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ON ARITHMETICAL NATURE OF TICHY-UITZ'S FUNCTION

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Abstract: In [10] R.F. Tichy and J. Uitz introduced a one parameter family g_λ, $\lambda \in (0,1)$ of singular functions. When $\lambda = 1/2$ function g_λ coincides with the famous Minkowski's question mark function. In this paper we describe the arithmetical nature of function g_λ when $\lambda = \frac{3-\sqrt{5}}{2}$.
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1. Stern-Brocot sequences

Let us remind the definition of Stern-Brocot sequences $\mathcal{F}_n$, $n = 0, 1, 2, \ldots$.

Consider two-point set $\mathcal{F}_0 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$. Let $n \geqslant 0$ and

$$\mathcal{F}_n = \left\{0 = x_{0,n} < x_{1,n} < \ldots < x_{N(n),n} = 1\right\},$$

where $x_{j,n} = p_{j,n}/q_{j,n}$, $(p_{j,n}, q_{j,n}) = 1$, $j = 0, \ldots, N(n)$ and $N(n) = 2^n + 1$. Then

$$\mathcal{F}_{n+1} = \mathcal{F}_n \cup Q_{n+1}$$

with

$$Q_{n+1} = \{x_{j-1,n} \oplus x_{j,n}, \quad j = 1, \ldots, N(n)\}.$$

Here

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+b}{c+d}$$

is the mediant of fractions $\frac{a}{b}$ and $\frac{c}{d}$.

Elements of Q_n can be characterized in the following way. Rational number $\xi \in [0,1]$ belongs to Q_n if and only if in continued fraction expansion of ξ

$$\xi = [0; a_1, a_2, \ldots, a_m] = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_m}}}, \qquad a_j \in \mathbb{N},\ a_m \geqslant 2. \qquad (1)$$

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sum of partial quotients is exactly $n+1$:

$$S(\xi) := a_1 + ... + a_m = n + 1.$$

So $\mathcal{F}_n$ consists of all rational $\xi \in [0,1]$ such that $S(\xi) \leqslant n+1$.

2. Tichy-Uitz's singular functions

In [10] R. F. Tichy and J. Uitz considered a one parameter family g_λ, $\lambda \in (0,1)$, of singular functions. In this section we describe the construction of g_λ from [10]. This construction is an inductive one.

Given $\lambda \in (0,1)$ put

$$g_\lambda(0) = g_\lambda(0/1) = 0, \qquad g_\lambda(1) = g_\lambda(1/1) = 1.$$

Suppose that $g_\lambda(x)$ is defined for all elements $x \in \mathcal{F}_n$. Then we define $g_\lambda(x)$ for $x \in Q_{n+1}$. Each $x \in Q_{n+1}$ is of the form $x = x_{j-1,n} \oplus x_{j,n}$ where $x_{j-1,n}$ and $x_{j,n}$ are consecutive elements from $\mathcal{F}_n$. Then

$$g_\lambda(x_{j-1,n} \oplus x_{j,n}) = g_\lambda(x_{j-1,n}) + (g_\lambda(x_{j,n}) - g_\lambda(x_{j-1,n}))\,\lambda.$$

So we have defined g_λ for all rational numbers from $[0,1]$. One can see that the function $g_\lambda(x)$ is a continuous function from $\mathbb{Q} \cap [0,1]$ to $[0,1]$. So it can be extended to a continuous function from the whole segment $[0,1]$ to $[0,1]$.

Similar functions $\kappa(x,\alpha)$, $x \in [0,\infty)$, $\alpha \in (0,1)$ were introduced in [2] by A. Denjoy. Definition of $\kappa(x,\alpha)$ is the following:

$$\kappa(0/1,\alpha) = 1, \qquad \kappa(1/0,\alpha) = 0,$$

and for p/q, p'/q' such that $pq' - qp' = 1$

$$\kappa(p/q \oplus p'/q', \alpha) = \alpha\kappa(p'/q', \alpha) + (1-\alpha)\kappa(p/q, \alpha).$$

For $x \in [0,1]$ functions $\kappa(x,\alpha)$ and $g_\lambda(x)$ are related in the following way:

$$\kappa(x,\alpha) = 1 - (1-\alpha)g_{1-\alpha}(x).$$

For every λ function $g_\lambda(x)$ increases in $x \in [0,1]$. By Lebesgue's theorem $g_\lambda(x)$ is a differentiable function almost everywhere. Moreover, it is easy to see that $g'_\lambda(x) = 0$ almost everywhere (in the sense of Lebesgue measure). Certain properties of functions $g_\lambda(x)$ were investigated in [10]. Some related topics can be found in [1] and [5]. Here we should note that in case $\lambda = 1/2$ function $g_{1/2}(x)$ coincides with the famous Minkowski's question mark function $?(x)$. This function may be considered as the limit distribution function for Stern-Brocot sequences $\mathcal{F}_n$. The purpose of the present paper is to explain the arithmetical nature of function $g_\lambda(x)$ when $\lambda = \frac{3-\sqrt{5}}{2}$.

3. Minkowski's function $?(x)$

Let's consider function $g_{1/2}(x) = ?(x)$. This function was introduced by Minkowski. As it follows from the definition of g_λ for $\lambda = 1/2$:

$$?(0) = ?(0/1) = 0, \qquad ?(1) = ?(1/1) = 1.$$

and for $x_{j-1,n}, x_{j,n} \in \mathcal{F}_n$

$$?(x_{j-1,n} \oplus x_{j,n}) = \frac{?(x_{j-1,n}) + ?(x_{j,n})}{2}.$$

The definition of $?(x)$ for irrational x follows by continuity.

R. Salem in [9] found a new presentation for $?(x)$. If $x \in (0,1)$ is represented in the form of regular continued fraction

$$x = [0; a_1, a_2, \ldots, a_m, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_m + \frac{1}{\ldots}}}}, \tag{2}$$

then

$$?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \frac{1}{2^{a_1+a_2+a_3-1}} - \ldots \tag{3}$$

For rational x representation (2) and consequently (3) is finite.

Minkowski's question mark function may be treated as limit distribution function for Stern-Brocot sequences in the following sense:

$$?(x) = \lim_{n\to\infty} \frac{\sharp\{\xi \in \mathcal{F}_n : \xi \leqslant x\}}{\sharp\mathcal{F}_n} = \lim_{n\to\infty} \frac{\sharp\{\xi \in \mathcal{F}_n : \xi \leqslant x\}}{2^n + 1}. \tag{4}$$

A finite formula for the right side of (4) was given by T. Rivoal in the preprint [8]. Various properties of Minkowski's question mark function were investigated in papers [2] by A. Denjoy, [11] by P. Viader, J. Paradis, L. Bibiloni and in [3] by A. A. Dushistova, I. D. Kan and N. G. Moshchevitin.

4. General form of formula (3)

Formula (3) can be generalized on the whole family of functions g_λ in the following way.

Proposition. *Let $x, \lambda \in (0,1)$ and $x = [0; a_1, \ldots, a_m, \ldots]$ be regular continued fraction expansion of x, then*

$$\begin{aligned} g_\lambda(x) &= \lambda^{a_1-1} - \lambda^{a_1-1}(1-\lambda)^{a_2} + \lambda^{a_1-1}(1-\lambda)^{a_2}\lambda^{a_3} - \ldots \\ &+ (-1)^{m+1}\lambda^{\left(\sum\limits_{1\leqslant i\leqslant m,\ i\equiv 1 (mod\, 2)} a_i - 1\right)}(1-\lambda)^{\left(\sum\limits_{1\leqslant i\leqslant m,\ i\equiv 0 (mod\, 2)} a_i\right)} + \ldots. \end{aligned} \tag{5}$$

Proof. By definition of g_λ

$$g_\lambda(0)=0, \qquad g_\lambda(1)=1$$

and

$$g_\lambda(x_{j-1,n}\oplus x_{j,n}) = g_\lambda(x_{j-1,n}) + \left(g_\lambda(x_{j,n}) - g_\lambda(x_{j-1,n})\right)\lambda, \tag{6}$$

where $x_{j-1,n}$ and $x_{j,n}$ are consecutive elements from $\mathcal{F}_n$. We can also rewrite formula (6) in the following form

$$g_\lambda(x_{j-1,n}\oplus x_{j,n}) = g_\lambda(x_{j,n}) - \left(g_\lambda(x_{j,n}) - g_\lambda(x_{j-1,n})\right)(1-\lambda). \tag{7}$$

We prove the proposition by induction on $S(x)$. The equality

$$g_\lambda(1/a_1) = \lambda^{a_1-1}$$

follows from formula (6) immediately since $1/a_1 = \underbrace{0\oplus\ldots\oplus 0\oplus}_{(a_1-1)\text{ times}}1$. Suppose that formula (5) is proved for $x=[0;a_1,\ldots,a_m]$, then it is enough to prove it for $y=[0;a_1,\ldots,a_m+1]$ and for $z=[0;a_1,\ldots,a_m-1,2]$.

Let m be odd, then by applying formula (6) we get

$$\begin{aligned} g_\lambda(y) &= g_\lambda([0;a_1,\ldots,a_{m-1}]\oplus[0;a_1,\ldots,a_m]) \\ &= g_\lambda([0;a_1,\ldots,a_{m-1}]) + \lambda(g_\lambda([0;a_1,\ldots,a_m]) - g_\lambda([0;a_1,\ldots,a_{m-1}])) \\ &= g_\lambda([0;a_1,\ldots,a_{m-1}]) + \lambda^{\sum\limits_{(1\leqslant i\leqslant m,\, i\equiv1(\mathrm{mod}2))} a_i-1}(1-\lambda)^{\sum\limits_{(1\leqslant i\leqslant m,\, i\equiv0(\mathrm{mod}2))} a_i}\lambda, \end{aligned} \tag{8}$$

and by applying formula (7) we get

$$\begin{aligned} g_\lambda(z) &= g_\lambda([0;a_1,\ldots,a_m]\oplus[0;a_1,\ldots,a_m-1]) \\ &= g_\lambda([0;a_1,\ldots,a_m-1]) - (1-\lambda)(g_\lambda([0;a_1,\ldots,a_m-1]) \\ &\quad - g_\lambda([0;a_1,\ldots,a_m])) \\ &= g_\lambda([0;a_1,\ldots,a_m-1]) - \lambda^{\sum\limits_{(1\leqslant i\leqslant m-1,\, i\equiv1(\mathrm{mod}2))} a_i+(a_m-1)-1} \\ &\quad\times(1-\lambda)^{\sum\limits_{(1\leqslant i\leqslant m,\, i\equiv0(\mathrm{mod}2))} a_i}(1-\lambda)^2. \end{aligned} \tag{9}$$

For even m the proof is similar. ∎

Similar formula for $\kappa(x,\alpha)$ was proved by A. Denjoy in [2].

5. Regular reduced continued fractions and the main result

Any real number x can be expressed uniquely in the form

$$x = [[b_0;b_1,b_2,\ldots,b_l,\ldots]] = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \ldots - \cfrac{1}{b_l - \frac{1}{\ldots}}}}, \qquad b_i\geqslant 2, \tag{10}$$

which is known as regular reduced continued fraction (ein reduziert-regalmaessiger Kettenbruch [4], [7]).

For a rational number $x \in (0,1)$ representation (10) takes the form:

$$x = [[1; b_1, \ldots, b_l]]. \tag{11}$$

For such x we denote $L(x) = b_1 + \ldots + b_l$.

Similarly to the sequence $\mathcal{F}_n$ we define the sequence Ξ_n:

$$\Xi_n := \{0,1\} \cup \left(\bigcup_{1 \leqslant k \leqslant n} \Theta_k \right),$$

where $\Theta_k = \{x \in Q : L(x) = k+1\}$, $k \geqslant 1$.

We arrange elements of Ξ_n in the increasing order:

$$\Xi_k = \{0 = \xi_{1,n} < \xi_{2,n} < \ldots < \xi_{\sharp\Xi_n,n} = 1\}.$$

The Theorem 1 stated below is the main result of present paper. It generalizes formula (4) on regular reduced continued fractions.

Theorem 1. Function g_λ, where $\lambda = \tau^2 = \frac{3-\sqrt{5}}{2}$, $\tau = \frac{\sqrt{5}-1}{2}$ coincides with the distributional function of the sequence Ξ_n, that is

$$g_{\tau^2}(x) = \lim_{n\to\infty} \frac{\sharp\{\xi \in \Xi_n : \xi \leqslant x\}}{\sharp\Xi_n}, \qquad x \in (0,1).$$

Now we consider the function

$$\mathcal{M}(x) := \lim_{n\to\infty} \frac{\sharp\{\xi \in \Xi_n : \xi \leqslant x\}}{\sharp\Xi_n}, \qquad x \in (0,1).$$

Our purpose is to prove that $\mathcal{M}(x) = g_{\tau^2}$. Function $\mathcal{M}(x)$ is increasing as a distribution function, so it is enough to prove that $\mathcal{M}(x)$ coincides with $g_{\tau^2}(x)$ for rational x, that is

$$\mathcal{M}(x \oplus y) = \mathcal{M}(x) + (\mathcal{M}(y) - \mathcal{M}(x))\,\tau^2. \tag{12}$$

for any two consecutive elements of Ξ_n for any n.

We would like to note that in special case $\lambda = \tau^2$ formula (5) gives:

$$\begin{aligned} g_{\tau^2}(x) = \tau^{2a_1-2} - \tau^{2a_1+a_2-2} + \tau^{2a_1+a_2+2a_3-2} - \ldots \\ + (-1)^{m+1}\tau^{\sum_{i=1}^{m}\alpha_i a_i - 2} + \ldots, \end{aligned} \tag{13}$$

where

$$\alpha_m = \begin{cases} 1, & \text{if } m \text{ is even}, \\ 2, & \text{if } m \text{ is odd}. \end{cases}$$

For rational x representation (13) is finite.

6. Auxiliary results

Lemma 1. Let x be represented in the form (1) and in the form (11). To get the set $(b_1, \ldots, b_l)$ from $(a_1, \ldots, a_m)$ we should replace a_i by

1. $\underbrace{2\ldots2}_{a_i-1}$ if i is odd (empty string if $a_i = 1$).
2. $a_i + 2$ if i is even and $i \neq m$.
3. $a_i + 1$ if i is even and $i = m$.

Lemma 1 can be found in [7].

Lemma 2. For a number of elements in Θ_n one has

$$\sharp\Theta_1 = 1, \qquad \sharp\Theta_2 = 1, \qquad \sharp\Theta_{n+1} = \sharp\Theta_n + \sharp\Theta_{n-1}.$$

It follows immediately from Lemma 2 that $\sharp\Theta_n$ is the nth Fibonacci number F_n, that is the nth member of the sequence

$$\{F_n\}_{n=1}^{\infty} = \{1,\, 1,\, 2,\, 3,\, 5,\, 8,\, \ldots\}$$

in which the first two terms are equal 1, and each following term is the sum of the two preceding ones. The Fibonacci numbers have a closed-form solution

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Proof of this fact can be found in [6].

Proof. We prove the lemma by induction. Since $\Theta_1 = \{1/2\}$, $\Theta_2 = \{2/3\}$, then the base of induction is true. Let us suppose that the lemma is true for $k \leqslant n$ and $x = [[1; b_1, \ldots, b_l]] \in \Theta_{n+1}$, then $b_1 + \ldots + b_l = n + 2$. There are two cases: either $b_l = 2$ or $b_l \geqslant 2$. In the first case $b_1 + \ldots + b_{l-1} = n$, so $[[1; b_1, \ldots, b_{l-1}]] \in \Theta_{n-1}$, in the second case $b_1 + \ldots + b_l - 1 = n + 1$, so $[[1; b_1, \ldots, b_l - 1]] \in \Theta_n$. Thus we have one-to-one correspondence between $\Theta_{n-1} \cup \Theta_n$ and Θ_{n+1}, and so $\sharp\Theta_{n+1} = \sharp\Theta_n + \sharp\Theta_{n-1}$. ■

Definition 1. Let x, y, z be consecutive elements of Ξ_n, $y \in \Theta_n$. We denote the mediant $x \oplus y$ by y^l and the mediant $y \oplus z$ by y^r.

Lemma 3. Let x, y, z be consecutive elements of Ξ_n, $y \in \Theta_n$, then $y^l \in \Theta_{n+2}$, $y^r \in \Theta_{n+1}$.

Proof. Let $y = [[1; b_1, \ldots, b_s]]$. Then $y^l = [[1; b_1, \ldots, b_s, 2]]$, $y^r = [[1; b_1, \ldots, b_s + 1]]$. ■

Now let us construct an infinite tree D whose nodes are labeled by rationals from $(0, 1)$. We identify nodes with rationals they are labeled by. The root is labeled by $1/2$. From the node x come two arrows: left arrow goes to x^l and right

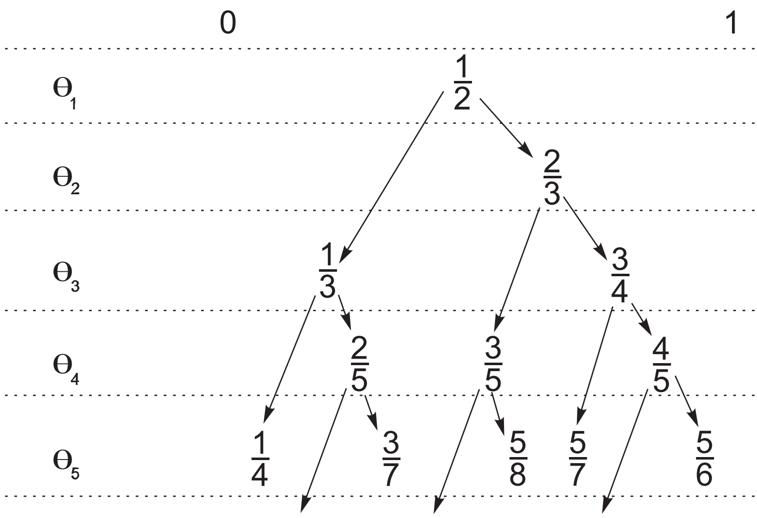

Figure 1.

arrow goes to x^r. Nodes of the tree D are partitioned into levels. $1/2$ belongs to the level 1. If x belongs to the level n, then x^r belongs to the level $n+1$, and x^l belongs to the level $n+2$ (figure 1).

It follows from the construction of the tree that nodes from level n of D are marked by numbers from Θ_n. So x belongs to the level n if and only if $x \in \Theta_n$.

We denote the subtree of D with root in the node x by $D^{(x)}$ and the set of nodes of D from level 1 to level n by D_n. Moreover, we denote the set of nodes of $D^{(x)} \cap D_n$ by $D_n^{(x)}$. Note that there exist levels preserving isomorphism between D and $D^{(x)}$. If x belongs to the level n, then

$$\sharp D_m^{(x)} = \sharp D_{m-n+1}.$$

Besides

$$\sharp D_n = \sharp\Theta_1 + \sharp\Theta_2 + \ldots + \sharp\Theta_n = F_1 + F_2 + \ldots + F_n = F_{n+2} - 1.$$

7. Proof of Theorem 1

We remind that it is enough to prove (12) for any consecutive elements of Ξ_n x and y.

To prove the equality (12) we consider the subtree $D^{(x\oplus y)}$ of D. Note that

$$\left\{\xi \in D^{(x\oplus y)}\right\} \cup \{y\} = \{\xi \in Q : x < \xi \leqslant y\}.$$

Consequently

$$\mathcal{M}(y) - \mathcal{M}(x) = \lim_{m\to\infty} \frac{\sharp\{\xi \in \Xi_m : x < \xi \leqslant y\}}{\sharp\Xi_m} = \lim_{m\to\infty} \frac{\sharp D_m^{(x\oplus y)}}{\sharp D_m}.$$

On the other hand

$$\mathcal{M}(x \oplus y) - \mathcal{M}(x) = \lim_{m\to\infty} \frac{\sharp\{\xi \in \Xi_m : x < \xi \leqslant x \oplus y\}}{\sharp\Xi_m} = \lim_{m\to\infty} \frac{\sharp D_m^{(x\oplus y)^l}}{\sharp D_m}.$$

Let $x \oplus y \in \Theta_k$, then $(x \oplus y)^l \in \Theta_{k+2}$. Therefore

$$\frac{\mathcal{M}(x \oplus y) - \mathcal{M}(x)}{\mathcal{M}(y) - \mathcal{M}(x)} = \lim_{m\to\infty} \frac{\sharp D_m^{(x\oplus y)^l}}{\sharp D_m^{(x\oplus y)}} = \lim_{m\to\infty} \frac{\sharp D_{m-k-1}}{\sharp D_{m-k+1}} = \lim_{m\to\infty} \frac{F_{m-k+1}}{F_{m-k+3}} = \tau^2. \quad \blacksquare$$

References

[1] J.C. Alexander and D.B. Zagier, *The entropy of a certain infinitely convolved Bernoulli measures*, J. London Math. Soc. **44** (1991), 121–134.

[2] A. Denjoy, *Sur une fonction reele de Minkowski*, J. Math. Pures Appl. **17** (1938), 105–151.

[3] A.A. Dushistova, I.D. Kan and N.G. Moshchevitin, *Differentiability of the Minkowski question mark function*, Preprint available at arXiv:0903.5537v1.pdf (2009)

[4] Yu. Yu Finkel'shtein, *Klein polygons and reduced regular continued fractions*, Russian Mathematical Surveys **48**(3) (1993), 198.

[5] J.P. Graber, P. Kirschenhofer and R.F. Tichy, *Combinatorial and arithmetical properties of linear numeration systems*, Combinatorica **22**(2) (2002), 245–267.

[6] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, Oxford, 1980.

[7] O. Perron, *Die Lehre von den Kettenbruchen*, Bd.I.Teuber, 1954.

[8] T. Rivoal, *Suites de Stern-Brocot et fonction de Minkowski*, Preprint available at http ://www-fourier.ujf-grenoble.fr/ rivoal

[9] R. Salem, *On some singular monotonic functions which are strictly increasing*, Trans. Amer. Math. Soc. **53** (1943), 427–439.

[10] R.F. Tichy and J. Uitz, *An extension of Minkowski's singular function*, Appl. Math. Lett. **8** (1995), 39–46.

[11] P. Viader, J. Paradis and L. Bibiloni, *A new light of Minkowski's ?(x) function*, J. Number Theory. **73** (1998), 212–227.

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