Functiones et Approximatio
43.1 (2010), 7–14

ON THE AVERAGE NUMBER OF UNITARY FACTORS OF FINITE ABELIAN GROUPS

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Abstract: We give a slight improvement on a well-known problem.
Keywords: exponential sums, abelian groups.

1. Introduction

For a finite abelian group G, let $t(G)$ be the number of unitary factors of G. For the group-theoritic description of this quantity, the reader is referred to the paper [K], in which the asymptotic behaviuor of $T^*(x)$ was first considered, where x is a sufficiently large positive number, and

$$T^*(x) = \sum t(G),$$

the summation is taken over all abelian groups of order not exceeding x. In [K] the author succeeded in reducing the problem to a problem of estimating exponential sums, by showing that if there holds (ϵ is any small positive constant)

$$\sum_{n \leqslant x} d(1,1,2;n) = \text{Main terms} + O\left(x^{\theta+\epsilon}\right), \qquad 0 < \theta < \frac{1}{2}, \tag{1.1}$$

where $d(1,1,2;n)$ is the number of ordered triple lattice points (r,s,t) such that $rst^2 = n$ (r, s and t are positive integers). The interest comes from the fact that the error term of (1.1) can be estimated by techniques of exponential sums. Thus in [K] the author first got a permissible value $\theta = \frac{11}{29}$. Subsequently this was improved in [S] ($\theta = \frac{3}{8}$), [L1] ($\theta = \frac{77}{208}$), [L2]($\theta = \frac{29}{80}$), [W1] ($\theta = \frac{47}{131}$) and [W2] ($\theta = \frac{45}{127} = 0.354\ldots$). In an unpublished manuscript, we once got a result $\theta = \frac{101}{283} = 0,356\ldots$. In this paper, we first derive a new bound for exponential sums of the shape

$$S = \sum_{w \sim W} \sum_{n \sim N} \phi_w e\left(A w^{\frac{1}{2}} n^{-1}\right) \qquad (|\phi_w| \leqslant 1), \tag{1.2}$$

2000 Mathematics Subject Classification: primary: 11L07; secondary: 11N45

by using Theorem 1 of the recent work of [L3] instead of Lemma 2.1 of [W2], and then we shall use it in the method of [W2] to get a better θ in (1.1).

Theorem 1.1. $\theta = \frac{63}{178} = 0.353\ldots$ *is a permissible value.*

2. An estimate for the special double exponential sum S

Let $F = |A|W^{\frac{1}{2}}N^{-1} \ll 1, M \gg 1, N \gg 1$. We proceed to estimate the exponential sum of (1.2). By Weyl's inequality and a fair splitting of range, we have

$$L^{-1}S^2 \ll (WN)^2Q^{-1} + W^{\frac{3}{2}}NQ^{-1} \sum_{q\sim Q_1} \sum_{n-q,n+q\sim N} \sum_{w\sim W} w^{-\frac{1}{2}} e\left(Aw^{\frac{1}{2}}t(n,q)\right), \tag{2.1}$$

$Q \in \left(10, NL^{-1}\right)$ is a parameter, $L = \log(FMN+2), 1 \leqslant Q_1 \leqslant Q, t(n,q) = (n-q)^{-1} - (n+q)^{-1}$. Denote the multiple sum on the right of (2.1) by S_1. If $FQ_1N^{-1} \leqslant \epsilon W$, using a familiar inequality we have (first removing the factor $w^{-1/2}$ by a partial summation)

$$S_1 \ll Q_1NW^{-\frac{1}{2}} \left(\frac{FQ_1}{WN}\right)^{-1} := Q_1NW^{-\frac{1}{2}}V^{-1}, \tag{2.2}$$

where $V = \frac{FQ_1}{WN}$. Assume that $V > \epsilon$. Using Theorem 1 of [RS] we get

$$\begin{aligned}\sum_{w\sim W} w^{-\frac{1}{2}} e\left(Aw^{\frac{1}{2}}t(n,q)\right) &= i \sum_{V_1<v<V_2} v^{-\frac{1}{2}} e\left(F_1(n,q,v)\right) \\ &\quad + O\left(W^{-\frac{1}{2}}\left(L + \left(\frac{W}{V}\right)^{\frac{1}{2}} + V^{-1}\right)\right),\end{aligned} \tag{2.3}$$

(the last term on the right side seems unnecessary when $V > \epsilon$, but it will be clear that it can be used to absorb the contribution of (2.2) in case $V \leqslant \epsilon$) here

$$F_2(n,q,v) = \left(\frac{1}{2}At(n,q)\right)^2 v^{-1}, \quad V_1 = \frac{1}{2}A(2W)^{-\frac{1}{2}}t(n,q), \quad V_2 = \frac{1}{2}AW^{-\frac{1}{2}}t(n,q).$$

Since

$$t(n,q) = 2qn^{-2}\left(1 + O\left(Q_1N^{-2}\right)\right),$$

we have, for $V_3 = Aq(2W)^{-\frac{1}{2}}n^{-2}, V_4 = AqW^{-\frac{1}{2}}n^{-2}$,

$$\sum_{V_1<v<V_2} v^{-\frac{1}{2}}e(F_2) = \sum_{V_3<v<V_4} v^{-\frac{1}{2}}e(F_2) + O\left(V^{-\frac{1}{2}}\left(1 + V\left(\frac{Q_1}{N}\right)^2\right)\right). \tag{2.4}$$

From (2.3) and (2.4), exchanging the order of summation we have (when $V > \epsilon$)

$$\begin{aligned}S_1 &\ll V^{-\frac{1}{2}} \sum_{q\sim Q_1} \sum_{v\approx V} \left| \sum_{n\in I(q,v)} e(F_2(n,q,v)) \right| \\ &\quad + Q_1NV^{-\frac{1}{2}}\left(L + \left(\frac{V}{W}\right)^{\frac{1}{2}} + V\left(\frac{Q_1}{N}\right)^2 + (WV)^{-\frac{1}{2}}\right),\end{aligned} \tag{2.5}$$

where $I(q,v)$ is a small interval (depending on q,v) contained in $[N+q, 2N-q]$. Denote the multiple sum on the right of (2.5) by S_2 (without the $V^{-\frac{1}{2}}$ factor). We relax the condition for n by a familiar tool (see Lemma 2.3 of [SW] for instance), and then use Weyl's inequality to get

$$L^{-2}S_2^2 \ll \frac{(Q_1NV)^2}{Q_2} + \frac{Q_1NV}{Q_2} \sum_{1\leqslant|q_1|\leqslant Q_2} \left| \sum_{q\sim Q_1} \sum_{v\approx V} \sum_{n\in I_1(q)} e(F_2(n,q,v,q_1)) \right|, \tag{2.6}$$

where $Q_2 \in (10, NL^{-1})$ is another parameter, $I_1(q) = (N+q, 2N-q)$ and

$$F_2(n,q,v,q_1) = F_1(n-q_1,q,v) - F_1(n+q_1,q,v) = \frac{1}{4}A^2v^{-1}(t^2(n-q_1,q) - t^2(n+q_1,q)).$$

Using the easily obtained expansion

$$\left(\frac{(1-X)^{-1}-(1+X)^{-1}}{2X}\right)^2 = 1 + \sum_{k\geqslant 1} D_k X^{2k} \quad \left(0 < X < \frac{1}{2}\right),$$

$$F_2(n,q,v,q_1) = A^2v^{-1}q^2 \left\{ (n+q_1)^{-4} - (n-q_1)^{-4} + \sum_{k\geqslant 1} D_k q^{2k} T_k(n,q_1) \right\}, \tag{2.7}$$

where $T_k(n,q_1) = (n+q_1)^{-4-2k} - (n-q_1)^{-4-2k}$. Without loss the generality we may suppose that $q_1 > 0$. Denote the inner triple sum of (2.6) as S_3. If $FQ_1q_1N^{-3} \leqslant \epsilon$, we can use a familiar estimate to get

$$S_3 \ll Q_1V(FQ_1q_1N^{-3})^{-1}. \tag{2.8}$$

Assume $FQ_1q_1N^{-3} > \epsilon$. Then by Theorem 1 of [RS] we get

$$\begin{aligned} \sum_{n\in I_1(q)} e(F_2(n,q,v,q_1)) = -i \left(\sum_{U_1<u<U_2} G_1(u,q,q_1,v) e(G_2(u,q,q_1,v)) \right) \\ + O\left((FQ_1q_1N^{-4})^{-\frac{1}{2}} + L \right), \end{aligned} \tag{2.9}$$

where (to avoid confusion we use the same letter to denote also a real variable)

$$\begin{aligned} G_1(u,q,q_1,v) &= \left(-\frac{\partial^2 F_2}{\partial n^2}(n(u,q,q_1,v),q,q_1,v) \right)^{-\frac{1}{2}}, \\ G_2(u,q,q_1,v) &= F_2(n(u,q,q_1,v),q,q_1,v) - un(u,q,q_1,v), \end{aligned}$$

$n(u,q,q_1,v)$ is the solution of

$$\frac{\partial F_2}{\partial n}(n,q,q_1,v) = u,$$

and $U_1 = \frac{\partial F_2}{\partial n}(2N - q_1, q, q_1, v), U_2 = \frac{\partial F_2}{\partial n}(N + q_1, q, q_1, v)$. From (2.7) we get

$$\frac{\partial F_2}{\partial n}(n, q, q_1, v) = 40A^2 v^{-1} q^2 q_1 n^{-6} \left(1 + O\left(\frac{Q_1^2 + q_1^2}{N^2}\right)\right), \tag{2.10}$$

thus we have

$$\begin{aligned}\sum_{U_1<u<U_2} G_1 e(G_2) = &\sum_{U_3<u<U_4} G_1 e(G_2) \\ &+ O\left(\left(\frac{FQ_1q_1}{N^4}\right)^{-\frac{1}{2}} \left(1 + \frac{FQ_1q_1\left(Q_1^2+q_1^2\right)}{N^5}\right)\right),\end{aligned} \tag{2.11}$$

where $U_3 = 40A^2v^{-1}q^2q_1(2N-q_1)^{-6}, U_4 = 40A^2v^{-1}q^2q_1(N+q_1)^{-6}$. From (2.9) and (2.11), exchanging the order of summation, we have

$$\begin{aligned}S_3 \ll &\sum_{u \approx U} \left| \sum_{(q,v)\in E} G_1 e(G_2) \right| \\ &+ Q_1 V \left(\left(FQ_1q_1N^{-4}\right)^{-\frac{1}{2}} + L + N^3\left(FQ_1q_1\right)^{-1} + \frac{\left(FQ_1q_1\right)^{\frac{1}{2}}}{N^3}\left(Q_1^2+q_1^2\right)\right),\end{aligned}$$

where $U = FQ_1q_1N^{-3}$ and $E = \{(q,v) | q \sim Q_1, v \approx V, U_3(q,v) < u < U_4(q,v)\}$. Denote the above inner double summation for $(q,v) \in E$ by S_4. We use the partial summation for two variables (see [B] or [M] for such a result) to remove the factor G_1 and get

$$S_4 \ll \left(NU^{-1}\right)^{\frac{1}{2}} \left| \sum_{(q,v)\in E_1} e(G_2(u,q,q_1,v)) \right|, \tag{2.12}$$

where $E_1 = E \cap \{(q,v) | Q' < q < Q'', V' < v < V''\}$ for certain Q', $Q'' \approx Q_1$, V', $V'' \approx V$. From (2.10) we have

$$n(u,q,q_1,v) = \left(40A^2v^{-1}q^2q_1u^{-1}\right)^{\frac{1}{6}}(1+O(\Delta)), \Delta = \frac{Q_1^2+q_1^2}{N^2}.$$

Thus we find that for $P(v,q_1) = CA^{\frac{1}{6}}v^{-\frac{1}{6}}q_1^{\frac{1}{6}} \quad (C \neq 0)$

$$F_2(n(u,q,q_1,v),q,q_1,v) = P(v,q_1)q^{\frac{2}{6}}u^{\frac{5}{6}}(1+O(\Delta)).$$

Similarly we get for $1 \leqslant i+j \leqslant 5$ (here $(\xi)_i = \xi(\xi-1)\dots(\xi-i+1)$)

$$\frac{\partial^{i+j}F_2}{\partial q^i \partial u^j}(n(u,q,q_1v),q,q_1,v) = P(v,q_1)\left(\frac{2}{6}\right)_i\left(\frac{5}{6}\right)_j q^{\frac{2}{6}-i}u^{\frac{5}{6}-j}(1+O(\Delta)).$$

We use Weyl's inequality (after relaxing the summation restriction for v) to get

$$L^{-2}S_4^2 \ll NU^{-1}\left\{(VQ_1)^2R^{-1}+VQ_1R^{-1}\sum_{1\leqslant|r|\leqslant R}\left|\sum_{(q,v)\in E_r} e(G_3(u,q,q_1,v,r))\right|\right\}, \tag{2.13}$$

where $R\in(10,VL^{-1})$ is a parameter, $E_r=\{(q,v)|q\sim Q_1,\,(v+r)\approx V,\,(v-r)\approx V\}$, and

$$G_3(u,q,q_1,v,r)=G_2(u,q,q_1,v+r)-G_2(u,q,q_1,v-r).$$

Now we are in a position of using Theorem 1 of [L3] to estimate the inner double exponential sum of (2.13). We assume

$$Q_2\leqslant Q_1. \tag{2.14}$$

Then we have, with

$$F_1=\frac{FQ_1|q_1|R}{N^2V},\qquad \Delta=Q_1^2N^{-2}+R^2V^{-2},\qquad \Phi=0,$$

$$L^{-6}S_4^2\ll NU^{-1}\Big\{(VQ_1)^2R^{-1}+\sqrt[6]{F_1^2(VQ_1)^9}+\sqrt[6]{F_1^{-2}(VQ_1)^{13}}+\sqrt[10]{F_1^2(V^{19}Q_1^{15}+V^{15}Q_1^{19})\Delta^4}\Big\}=\sum_{1\leqslant i\leqslant 4}T_i.$$

Note that if we take $R=\frac{N^2V^2}{F|q_1|}$, then $F_1=VQ_1, T_2=T_3$. Thus we choose

$$R=\min\left(\frac{N^2V^2}{F|q_1|},VL^{-1}\right),$$

and thus in (2.6) we get

$$\begin{aligned}L^{-8}S_2^2\ll{}&\frac{(NVQ_1)^2}{Q_2}+\sqrt{FV^3Q_1^5Q_2}+\sqrt{F^2V^2N^{-2}Q_1^5Q_2^2}\\&+\sqrt[12]{F^6V^{23}Q_1^{29}Q_2^6}+\sqrt[12]{F^4V^{25}N^4Q_1^{29}Q_2^4}\\&+\sqrt[20]{(V^{41}Q_1^{35}+V^{37}Q_1^{39})F^{10}N^{-8}Q_2^{10}}\\&+\sqrt[20]{(V^{49}Q_1^{47}+V^{45}Q_1^{51})F^2N^{16}Q_2^2}\\&+Q_1^2NV^2+\sqrt{FQ_1^9N^{-4}V^4Q_2}\\&+\sqrt{N^6V^4F^{-1}Q_1^3Q_2^{-1}}+Q_1N^4V^2F^{-1}Q_2^{-1}.\end{aligned} \tag{2.15}$$

(Note that in view of (2.8), the bound between (2.11) and (2.12) holds also in case $FQ_1q_1N^{-3}<\epsilon$). We impose the conditions $Q_2\leqslant FQ_1N^{-2}$ and $FQ_1N^{-2}>1$, thus in (2.15) the last two terms can be neglected as compared with the first term. We then use Lemma 2.4 of [SW] to choose (see (2.14)) $Q_2 \in (0, \min(FQ_1N^{-2}, Q_1L^{-1}))$ optimally in (2.15) and get

$$\begin{aligned} L^{-5}S_2 \ll\ & \sqrt[6]{FV^5Q_1^7N^2} + \sqrt[8]{F^2Q_1^9N^2V^6} + \sqrt[36]{F^6V^{35}Q_1^{41}N^{12}} \\ & + \sqrt[32]{F^4Q_1^{37}V^{33}N^{12}} + \sqrt[60]{(V^{61}Q_1^{55} + V^{57}Q_1^{59})F^{10}N^{12}} \\ & + \sqrt[44]{(V^{53}Q_1^{51} + V^{49}Q_1^{55})F^2N^{20}} + \sqrt[6]{FQ_1^{11}N^{-2}V^6} \\ & + \sqrt{N^2V^2Q_1} + \sqrt{N^4V^2Q_1F^{-1}}. \end{aligned} \tag{2.16}$$

In view of the last term of (2.16), we find that (2.16) holds also in case $FQ_1N^{-2} \leqslant 1$. Put this in (2.5), we get

$$\begin{aligned} L^{-6}S_1 \ll\ & \sqrt[6]{FV^2Q_1^7N^2} + \sqrt[8]{F^7V^2Q_1^9N^2} + \sqrt[36]{F^6V^{17}Q_1^{41}N^{12}} \\ & + \sqrt[32]{F^4Q_1^{37}V^{17}N^{12}} + \sqrt[60]{(V^{31}Q_1^{55} + V^{27}Q_1^{59})F^{10}N^{12}} \\ & + \sqrt[44]{(V^{31}Q_1^{29} + V^{27}Q_1^{33})F^2N^{20}} + \sqrt[6]{FQ_1^{11}N^{-2}V^3} \\ & + \sqrt{N^2VQ_1} + \sqrt{N^4VQ_1F^{-1}} + Q_1NV^{-\frac{1}{2}} \\ & + Q_1W^{-\frac{1}{2}}N + Q_1^3N^{-1}V^{\frac{1}{2}} + Q_1NV^{-1}W^{-\frac{1}{2}}. \end{aligned} \tag{2.17}$$

In view of (2.2) we see that (2.17) holds also in case $V \leqslant \epsilon$. Thus by (2.1) and (2.17) we get

$$\begin{aligned} L^{-7}S^2 \ll\ & (NW)^2Q^{-1} + \sqrt[6]{F^3Q^3W^7N^6} + \sqrt[8]{F^4Q^3W^{10}N^8} \\ & + \sqrt[36]{F^{23}Q^{22}W^{37}N^{31}} + \sqrt[32]{F^{21}Q^{18}W^{31}N^{27}} \\ & + \sqrt[60]{F^{41}Q^{42}W^{61}N^{43}} + \sqrt[60]{F^{37}Q^{42}W^{65}N^{47}} \\ & + \sqrt[44]{F^{33}Q^{16}W^6N} + \sqrt[44]{F^{29}Q^{16}W^{39}N^{37}} \\ & + \sqrt[6]{F^4Q^8NW^6} + \sqrt{FN^3W^2} + \sqrt{N^5W^2} \\ & + \sqrt{FQ^5W^2N^{-1}} + N^3W^2F^{-1}Q^{-1} + \sqrt{N^5W^4F^{-1}Q^{-1}}. \end{aligned} \tag{2.18}$$

Assume that $F \geqslant N$ and $Q \leqslant FN^{-1}$. Then the last two terms of (2.18) can be neglected as compared with the first term. We choose optimally a parameter $Q \in (0, \min(NL^{-1}, FN^{-1}))$ to infer

$$\begin{aligned} L^{-4}S \ll\ & \sqrt[18]{F^3W^{13}N^{12}} + \sqrt[22]{F^4W^{16}N^{14}} + \sqrt[116]{F^{23}W^{81}N^{75}} \\ & + \sqrt[100]{F^{21}W^{67}N^{65}} + \sqrt[172]{F^{41}W^{111}N^{93}} + \sqrt[172]{F^{37}W^{115}N^{97}} \\ & + \sqrt[120]{F^{33}W^{67}N^{65}} + \sqrt[120]{F^{29}W^{71}N^{69}} + \sqrt[4]{FW^2N^3} \\ & + \sqrt[4]{N^5W^2} + \sqrt[14]{FN^9W^{12}} + \sqrt{NW^2} + \sqrt{F^{-1}N^3W^2}. \end{aligned} \tag{2.19}$$

In view of the last term of (2.19), we see that (2.19) holds also in case $F < N$.

3. Proof of Theorem 1.1

We use (2.19) to replace Lemma 2.1 of [W2] in the proof of (3.1) of [W2] for estimating the three-dimensional exponential sum of the shape

$$\sum_{h\sim H}\sum_{m\sim M}\sum_{n\sim N} a_h e\left(\frac{hx}{mn^2}\right).$$

Thus we can obtain, instead of (4.4) of [W2], the following estimate

$$\begin{aligned} L^{-5}S(H_0,\mathbf{N}) \ll \Big(& \sqrt[18]{H_0^2G^7N_1^{12}N_2^5} + \sqrt[22]{H_0^3G^9N_1^{14}N_2^6} + \sqrt[116]{H_0^{11}G^{46}N_1^{75}N_2^{35}} \\ & + \sqrt[100]{H_0^5G^{38}N_1^{65}N_2^{33}} + \sqrt[172]{H_0^5G^{66}N_1^{93}N_2^{61}} \\ & + \sqrt[172]{H_0^9G^{66}N_1^{97}N_2^{57}} + \sqrt[120]{H_0^{-13}G^{40}N_1^{65}N_2^{53}} \\ & + \sqrt[120]{H_0^{-9}G^{40}N_1^{69}N_2^{49}} + \sqrt[28]{H_0^6G^{12}N_1^{17}N_2^6} \\ & + \sqrt[4]{H_0^{-1}GN_1^3N_2^2} + \sqrt[4]{H_0^{-2}N_1^5N_2^2} + \sqrt[14]{H_0^4G^6N_1^9N_2^2} \\ & + \sqrt{H_0GN_1} + N_1^{\frac{3}{2}}\Big) H_0 := \left(\sum_{1\leqslant i\leqslant 14} G_i\right) H_0. \end{aligned}$$

Using this and the bound for $S(H_0,\mathbf{N})$ given at the last lines in proving Lemma 4.1 of [W2], we thus get ($\delta=\epsilon^2$)

$$x^{-\delta}S(H_0,\mathbf{N}) \ll \left(\sum_{i\neq 7,8,10,11} G_i + \sum_{i=7,8,10,11} G_i^* + \sqrt[8]{x^3N_1^{-2}} + x^{1/3}\right) H_0, \quad (3.1)$$

where (by letting $\sigma = \sqrt[35]{H_0^3G^{14}N_1^{22}N_2^{11}}$) $G_i^* = \min(G_i,\sigma)$. Using the usual manner to diminish the power of H_0, we can get (using $G := \left(xN_1^{-a}N_2^{-b}\right)^{\frac{1}{c}} \ll xN_1^{-2}N_2^{-1}$ for any permutation (a,b,c) of $(1,1,2)$)

$$\begin{aligned} G_7^* &\ll \sqrt[815]{G^{302}N_1^{481}N_2^{302}} \ll \sqrt[815]{x^{302}N_1^{-123}} := T_1, \\ G_8^* &\ll \sqrt[225]{G^{82}N_1^{135}N_2^{82}} \ll \sqrt[225]{x^{82}N_1^{-29}} := T_2, \\ G_{10}^* &\ll \sqrt[82]{G^{28}N_1^{59}N_2^{28}} \ll \sqrt[82]{x^{28}N_1^{3}} := T_3, \\ G_{11}^* &\ll \sqrt[47]{G^{17}N_1^{31}N_2^{17}} \ll \sqrt[47]{x^{17}N_1^{-3}} := T_4. \end{aligned}$$

We then insert (3.1) into (4.3) of [W2] and choose H_0 optimally to get

$$\begin{aligned} x^{-\delta}S(\mathbf{u},\mathbf{N};x) \ll & \sqrt[25]{x^9N_1^{-1}} + \sqrt[127]{x^{46}N_1^{-6}} + \sqrt[105]{x^{38}N_1^{-5}} + \sqrt[177]{x^{66}N_1^{-34}} \\ & + \sqrt[181]{x^{66}N_1^{-26}} + \sqrt[34]{x^{12}N_1^{-1}} + \sqrt[18]{x^6N_1} + N_1^{3/2} \\ & + \sqrt[8]{x^3N_1^{-2}} + \sum_{1\leqslant i\leqslant 4} T_i + x^{\theta}, \end{aligned} \tag{3.2}$$

where $\theta = \frac{63}{178}$. Now if $N_1 > x^{\frac{98}{534}} =: x^{\rho}$ our result follows from Lemma 4.1 of [W2], and if $N_1 < x^{\frac{187}{1068}} := x^{\eta}$ our result follows from Lemma 4.2 of [W2]. Thus we can assume $x^{\eta} \leqslant N_1 \leqslant x^{\rho}$, and our result follows from (3.2). The proof is finished.

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Received: 13 July 2009; **revised:** 19 August 2009