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ON MAXIMAL WEIGHT SOLUTIONS IN A TRUNCATED TRIGONOMETRIC MATRIX MOMENT PROBLEM

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Abstract: The truncated trigonometric matrix moment problem is a well studied object. In particular, one can find in the literature the extremal value concerning the weight assigned to some point of the unit circle within the solution set of that matrix moment problem for the so-called non-degenerate situation and some special measure which realizes the extremal value. The primary concern of the paper is to give a basic proof for the fact that this distinguished measure is uniquely determined by that extremal feature.

Keywords: trigonometric matrix moment problem, maximal weight, unique solution.

1. Introduction

Moment problems or associated interpolation problems for holomorphic functions have been earned a fixed position in mathematical analyses and applications in engineering. There is an extensive literature on several types of such problems (see, e.g., the books [1], [3], [5], [8], [9], [11], [18], and the references therein). The present paper deals with an extremal question concerning the weight assigned to some point of the open unit disk within the solution set of a truncated trigonometric matrix moment problem.

Throughout the paper, let n be a non-negative integer, let q be a positive integer, let $\mathbb{C}$ be the set of all complex numbers, and let $\mathbb{C}^{q\times q}$ be the set of all complex $q \times q$ matrices. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and let $\mathfrak{B}_{\mathbb{T}}$ be the σ-algebra of all Borel subsets of $\mathbb{T}$. The matricial version of the classical truncated trigonometric moment problem consists of the following:

Problem (M). *Let $\mathbf{C}_0, \mathbf{C}_1, \dots, \mathbf{C}_n \in \mathbb{C}^{q\times q}$. Describe the set $\mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$ of all non-negative Hermitian $q \times q$ measures F defined on $\mathfrak{B}_{\mathbb{T}}$ such that*

$$\mathbf{C}_\ell = \int_{\mathbb{T}} z^{-\ell}\, F(\mathrm{d}z), \quad \ell = 0, 1, \dots, n.$$

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As is generally known (see, e.g., [9, Theorems 2.2.1 and 3.4.1]), the solution set $\mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$ is nonempty (with $\mathbf{C}_{-s} := \mathbf{C}_s^*$ for $s = 1, 2, \ldots, n$) if and only if the block Toeplitz matrix $(\mathbf{C}_{j-k})_{j,k=0}^n$ is non-negative Hermitian. The considerations below on Problem (M) are restricted to the so-called non-degenerate situation, i.e. to the circumstance that the corresponding block Toeplitz matrix $(\mathbf{C}_{j-k})_{j,k=0}^n$ is positive Hermitian. In doing so, for some fixed point $u \in \mathbb{T}$, the studies here tie in with those in [14, Section 9] on the extremal question regarding the maximal value of the matrix $F(\{u\})$ with respect to the Löwner semiordering of Hermitian matrices when F varies over the solution set $\mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$.

Concerning the truncated trigonometric moment problem in the scalar situation $q = 1$, a result closely related to the extremal feature stated in [14, Theorem 9.2] was obtained by Geronimus in [15, Theorem 20.1]. From today's point of view, this treatment of Geronimus is connected with those about para-orthogonal polynomials on $\mathbb{T}$ (see, e.g., [6], [7], and [16] as well as [4] and [5, Chapter 5] for a rational extension of that topic). For a comprehensive exposition of the discussion of extremal questions (similar to [15, Theorem 20.1]) associated with several scalar power moment problems we refer to Kreĭn [17, Section 2 in Chapter I] as well as to Kreĭn and Nudelman [18, Section 3 in Chapter III].

With a view to former considerations of extremal questions in the matrix case like those in the present paper we point particularly at Arov [2], where underlying sets given by certain linear fractional transformations were analyzed in this regard. Based on [2], the extremal question on maximal weight assigned to a point $u \in \mathbb{T}$ was somewhat more explicitly discussed in [14, Section 9] with respect to the matricial Carathéodory problem for the non-degenerate situation. In particular, some information about the structure of a distinguished solution which realizes the extremal value was given by using the theory of orthogonal matrix polynomials on $\mathbb{T}$. We will also mention [10], where by taking advantage of the theory of orthogonal matrix polynomials on the real line results on maximal weight solutions concerning a truncated matrix moment problems in that context were obtained.

In comparison with the classical investigations, the matrix case is more complicated. The extremal problems in question offer in the scalar case $q = 1$ solutions which are uniquely determined. For instance, this fact can be deduced from elementary results on para-orthogonal polynomials on $\mathbb{T}$. In the matrix case, it is not that simple to catch on whether this uniqueness is available or not. The bottom line is, the question concerning uniqueness in the matrix case seems to be not completely answered so far. In fact, that question is not touched in [2] (or in [10]). Moreover, in [14] just some special situations are discussed, where uniqueness can be met. In the final analysis, these special situations are closely related to the scalar case $q = 1$ in a way and far from the general state of affairs. However, we will verify that the uniqueness is generally on hand. This will be done in Section 3, based on a suitable study of the rank of associated block Toeplitz matrices. Before, in Section 2, we will briefly recall some convenient results on the distinguished solutions already come by in [14, Section 9].

2. Preliminaries

In the following (cf. [9, Section 1.2]), if P is a complex $q \times q$ matrix polynomial of degree not greater than k (with some non-negative integer k), then $P^{[k]}$ stands for the complex $q \times q$ matrix polynomial which is uniquely determined by P via

$$P^{[k]}(z) = z^k \big(P(z)\big)^*, \qquad z \in \mathbb{T}.$$

Henceforth, with a view to Problem (M), let $\mathbf{C}_0, \mathbf{C}_1, \dots, \mathbf{C}_n \in \mathbb{C}^{q\times q}$ be such that the block Toeplitz matrix $(\mathbf{C}_{j-k})_{j,k=0}^n$ is positive Hermitian (where $\mathbf{C}_{-\ell} = \mathbf{C}_\ell^*$ for $\ell = 0, 1, \dots, n$). Thus, the matrix $\mathbf{T}_\ell := (\mathbf{C}_{j-k})_{j,k=0}^\ell$ and, if $v \in \mathbb{C}$, then the values

$$A_\ell(v) := \Big(v^0\mathbf{I}_q\ v^1\mathbf{I}_q\ \cdots\ v^\ell\mathbf{I}_q\Big)\, \mathbf{T}_\ell^{-1} \begin{pmatrix} \overline{v^0}\mathbf{I}_q \\ \overline{v^1}\mathbf{I}_q \\ \vdots \\ \overline{v^\ell}\mathbf{I}_q \end{pmatrix},$$

$$C_\ell(v) := \Big(\overline{v^\ell}\mathbf{I}_q\ \overline{v^{\ell-1}}\mathbf{I}_q\ \cdots\ \overline{v^0}\mathbf{I}_q\Big)\, \mathbf{T}_\ell^{-1} \begin{pmatrix} v^\ell\mathbf{I}_q \\ v^{\ell-1}\mathbf{I}_q \\ \vdots \\ v^0\mathbf{I}_q \end{pmatrix}$$

are positive Hermitian for $\ell = 0, 1, \dots, n$, where $\mathbf{I}_q$ stands for the identity matrix of size $q \times q$. In particular, the matrix $\mathbf{C}_0$ is positive Hermitian and the matrix $\sqrt{\mathbf{C}_0}^{\,-1}$, i.e. the inverse of the non-negative Hermitian square root of $\mathbf{C}_0$, is well-defined. Moreover, the set $\mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$ is nonempty and there exist orthonormal systems $(P_\ell)_{\ell=0}^n$ and $(Q_\ell)_{\ell=0}^n$ corresponding to some $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$, where P_ℓ and Q_ℓ are complex $q \times q$ matrix polynomials of degree not greater than ℓ for $\ell = 0, 1, \dots, n$ and where

$$\left(\int_{\mathbb{T}} P_j(z)\, F(\mathrm{d}z)\, \big(P_k(z)\big)^*\right)_{j,k=0}^n = \mathbf{I}_{(n+1)q}$$

and

$$\left(\int_{\mathbb{T}} \big(Q_j(z)\big)^* F(\mathrm{d}z)\, Q_k(z)\right)_{j,k=0}^n = \mathbf{I}_{(n+1)q}$$

(cf. [9, Lemma 3.6.4] and [14, Section 4]). In fact, one can fix a distinguished pair of such orthonormal systems $[(P_\ell)_{\ell=0}^n, (Q_\ell)_{\ell=0}^n]$ which is uniquely determined by $(\mathbf{C}_\ell)_{\ell=0}^n$ via the initial condition that P_0 and Q_0, respectively, is the constant function on $\mathbb{C}$ with value $\sqrt{\mathbf{C}_0}^{\,-1}$ as well as the recurrence relations

$$P_\ell(v) = \sqrt{\mathbf{I}_q - \mathbf{E}_\ell\mathbf{E}_\ell^*}^{\,-1} \Big(vP_{\ell-1}(v) + \mathbf{E}_\ell Q_{\ell-1}^{[\ell-1]}(v)\Big)$$

and

$$Q_\ell(v) = \Big(vQ_{\ell-1}(v) + P_{\ell-1}^{[\ell-1]}(v)\mathbf{E}_\ell\Big) \sqrt{\mathbf{I}_q - \mathbf{E}_\ell^*\mathbf{E}_\ell}^{\,-1}$$

for $v \in \mathbb{C}$ and $\ell = 1, 2, \dots, n$, where the strictly contractive $q \times q$ matrix $\mathbf{E}_\ell$ is given by

$$\mathbf{E}_\ell := \pi_\ell \, \mathbf{E}_{0\ell} \, \rho_\ell$$

in what $\mathbf{E}_{0\ell}$ stands for the upper right $q \times q$ block in $\mathbf{T}_\ell^{-1}$ as well as the complex $q \times q$ matrices π_ℓ and ρ_ℓ are successively chosen such that $\pi_\ell \pi_{\ell-1}^{-1}$ and $\rho_{\ell-1}^{-1} \rho_\ell$ are positive Hermitian matrices with $\pi_0 := \sqrt{\mathbf{C}_0}$ and $\rho_0 := \sqrt{\mathbf{C}_0}$ as well as $\pi_\ell^* \pi_\ell = \big(A_\ell(0)\big)^{-1}$ and $\rho_\ell \rho_\ell^* = \big(C_\ell(0)\big)^{-1}$ (cf. [9, Section 3.6] and [14, Section 5]). Corresponding to $[(P_\ell)_{\ell=0}^n, (Q_\ell)_{\ell=0}^n]$ there is a dual pair of orthonormal systems of $q \times q$ matrix polynomials $[(P_\ell^{\#})_{\ell=0}^n, (Q_\ell^{\#})_{\ell=0}^n]$ which is given by the slightly modified recursions

$$P_\ell^{\#}(v) = \sqrt{\mathbf{I}_q - \mathbf{E}_\ell \mathbf{E}_\ell^*}^{\,-1} \Big(v P_{\ell-1}^{\#}(v) - \mathbf{E}_\ell (Q_{\ell-1}^{\#})^{[\ell-1]}(v) \Big)$$

and

$$Q_\ell^{\#}(v) = \Big(v Q_{\ell-1}^{\#}(v) - (P_{\ell-1}^{\#})^{[\ell-1]}(v) \mathbf{E}_\ell \Big) \sqrt{\mathbf{I}_q - \mathbf{E}_\ell^* \mathbf{E}_\ell}^{\,-1}$$

for $v \in \mathbb{C}$ and $\ell = 1, 2, \dots, n$, where $P_0^{\#}$ and $Q_0^{\#}$, respectively, is the constant function on $\mathbb{C}$ with value $\sqrt{\mathbf{C}_0}$ (cf. [9, Section 3.6] and [14, Section 6]).

Recall that a matrix function $\Omega : \mathbb{D} \to \mathbb{C}^{q \times q}$ which is holomorphic in the open unit disk $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ and for which $\Omega(w) + \big(\Omega(w)\big)^*$ is non-negative Hermitian for each $w \in \mathbb{D}$ is called $q \times q$ Carathéodory function (in $\mathbb{D}$). In particular, if F is a non-negative Hermitian $q \times q$ measure defined on $\mathfrak{B}_{\mathbb{T}}$, then $\Omega : \mathbb{D} \to \mathbb{C}^{q \times q}$ given by

$$w \mapsto \int_{\mathbb{T}} \frac{z+w}{z-w} \, F(\mathrm{d}z)$$

is a $q \times q$ Carathéodory function (see, e.g., [9, Theorem 2.2.2]). We will call this function Ω the Riesz–Herglotz transform of (the matrix measure) F.

Based on the interplay between matrix measures and matricial Carathéodory functions one can get descriptions of $\mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$ in terms of the associated Riesz–Herglotz transforms via linear fractional transformations, where the orthonormal matrix polynomials P_n, Q_n, $P_n^{\#}$, and $Q_n^{\#}$ of the pairs $[(P_\ell)_{\ell=0}^n, (Q_\ell)_{\ell=0}^n]$ and $[(P_\ell^{\#})_{\ell=0}^n, (Q_\ell^{\#})_{\ell=0}^n]$ are involved (see, e.g., [14, Section 6]). Following this train of thoughts, if $u \in \mathbb{T}$ and if F_u is the non-negative Hermitian $q \times q$ measure defined on $\mathfrak{B}_{\mathbb{T}}$ with Riesz–Herglotz transform $\Omega_u : \mathbb{D} \to \mathbb{C}^{q \times q}$ given by

$$\begin{aligned} w \mapsto & \Big((P_n^{\#})^{[n]}(w) \big(P_n^{[n]}(u)\big)^* + w\overline{u} Q_n^{\#}(w) \big(Q_n(u)\big)^* \Big) \\ & \times \Big(P_n^{[n]}(w) \big(P_n^{[n]}(u)\big)^* - w\overline{u} Q_n(w) \big(Q_n(u)\big)^* \Big)^{-1} \end{aligned}$$

then in view of [14, Corollary 6.6, Remark 6.8, and Proposition 9.2] and [9, Theorem 2.2.2] we know that F_u belongs to $\mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$ and that the maximal value of the matrix $F(\{u\})$ with respect to the Löwner semiordering of Hermitian matrices is precisely $F_u(\{u\})$ when F varies over the set $\mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$.

Let $u \in \mathbb{T}$. The structure of Ω_u implies that F_u admits the representation

$$F_u = \varepsilon_u \big(A_n(u)\big)^{-1} + \sum_{s=1}^{r} \varepsilon_{u_s} \mathbf{A}_s \tag{2.1}$$

with some positive integer r, pairwise different points $u_1, u_2, \ldots, u_r \in \mathbb{T}$, and non-negative Hermitian matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r \in \mathbb{C}^{q\times q}$, where ε_z denotes the Dirac measure defined on $\mathfrak{B}_{\mathbb{T}}$ for certain $z \in \mathbb{T}$ (cf. [14, Remark 9.10]). In fact, since $H_u \colon \mathbb{C} \to \mathbb{C}^{q\times q}$ given by

$$v \mapsto P_n^{[n]}(v)\big(P_n^{[n]}(u)\big)^* - v\overline{u} Q_n(v)\big(Q_n(u)\big)^*$$

is a $q \times q$ matrix polynomial of degree not greater than $n+1$, it follows that

$$\sum_{s=1}^{r} \operatorname{rank} \mathbf{A}_s \leqslant nq \tag{2.2}$$

(q.v. [14, Lemma 6.4, Lemma 9.7, Proposition 9.8, and Corollary 9.9]). Particularly (cf. [14, Remark 6.8]), in the special case $n = 0$ we get $F_u = \varepsilon_u \big(A_n(u)\big)^{-1}$.

Note casually that, since $u \in \mathbb{T}$, in any case we have apparently $A_n(u) = C_n(u)$.

3. The principal item

Somewhat surprising, beside taking advantage of some information about the structure of the distinguished solution F_u recapitulated in the preceding section and a well-known statement concerning uniqueness in the truncated trigonometric matrix moment problem, we only need the following auxiliary result. This delivers some insight into the regularity of block Toeplitz matrices corresponding to non-negative Hermitian matrix measures on $\mathfrak{B}_{\mathbb{T}}$ which are molecular (q.v. [12, Section 6]). In doing so, with a view to Problem (M), if F is a non-negative Hermitian $q \times q$ measure defined on $\mathfrak{B}_{\mathbb{T}}$, we use the notations

$$\mathbf{C}_\ell^{(F)} := \int_{\mathbb{T}} z^{-\ell} F(\mathrm{d}z) \qquad \text{and} \qquad \mathbf{T}_m^{(F)} := \left(\int_{\mathbb{T}} z^{k-j} F(\mathrm{d}z)\right)_{j,k=0}^{m} \left(= \big(\mathbf{C}_{j-k}^{(F)}\big)_{j,k=0}^{m}\right)$$

for some integer ℓ and some non-negative integer m.

Lemma 3.1. *Let m be a positive integer, let $u_0, u_1, \ldots, u_r \in \mathbb{T}$ be pairwise different points with some non-negative integer r, let $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_r \in \mathbb{C}^{q\times q}$ be non-negative Hermitian, where $\mathbf{A}_0$ is non-singular, and let the matrix measure*

$$F := \sum_{s=0}^{r} \varepsilon_{u_s} \mathbf{A}_s \tag{3.1}$$

be such that the block Toeplitz matrix $\mathbf{T}_m^{(F)}$ is non-singular. Then $r \geqslant 1$ and the matrix $\mathbf{T}_{m-1}^{(\widetilde{F})}$ is non-singular, where $\widetilde{F}$ is the matrix measure given by

$$\widetilde{F} := \sum_{s=1}^{r} \varepsilon_{u_s} \mathbf{A}_s. \tag{3.2}$$

Proof. A straightforward calculation (see, e.g., [12, Example 3.11]) yields that the rank of the block Toeplitz matrix $\mathbf{T}_m^{(\varepsilon_{u_0}\mathbf{A}_0)}$ coincides with the rank of $\mathbf{A}_0$, i.e. that rank is q. Therefore, the choice of F implies $r \geqslant 1$. We prove now by contraposition that the matrix $\mathbf{T}_{m-1}^{(\widetilde{F})}$ is non-singular. So, we suppose that $\mathbf{T}_{m-1}^{(\widetilde{F})}$ is a singular matrix. Hence (see, e.g., [12, Theorem 5.8]), we find a complex $q \times q$ matrix polynomial Q of degree not greater than $m-1$ such that $Q(v)$ is not equal to the zero matrix $0_{q\times q}$ of size $q \times q$ for some $v \in \mathbb{C}$, but

$$\int_{\mathbb{T}} \big(Q(z)\big)^* \widetilde{F}(\mathrm{d}z)\, Q(z) = 0_{q\times q}.$$

By using (3.2) and some elementary facts from integration theory with respect to non-negative Hermitian $q \times q$ measures on $\mathfrak{B}_{\mathbb{T}}$ we get

$$\sum_{s=1}^{r} \big(Q(u_s)\big)^* \mathbf{A}_s Q(u_s) = 0_{q\times q}.$$

Since the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_r$ are non-negative Hermitian, we obtain

$$\big(Q(u_s)\big)^* \mathbf{A}_s Q(u_s) = 0_{q\times q}, \qquad s = 1, 2, \dots, r.$$

Thus, $P : \mathbb{C} \to \mathbb{C}^{q\times q}$ defined by $(v - u_0)Q(v)$ for $v \in \mathbb{C}$ is a complex $q \times q$ matrix polynomial of degree not greater than m not equal to the constant function with value $0_{q\times q}$ such that

$$\begin{aligned}\int_{\mathbb{T}} \big(P(z)\big)^* F(\mathrm{d}z)\, P(z) &= \sum_{s=0}^{r} \big(P(u_s)\big)^* \mathbf{A}_s P(u_s) \\ &= \sum_{s=0}^{r} |u_s - u_0|^2 \big(Q(u_s)\big)^* \mathbf{A}_s Q(u_s) = 0_{q\times q}.\end{aligned}$$

Admittedly (see again [12, Theorem 5.8]), this is contrary to the assumption that the matrix $\mathbf{T}_m^{(F)}$ is non-singular. ■

In the following, we will write $\mathbf{A} \geqslant \mathbf{B}$ (respectively, $\mathbf{A} > \mathbf{B}$) when $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices of the same size such that $\mathbf{A} - \mathbf{B}$ is a non-negative Hermitian (respectively, positive Hermitian) matrix.

Theorem 3.2. *Let $\mathbf{C}_0, \mathbf{C}_1, \dots, \mathbf{C}_n \in \mathbb{C}^{q\times q}$ be such that the block Toeplitz matrix $(\mathbf{C}_{j-k})_{j,k=0}^n$ with $\mathbf{C}_{-s} := \mathbf{C}_s^*$ for $s = 1, 2, \dots, n$ is positive Hermitian. Furthermore, let $u \in \mathbb{T}$ as well as let $A_n(u)$ and F_u be given as in* Section 2. *Then*

$$\big(A_n(u)\big)^{-1} \geqslant F(\{u\}) \tag{3.3}$$

for each $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$, where equality holds in (3.3) *if and only if $F = F_u$.*

Proof. In view of [14, Theorem 9.2] and the matricial version of the Riesz–Herglotz Theorem (see, e.g., [9, Theorem 2.2.2]) we already know that (3.3) is fulfilled for each $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$ and that equality holds in (3.3) when F coincides with the measure F_u. It remains to prove that the equality $F(\{u\}) = \big(A_n(u)\big)^{-1}$ implies $F = F_u$ as well. In doing so, we comment firstly that the setting

$$\widetilde{F}_u := F_u - \varepsilon_u \big(A_n(u)\big)^{-1} \tag{3.4}$$

leads to a well-defined non-negative Hermitian $q \times q$ measure on $\mathfrak{B}_{\mathbb{T}}$ because of (2.1). Note that (3.4) entails particularly the relation

$$\mathbf{T}_n^{(F_u)} = \mathbf{T}_n^{(\varepsilon_u (A_n(u))^{-1})} + \mathbf{T}_n^{(\widetilde{F}_u)}. \tag{3.5}$$

By virtue of (3.4), (3.5), and $F_u \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$ one can conclude that F_u is the unique element within the set $\mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$ such that equality holds in (3.3) if and only if the solution set $\mathcal{M}[(\mathbf{C}_\ell^{(\widetilde{F}_u)})_{\ell=0}^n]$ consists of exactly one element, namely $\widetilde{F}_u$. As is known (use, e.g., [9, Lemma 1.1.7, Lemma 1.1.9, Theorem 2.2.1, Theorem 3.4.1, and Remark 3.4.3]), the latter condition is equivalent to

$$\operatorname{rank} \mathbf{T}_n^{(\widetilde{F}_u)} = \operatorname{rank} \mathbf{T}_{n-1}^{(\widetilde{F}_u)}, \tag{3.6}$$

where $\mathbf{T}_{n-1}^{(\widetilde{F}_u)}$ stands for the zero matrix in the case of $n = 0$. Therefore, to complete the proof, we will verify (3.6). If $n = 0$, then (3.6) is obviously fulfilled, since $F_u = \varepsilon_u \big(A_n(u)\big)^{-1}$ in that situation (see Section 2). We suppose now that n is a positive integer. Since $F_u \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$ and since $(\mathbf{C}_{j-k})_{j,k=0}^n$ is a non-singular $(n+1)q \times (n+1)q$ matrix, we obtain

$$\operatorname{rank} \mathbf{T}_n^{(F_u)} = \operatorname{rank} (\mathbf{C}_{j-k})_{j,k=0}^n = (n+1)q. \tag{3.7}$$

Because of (3.4) and (2.1) we get that the measure $\widetilde{F}_u$ admits the representation

$$\widetilde{F}_u = \sum_{s=1}^r \varepsilon_{u_s} \mathbf{A}_s \tag{3.8}$$

with some positive integer r, pairwise different points $u_1, u_2, \ldots, u_r \in \mathbb{T}$, and non-negative Hermitian matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r \in \mathbb{C}^{q \times q}$ such that (2.2) is satisfied. By (3.8), (2.2), and [12, Remark 3.9 and Theorem 6.6] we have on the one hand

$$\operatorname{rank} \mathbf{T}_n^{(\widetilde{F}_u)} \leqslant \sum_{s=1}^r \operatorname{rank} \mathbf{A}_s \leqslant nq$$

and, since [12, Remark 3.9 and Example 3.11] results in $\operatorname{rank} \mathbf{T}_n^{(\varepsilon_u (A_n(u))^{-1})} = q$, by using (3.7) and (3.5) we catch on the other hand

$$nq = \operatorname{rank} \mathbf{T}_n^{(F_u)} - \operatorname{rank} \mathbf{T}_n^{(\varepsilon_u (A_n(u))^{-1})} \leqslant \operatorname{rank} \mathbf{T}_n^{(\widetilde{F}_u)}.$$

Accordingly, it follows

$$\operatorname{rank} \mathbf{T}_n^{(\widetilde{F}_u)} = \sum_{s=1}^{r} \operatorname{rank} \mathbf{A}_s = nq. \tag{3.9}$$

In addition, from (2.1), (3.7), and (3.8) along with Lemma 3.1 we perceive that

$$\operatorname{rank} \mathbf{T}_{n-1}^{(\widetilde{F}_u)} = nq.$$

Finally, this equality and (3.9) give rise to (3.6) in that case. ■

Corollary 3.3. *Let $\mathbf{C}_0, \mathbf{C}_1, \ldots, \mathbf{C}_n \in \mathbb{C}^{q \times q}$ be such that the matrix $(\mathbf{C}_{j-k})_{j,k=0}^{n}$ with $\mathbf{C}_{-s} := \mathbf{C}_s^*$ for $s = 1, 2, \ldots, n$ is positive Hermitian. If $u \in \mathbb{T}$, then F_u admits (2.1) with some positive integer r, pairwise different points $u_1, u_2, \ldots, u_r \in \mathbb{T}$, and non-negative Hermitian matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r \in \mathbb{C}^{q \times q}$, where*

$$\sum_{s=1}^{r} \operatorname{rank} \mathbf{A}_s \; = \; nq. \tag{3.10}$$

In particular, if $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^{n}]$ such that $F(\mathbb{T} \setminus \Delta) = 0_{q \times q}$ and

$$\sum_{z \in \Delta} \operatorname{rank} F(\{z\}) \neq (n+1)q$$

for some set $\Delta \subseteq \mathbb{T}$, then equality does not hold in (3.3) for each $u \in \mathbb{T}$.

Proof. Let $u \in \mathbb{T}$. As already explained in Section 2, the measure F_u admits the representation (2.1) with some positive integer r, pairwise different points $u_1, u_2, \ldots, u_r \in \mathbb{T}$, and non-negative Hermitian matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r \in \mathbb{C}^{q \times q}$, where the rank estimate subject to (2.2) holds. In the special case $n = 0$, from (2.2) it follows immediately (3.10). If n is a positive integer, then the argumentation in the proof of Theorem 3.2 (see particularly (3.9)) yields (3.10). Since equality in (3.3) for some $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^{n}]$ implies $F = F_u$ according to Theorem 3.2, the remaining part of the assertion is an easy consequence of (3.10). ■

Corollary 3.4. *Let $\mathbf{C}_0, \mathbf{C}_1, \ldots, \mathbf{C}_n \in \mathbb{C}^{q \times q}$ be such that the corresponding block Toeplitz matrix $(\mathbf{C}_{j-k})_{j,k=0}^{n}$ with $\mathbf{C}_{-s} := \mathbf{C}_s^*$ for $s = 1, 2, \ldots, n$ is positive Hermitian and let $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^{n}]$. If the matrix $\mathbf{T}_{n+1}^{(F)}$ is non-singular, then*

$$\big(A_n(u)\big)^{-1} \; > \; F(\{u\}), \qquad u \in \mathbb{T}. \tag{3.11}$$

In particular, if there exist some $n+2$ pairwise different points $u_0, u_1, \ldots, u_{n+1} \in \mathbb{T}$ such that the value $F(\{u_s\})$ is non-singular for $s = 0, 1, \ldots, n+1$, then (3.11) holds.

Proof. Let $u \in \mathbb{T}$. Moreover, suppose that $\mathbf{T}_{n+1}^{(F)}$ is non-singular. Hence, we get

$$A_{n+1}(u) > A_n(u),$$

where the value $A_{n+1}(u)$ is given as in Section 2 concerning $(\mathbf{C}_\ell)_{\ell=0}^{n+1}$ in what $\mathbf{C}_{n+1} := \mathbf{C}_{n+1}^{(F)}$ (cf. [14, Remarks 4.1 and 4.2]). Therefore, Theorem 3.2 with respect to $\mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^{n+1}]$ yields

$$\big(A_n(u)\big)^{-1} > \big(A_{n+1}(u)\big)^{-1} \geqslant F(\{u\}).$$

Thus, the estimate in (3.11) is satisfied when $\mathbf{T}_{n+1}^{(F)}$ is a non-singular matrix. Finally, we consider the case that there exist $n+2$ pairwise different points $u_0, u_1, \dots, u_{n+1} \in \mathbb{T}$ such that the value $F(\{u_s\})$ is a non-singular $q \times q$ matrix for $s = 0, 1, \dots, n+1$. Since this choice implies in view of [12, Remark 5.12 and Theorem 6.11] that the matrix $\mathbf{T}_{n+1}^{(F)}$ is non-singular, from the already proven part one can reason that (3.11) is fulfilled in that situation. ■

Remark 3.5. The case $n = 0$ in Theorem 3.2 represents that, if $\mathbf{C}_0$ is a positive Hermitian $q \times q$ matrix and if $u \in \mathbb{T}$, then $\mathbf{C}_0 \geqslant F(\{u\})$ for each $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^{0}]$, where equality holds if and only if $F = \varepsilon_u \mathbf{C}_0$. The case $n = 0$ is a special situation of the given data in Problem (M) which is in a way related to that in [14, Proposition 9.13]. In particular, the uniqueness part of Theorem 3.2 for $n = 0$ is somewhat easier to catch (cf. [14, Remark 9.16]).

In addendum to Corollary 3.3, the next result emphasizes that the rank identity in (3.10) can be actually used to characterize equality in (3.3) for some $u \in \mathbb{T}$.

Proposition 3.6. *Let $\mathbf{C}_0, \mathbf{C}_1, \dots, \mathbf{C}_n \in \mathbb{C}^{q\times q}$ be such that the block Toeplitz matrix $(\mathbf{C}_{j-k})_{j,k=0}^{n}$ with $\mathbf{C}_{-s} := \mathbf{C}_s^*$ for $s = 1, 2, \dots, n$ is positive Hermitian. Furthermore, let $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^{n}]$ fulfilling* (3.1) *for some non-negative integer r, pairwise different points $u_0, u_1, \dots, u_r \in \mathbb{T}$, and non-negative Hermitian matrices $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_r \in \mathbb{C}^{q\times q}$, where $\mathbf{A}_0$ is non-singular. Then equality holds in* (3.3) *with respect to $u = u_0$ if and only if* (3.10).

Proof. Let $u = u_0$. From Theorem 3.2 and Corollary 3.3 we already know that equality in (3.3) implies (3.10). Conversely, we suppose now that (3.10) is satisfied. By virtue of [9, Lemma 1.1.7 and Lemma 1.1.9], [12, Remark 3.9 and Theorem 6.6], and (3.10) we get

$$\operatorname{rank} \mathbf{T}_n^{(F)} \leqslant \operatorname{rank} \mathbf{T}_{n+1}^{(F)} \leqslant \sum_{s=0}^{r} \operatorname{rank} \mathbf{A}_s = (n+1)q.$$

Moreover, because of $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^{n}]$ it follows that $\operatorname{rank} \mathbf{T}_n^{(F)} = (n+1)q$. Therefore, we obtain $\operatorname{rank} \mathbf{T}_n^{(F)} = \operatorname{rank} \mathbf{T}_{n+1}^{(F)}$. By using this fact in combination with some well-known results on the extension of block Toeplitz matrices (see, e.g., [9, Theorem 2.2.1, Theorem 3.4.1, and Remark 3.4.3]), the matricial version of the Riesz–Herglotz Theorem (see, e.g., [9, Theorem 2.2.2]), and some insight on descriptions of the matricial Carathéodory problem (see, e.g., [13, Corollary 2.7 and Section 5] and [14, Lemma 10.1]) one can realize that the Riesz–Herglotz transformation Ω of the measure F admits the representation

$$\Omega(w) = \Big((P_n^{\#})^{[n]}(w) - wQ_n^{\#}(w)\mathbf{U}\Big)\Big(P_n^{[n]}(w) + wQ_n(w)\mathbf{U}\Big)^{-1}, \qquad w \in \mathbb{D},$$

with some unitary $q \times q$ matrix $\mathbf{U}$. Since the choice of F subject to (3.1) yields

$$\Omega(w) = \sum_{s=0}^{r} \frac{u_s + w}{u_s - w}\mathbf{A}_s, \qquad w \in \mathbb{D},$$

where $u = u_0$ and where $\mathbf{A}_0$ is a non-singular $q \times q$ matrix, one can conclude that the value $P_n^{[n]}(u) + uQ_n(u)\mathbf{U}$ is the zero matrix $0_{q\times q}$. Hence (cf. [14, Proposition 9.8]), the function Ω coincides with the Riesz–Herglotz transformation of the measure F_u, i.e. $F = F_u$. ■

Corollary 3.7. *Let $\mathbf{C}_0, \mathbf{C}_1, \ldots, \mathbf{C}_n \in \mathbb{C}^{q\times q}$ be such that the block Toeplitz matrix $(\mathbf{C}_{j-k})_{j,k=0}^n$ with $\mathbf{C}_{-s} := \mathbf{C}_s^*$ for $s = 1, 2, \ldots, n$ is positive Hermitian. Furthermore, let $u \in \mathbb{T}$, let $\Delta \subset \mathbb{T}$ be a finite set, and let $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^n]$ be such that $F(\mathbb{T} \setminus \Delta) = 0_{q\times q}$ and*

$$\sum_{z\in\Delta} \operatorname{rank} F(\{z\}) = (n+1)q.$$

Then equality holds in (3.3) *if and only if $F(\{u\})$ is a non-singular matrix.*

Proof. Use Theorem 3.2 in combination with Proposition 3.6. ■

In the scalar case $q = 1$, the distinguished solution F_u which realizes equality in (3.3) concerning the point $u \in \mathbb{T}$ actually fulfills that extremal property with respect to all of its mass points (see, e.g., [14, Proposition 9.17]). The following example shows particularly that this state of affairs does not be generally on hand in the matrix case (cf. [14, Proposition 9.13]).

Example 3.8. Let $E := \varepsilon_1\mathbf{A}_0 + \varepsilon_{\mathrm{i}}\mathbf{A}_1 + \varepsilon_{-\mathrm{i}}\mathbf{A}_2$, where

$$\mathbf{A}_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{A}_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, let

$$\mathbf{C}_0 := \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{C}_1 := \begin{pmatrix} 1-\mathrm{i} & 0 \\ 0 & 1+\mathrm{i} \end{pmatrix}, \quad \text{and} \quad \mathbf{C}_{-1} := \begin{pmatrix} 1+\mathrm{i} & 0 \\ 0 & 1-\mathrm{i} \end{pmatrix}.$$

Then $\mathbf{T}_1^{(E)} = (\mathbf{C}_{j-k})_{j,k=0}^1$, the matrix $\mathbf{T}_1^{(E)}$ is non-singular, $\mathbf{A}_1$ and $\mathbf{A}_2$ are singular matrices, and $\mathbf{A}_0 \geqslant F(\{1\})$ for each $F \in \mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^1]$, where $F(\{1\}) = \mathbf{A}_0$ holds if and only if $F = E$.

Proof. Clearly, $\mathbf{A}_1$ and $\mathbf{A}_2$ are singular matrices and $E(\{1\}) = \mathbf{A}_0$. Moreover, we have the identities

$$\int_{\mathbb{T}} E(\mathrm{d}z) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \text{and} \qquad \int_{\mathbb{T}} z^{-1}\, E(\mathrm{d}z) = \begin{pmatrix} 1-\mathrm{i} & 0 \\ 0 & 1+\mathrm{i} \end{pmatrix}$$

which implies that $\mathbf{T}_1^{(E)} = (\mathbf{C}_{j-k})_{j,k=0}^{1}$, i.e. that the measure E belongs to $\mathcal{M}[(\mathbf{C}_\ell)_{\ell=0}^{1}]$, and that (use, e.g., [9, Lemma 1.1.9]) the matrix $\mathbf{T}_1^{(E)}$ is non-singular. The remaining part of the assertion is then a consequence of Theorem 3.2 along with Proposition 3.6. ■

In view of Example 3.8 and Proposition 3.6 one can see that equality in (3.3) for the matrix measure F_u concerning $u \in \mathbb{T}$ does not generally imply the analogous equality with respect to the trace measure trace F_u, the corresponding complex numbers trace $\mathbf{C}_0$, trace $\mathbf{C}_1, \ldots,$ trace $\mathbf{C}_n$, and the point $u \in \mathbb{T}$ (note [12, Remark 3.15 and Remark 5.14]).

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