Z, u) = 1$. If $D = 10$ or 110, then (3.2) implies $Z \equiv 0 \pmod 5$, hence also $2y^2 \equiv 0 \pmod 5$. But $2y^2 = 5^{a_1} 11^{b_1} + 5^{a_2} 11^{b_2}$ and we have assumed that $a_1 \geqslant 1$; therefore, $a_2 \geqslant 1$, contradicting our remark that a_1, a_2 cannot both be positive.

With completely analogous arguments we prove that $b_1 = 0$, by distinguishing the cases $D = 2, 10$ and $D = 22, 110$ and taking into account that b_1, b_2 cannot both be positive.

Thus, $y^2 + x = 1$, which is impossible since x and y are positive integers. ∎

4. Equation (1.2) with $n \geqslant 5, n \neq 6$

In this section we prove the following result.

Proposition 4. *The equation*

$$x^2 + 5^a 11^b = y^n\,, \qquad (x,y)=1,\ n \geqslant 5\,, \tag{4.1}$$

is impossible if at least one among a and b is even or if ab is odd and x is even.

Proof. Since in the previous sections we have completed the study of the equation $x^2 + 5^a 11^b = y^n$ with $n = 3, 4$, we certainly can assume that n is a prime $\geqslant 5$.

We write (4.1) as

$$x^2 + dz^2 = y^n\,, \qquad d \in \{1, 5, 11, 55\}\,,\ \ z = 5^\alpha 11^\beta\,, \tag{4.2}$$

where the relation of α and β with a and b, respectively, is clear.

If at least one among a and b is even, then $d \in \{1, 5, 11\}$ and we see mod 8 that x is even. If both a and b are odd, then $d = 55$ and both cases, x even or odd can arrise. According to the announcement of the Proposition, we consider only the case that x is even.

We work in the field $\mathbb{Q}(\sqrt{-d})$. The algebraic integers in this number field are of the form $(u + v\sqrt{-d})/2$, where $u, v \in \mathbb{Z}$ with u, v both even, if $d = 1, 5$ and $u \equiv v (\text{mod } 2)$ if $d = 11, 55$.

Since x is even, the factors in the left-hand side of the equation $(x+z\sqrt{-d})(x-z\sqrt{-d}) = y^n$ are relatively prime and we obtain the ideal equation $\langle x + y\sqrt{-d}\rangle = \mathfrak{a}^n$. Then, since the ideal-class number is 1, 2, or 4, and n is odd, we conclude that the ideal $\mathfrak{a}$ is principal. Moreover, the units are ± 1 and, in case $d = 1$, also $\pm i$ $(i = \sqrt{-1})$. In any case, the units are always n-th powers, so that we can finally write

$$x + z\sqrt{-d} = \mu^n\,, \qquad \mu = \frac{u + v\sqrt{-d}}{2}\,,$$

where $u, v \in \mathbb{Z}$, with u, v both even, if $d = 1, 5$ and $u \equiv v \pmod 2$ if $d = 11, 55$. For any $\gamma \in \mathbb{Q}(\sqrt{-d})$ we denote by $\overline{\gamma}$ the conjugate of γ. Note that

$$\mu - \overline{\mu} = v\sqrt{-d}\,, \qquad \mu + \overline{\mu} = u\,, \qquad \mu\overline{\mu} = \frac{u^2 + dv^2}{4}\,.$$

We thus obtain

$$\frac{2 \cdot 5^\alpha 11^\beta}{v} = \frac{2z}{v} = \frac{\mu^n - \overline{\mu}^n}{\mu - \overline{\mu}} = (\text{by definition})\ L_n \in \mathbb{Z}\,. \tag{4.3}$$

Thus, $\frac{2z}{v}$ is the n-th term of *Lucas sequence* $(L_m)_{m\geqslant 0}$. Note that

$$L_0 = 0, \qquad L_1 = 1\,, \qquad L_m = uL_{m-1} - \frac{u^2 + dv^2}{4} L_{m-2}\,, \quad m \geqslant 2\,. \tag{4.4}$$

Following the nowadays standard strategy based on the important paper [12], we distinguish two cases according as L_n has or has not primitive divisors.

Suppose first that L_n has a primitive divisor, say q. By definition, this means that the prime q divides L_n and q does not divide $(\mu - \overline{\mu})^2 L_1 \cdots L_{n-1}$, hence

$$q \not| (\mu - \overline{\mu})^2 L_1 \cdots L_4 = (dv^2) \cdot u \cdot \frac{3u^2 - dv^2}{4} \cdot \frac{u^2 - dv^2}{2}. \tag{4.5}$$

If $q = 2$, then (4.5) implies that uv is odd, hence $d = 11$ or 55. If $d = 11$, then the third factor in the right-most side of (4.5) is even, a contradiction. If $d = 55$, then, from (4.4) we see that $L_m \equiv L_{m-1} \pmod 2$, hence L_m is odd for every $m \geqslant 1$, implying that 2 cannot be a primitive divisor of L_n.

If $q = 5$, then (4.5) implies that $d = 1, 11$ and 5 does not divide $uv(3u^2 - dv^2)(u^2 - dv^2)$. It follows easily then that $v^2 \equiv -u^2 \pmod 5$, so that, by (4.4), $L_m \equiv uL_{m-1} \pmod 5$ for every $m \geqslant 2$. Therefore, $5 \not| L_n$, so that 5 cannot be a primitive divisor of L_n.

If $q = 11$, then, by (4.5), $d = 1, 5$ and we write $u = 2u_1, v = 2v_1$ with $u_1, v_1 \in \mathbb{Z}$, so that $\mu = u_1 + v_1\sqrt{-d}$ and (4.5) becomes $q \not| u_1v_1(3u_1^2 - dv_1^2)(u_1^2 - dv_1^2)$. Moreover, $L_m = 2u_1L_{m-1} - (u_1^2 + dv_1^2)L_{m-2}$ for $m \geqslant 2$. Note that $\mu\overline{\mu} = u_1^2 + dv_1^2 \not\equiv 0 \pmod{11}$; therefore, by Corollary 2.2 of [12], there exists a positive integer m_{11} such that $11 | L_{m_{11}}$ and $m_{11} | m$ for every m such that $11 | L_m$. It follows then that $11 | \gcd(L_n, L_{m_{11}}) = L_{\gcd(n, m_{11})}$[1]. Because of the minimality property of m_{11}, we conclude that $\gcd(n, m_{11})$, hence, since n is prime, $m_{11} = n$. On the other hand, the Legendre symbol $\left(\frac{(\mu - \overline{\mu})^2}{11}\right) = -1$, hence, by Theorem XII of [18] (or by Theorem 2.2.4 (iv) of [30]), $11 | L_{12}$. Therefore $m_{11} | 12$, i.e. $n | 12$, a contradiction, since n is a prime $\geqslant 5$.

We therefore conclude that L_n has no primitive divisors. Then, by Theorem 1.4 of [12], $n < 30$. By (4.3), the prime divisors of L_n belong to $\{2, 5, 11\}$ and now, looking at the table 1 of [12], we se that the only possibility is $n = 5$ and $(u, -dv^2) = (1, -11)$, i.e. $\mu = (1 + \sqrt{-11})/2$. Going back to (4.3) we obtain no solution. ■

Remark (On the case when in (4.1) *a*, *b* and *x* are odd). We explain here why the method applied for the proof of Proposition 4 does not apply when abx is odd. In this case $d = 55$ and we work in the field $\mathbb{Q}(\theta)$, where $\theta^2 - \theta + 14 = 0$. The equation (4.1) is factorized as $(x - z + 2z\theta)(x + z - 2z\theta) = y^n$, where the factors in the left-hand side are *not* relatively prime. Then, using rather standard arguments of algebraic Number Theory, we are led to the equation

$$\frac{5^\alpha 11^\beta}{v} = z = \frac{1}{2^{2(n+1)}} \cdot \frac{(1+\theta)^{\frac{n+1}{2}} \mu^n - (2-\theta)^{\frac{n+1}{2}} \overline{\mu}^n}{\mu - \overline{\mu}},$$

which is the analogous to equation (4.3). Now, however, although the right-hand side is a term of a second order recurrence sequence, it is not a term of a *Lucas sequence* and consequently we cannot argue based on the results of [12] as we previously did.

[1] By the well-known property of Lucas sequences: $\gcd(L_m, L_k) = L_{\gcd(m,k)}$.

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Address: İsmail Naci Cangül and Musa Demirci: Department of Mathematics, Uludağ University, 16059 Bursa, Turkey;
Gökhan Soydan: Isiklar Air Force High School, 16039 Bursa, Turkey;
Nikos Tzanakis: Department of Mathematics, University of Crete, 71409 Iraklion-Crete, Greece.

E-mail: cangul@uludag.edu.tr, mdemirci@uludag.edu.tr, gsoydan@uludag.edu.tr, tzanakis@math.uoc.gr

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